## 2.1: limits of sequences.

- An infinite sequence (briefly, a sequence) is a function whose domain in $\mathbb{N}$.

- A sequence $x_n := f(n)$ will be donated by $x_1, x_2, \ldots$ or $\{x_n\}_{n\in\mathbb{N}}$ or $\{x_n\}$.

**Def 1:** A sequence of real numbers $\{x_n\}$ is said to be converges to $a \in \mathbb{R}$ iff $\varepsilon > 0$ $\exists$ an $K \in \mathbb{N}$ (in general $K(\varepsilon)$) s.t $n \geq K \Rightarrow |x_n - a| < \varepsilon$.

**RMK:** A sequence can have at most one limit. الـ قيمة بتكون unique

**Def 2:** A subsequence of a sequence of the form $\{x_{n_k}\}_{k\in\mathbb{N}}$ where each $n_k \in \mathbb{N}$ and $n_1 < n_2 < \cdots$ Thus, a subsequence $x_{n_1}, x_{n_2}, \ldots$ of $x_1, x_2, \ldots$ is obtained by deleting from $x_1, x_2, \ldots$ all $x_n$'s except those such that $n = n_k$ for some $k$.

**RMK:** If $\{x_n\}_{n\in\mathbb{N}}$ converges to $\alpha$ and $\{x_{n_k}\}_{k\in\mathbb{N}}$ is any subsequence of $\{x_n\}_{n\in\mathbb{N}}$ then $x_{n_k} \to \alpha$ as $k \to \infty$. أي seq. بتكون converge to α فلازم subseq. تبعها converge to α بس العكس غير صحيح

**Def 3:** let $\{x_n\}$ be a sequence of real number, Then

1. $\{x_n\}$ is said to be bounded above iff the set $\{x_n : n \in \mathbb{N}\}$ is bounded above, i.e iff $\exists$ an $M \in \mathbb{R}$ s.t $x_n \leq M$ $\forall n \in \mathbb{N}$.

2. $\{x_n\}$ is said to be bounded below iff the set $\{x_n : n \in \mathbb{N}\}$ is bounded below, iff $\exists$ an $m \in \mathbb{R}$ s.t $x_n \geq m$ $\forall n \in \mathbb{N}$.

3. $\{x_n\}$ is said to be bounded iff it is bounded both above and below, $\exists$ a $C > 0$ s.t $|x_n| \leq C$, $\forall n \in \mathbb{N}$.

**Thm:** every convergent sequence is bounded but the converse is not true.

## 2.2: limits Theorem.

[squeeze Theorem] Thm 1: suppose that $\{x_n\}$, $\{y_n\}$ and $\{w_n\}$ are real sequences.

1. If $x_n \to a$ and $y_n \to a$ (same a) as $n \to \infty$ and if $\exists$ an $N_0 \in \mathbb{N}$ s.t $x_n \leq w_n \leq y_n$ for $n \geq N_0$ then $w_n \to a$ as $n \to \infty$.

2. If $x_n \to 0$ as $n \to \infty$ and $\{y_n\}$ is bounded, then $x_n y_n \to 0$ as $n \to \infty$.

Thm 2: let $E \subseteq \mathbb{R}$ if $E$ has a finite supremum (respectively, a finite infimum) then there is a sequence $x_n \in E$ s.t $x_n \to \sup E$ (respectively, a seq. $y_n \in E$ s.t $y_n \to \inf E$) as $n \to \infty$.

Thm 3: suppose that $\{x_n\}$ and $\{y_n\}$ are real seq. and that $\alpha \in \mathbb{R}$ If $\{x_n\}$ and $\{y_n\}$ convergent then

1. $\lim_{n\to\infty}(x_n + y_n) = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n$.
2. $\lim_{n\to\infty}(\alpha x_n) = \alpha \lim_{n\to\infty} x_n$.
3. $\lim_{n\to\infty}(x_n y_n) = \lim_{n\to\infty} x_n \lim_{n\to\infty} y_n$
4. If in addition $y_n \neq 0$ and $\lim_{n\to\infty} y_n \neq 0$, then $\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{\lim_{n\to\infty} x_n}{\lim_{n\to\infty} y_n}$.

Def: let $\{x_n\}$ be a sequence of real numbers:

1. $\{x_n\}$ is said to be diverges to $+\infty$ ($x_n \to \infty$ as $n \to \infty$) iff for each $M \in \mathbb{R}$ there is an $N \in \mathbb{N}$ s.t $n \geq N$ implies $x_n > M$.

2. $\{x_n\}$ is said to be diverges to $-\infty$ ($x_n \to -\infty$ as $n \to \infty$) iff for each $M \in \mathbb{R}$ there is an $N \in \mathbb{N}$ s.t $n \geq N$ implies $x_n < M$.

