من للقدس إلا أنت

كلٌّ محاولاتك عند الله أجور

استعن بالله ولا تعجز

1. Let

$$A = \begin{bmatrix} 3 & 2 & 4 \\ 1 & -2 & 3 \\ 2 & 3 & 2 \end{bmatrix}$$

(a) Find the values of $\det(M_{21})$, $\det(M_{22})$, and $\det(M_{23})$.

(b) Find the values of A_{21}, A_{22}, and A_{23}.

(c) Use your answers from part (b) to compute $\det(A)$.

(a)

$\det(M_{21}) = \begin{vmatrix} 2 & 4 \\ 3 & 2 \end{vmatrix} = 4 - 12 = -8$

$\det(M_{22}) = \begin{vmatrix} 3 & 4 \\ 2 & 2 \end{vmatrix} = 6 - 8 = -2$

$\det(M_{23}) = \begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix} = 9 - 4 = 5$

(b) $A_{21} = (-1)^3 \cdot -8 = 8$

$A_{22} = (-1)^4 \cdot -2 = -2$

$A_{23} = (-1)^5 \cdot 5 = -5$

(c) $|A| = 8 + (-2)(-2) + 3(-5)$

$= 8 + 4 - 15 = -3$

Q. evaluate the following determinate :-

(d) $\begin{vmatrix} 4 & 3 & 0 \\ 3 & 1 & 2 \\ 5 & -1 & -4 \end{vmatrix}$

(d) $|A| = a_{13}A_{13} + a_{23}A_{23} + a_{33}A_{33}$

$= 0 + 2 \times (-1)^5 \times \begin{vmatrix} 4 & 3 \\ 5 & -1 \end{vmatrix} + (-4) \times (-1)^6 \times \begin{vmatrix} 4 & 3 \\ 3 & 1 \end{vmatrix}$

$= -2(-4 - 15) - 4(-5)$

$= 38 + 20$

$= 58$

(g) $\begin{vmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 6 & 2 & 0 \\ 1 & 1 & -2 & 3 \end{vmatrix}$

$|A| = a_{14}A_{14} + \cancel{a_{24}A_{24}}^{\,0} + \cancel{a_{34}A_{34}}^{\,0} + a_{44}A_{44}$

$= (1)(-1)^5 \begin{vmatrix} 0 & 1 & 0 \\ 1 & 6 & 2 \\ 1 & 1 & -2 \end{vmatrix} + 3 \times (-1)^8 \times \begin{vmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 6 & 2 \end{vmatrix}$

$= + \begin{vmatrix} 1 & 1 & -2 \\ 1 & 6 & 2 \\ 0 & 1 & 0 \end{vmatrix} + (3)(1)(2)(2)$ 12

$-R_1 + R_2$

$= \begin{vmatrix} 1 & 1 & -2 \\ 0 & 5 & 4 \\ 0 & 1 & 0 \end{vmatrix} + 12$

$= -\begin{vmatrix} 1 & -2 & 1 \\ 0 & 4 & 5 \\ 0 & 0 & 1 \end{vmatrix} + 12$

$= -(1)(4)(1) + 12 = \boxed{8}$

5. Evaluate the following determinant. Write your answer as a polynomial in x:

$$\begin{vmatrix} a-x & b & c \\ 1 & -x & 0 \\ 0 & 1 & -x \end{vmatrix}$$

$|A| = a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$

$= (a-x)(-1)^2 \times \begin{vmatrix} -x & 0 \\ 1 & -x \end{vmatrix} + (1)(-1)^3 \begin{vmatrix} b & c \\ 1 & -x \end{vmatrix} + 0$

$= (a-x)(x^2) - (-xb - c)$

$= ax^2 - x^3 + xb - c$

$= -x^3 + ax^2 + xb - c$

6. Find all values of λ for which the following determinant will equal 0:

$$\begin{vmatrix} 2-\lambda & 4 \\ 3 & 3-\lambda \end{vmatrix}$$

$\det(A) = 0 \rightarrow$ means that A is singular.

$(2-\lambda)(3-\lambda) - 12 = 0$

$6 - 2\lambda - 3\lambda + \lambda^2 - 12 = 0$

$\lambda^2 - 5\lambda - 6 = 0$

$(\lambda - 6)(\lambda + 1) = 0$

$\lambda = 6, \ \lambda = -1$

9. Prove that if a row or a column of an $n \times n$ matrix A consists entirely of zeros, then $\det(A) = 0$.

to evaluate the determinant of amatrix, $|A| = a_{1n}A_{1n} + a_{2n}A_{2n} + \dots + a_{in}A_{in}$