Thm 4: suppose that $\{x_n\}$ and $\{y_n\}$ are real seq. s.t $x_n \to \infty$ ($x_n \to -\infty$) as $n \to \infty$

1. If $y_n$ is bounded below (resp. $y_n$ is bounded above) then $\lim_{n\to\infty} (x_n + y_n) = \infty$ (resp. $\lim_{n\to\infty} (x_n + y_n) = -\infty$)

2. If $\alpha > 0$ then $\lim_{n\to\infty} (\alpha x_n) = +\infty$ (resp. $\lim_{n\to\infty} \alpha x_n \to -\infty$).

3. If $y_n > M_0$ for some $M_0 > 0$ and all $n \in \mathbb{N}$ then $\lim_{n\to\infty} (x_n y_n) = +\infty$ (resp. $\lim_{n\to\infty} (x_n y_n) = -\infty$).

4. If $\{y_n\}$ is bounded and $x_n \neq 0$ then $\lim_{n\to\infty} \frac{y_n}{x_n} = 0$

RMK:

1. $x + \infty = \infty$ , $x - \infty = -\infty$ $\forall x \in \mathbb{R}$.
2. $x \cdot \infty = \infty$ , $x \cdot -\infty = -\infty$ $x > 0$
3. $x \cdot \infty = -\infty$ , $x \cdot -\infty = \infty$ $x < 0$.
4. $\infty + \infty = \infty$ , $-\infty - \infty = -\infty$
5. $\infty \cdot \infty = (-\infty) \cdot (-\infty) = \infty$ And $(-\infty) \cdot \infty = \infty(-\infty) = -\infty$.

Corollary: let $\{x_n\}, \{y_n\}$ be real sequences and $\alpha, x, y$ be extended real numbers

If $x_n \to x$ and $y_n \to y$ as $n \to \infty$ then $\lim_{n\to\infty} (x_n + y_n) = x + y$.

Thm5: Comparison Theorem: suppose that $\{x_n\}$ and $\{y_n\}$ are convergent sequences. If there is an $N_0 \in \mathbb{N}$ s.t $x_n \leq y_n$ for $n \geq N_0$ Then $x = \lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n = y$.

- In particular, If $x_n \in [a,b]$ converge to some point $c$ then $c$ must be belong to $[a,b]$.

RMK: $x_n < y_n$ , $n \geq N_0$ does not imply that $\lim_{n\to\infty} x_n < \lim_{n\to\infty} y_n$

## 2.3: Bolzano - Weirstrass Theorem:

Def: let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of Real number

i. $\{x_n\}$ is said to be increasing (resp. strictly increasing) iff $x_1 \le x_2 \le \dots$ (resp. $x_1 < x_2 < \dots$)

ii. $\{x_n\}$ is said to be decreasing (resp. strictly decreasing) iff $x_1 \ge x_2 \ge \dots$ (resp. $x_1 > x_2 > \dots$)

iii. $\{x_n\}$ said Monotone iff it is either increasing or decreasing.

RMK:

1. some times, we call decreasing seq. Non increasing and increasing seq. nondecreasing.
2. If $\{x_n\}$ is increasing (resp. decreasing) and $x_n \to a$ as $n \to \infty$, write $x_n \uparrow a$ (resp. $x_n \downarrow a$) as $n \to \infty$.
3. every strictly increasing seq. is increasing and every strictly decreasing seq. is decreasing.
4. $\{x_n\}$ is increasing iff the sequence $\{-x_n\}$ is decreasing.

Thm 1: Monotone convergence Thereom (MCT):

If $\{x_n\}$ is increasing and bounded above, or $\{x_n\}$ is decreasing and bounded below then $\{x_n\}$ converges to a finite limit.

bdd above + increasing $\to$ converge.

bdd below + decreasing $\to$ converge.

Def: A sequence of sets $\{I_n\}_{n\in\mathbb{N}}$ is said to be nested iff $I_1 \supseteq I_2 \supseteq \dots$.

Thm 2: Nested Interval property:

If $\{I_n\}_{n\in\mathbb{N}}$ is a nested sequence of nonempty close bdd intervals, then $E := \bigcap_{n=1}^{\infty} I_n \ne \emptyset$. Moreover, if the lengths of these intervals satisfy $|I_n| \to 0$ as $n \to \infty$ then $E$ is a single point.

Thm 3: every bounded sequence of real numbers has a convergent subsequence.

## 2.4: Cauchy sequences.

Def: A sequence of points $x_n \in \mathbb{R}$ is said to be cauchy in $\mathbb{R}$ iff $\forall \varepsilon > 0$ $\exists$ an $N \in \mathbb{N}$ s.t $m, n \geq N$ implies $|x_n - x_m| < \varepsilon$.

RMK: If $\{x_n\}$ is convergent then $\{x_n\}$ is cauchy.
The converse of the above is also true for real seq.

Thm1: cauchy:
let $\{x_n\}$ be a seq. of real numbers Then $\{x_n\}$ is cauchy iff $\{x_n\}$ converges.

RMK: every cauchy seq. is bounded But converse not true.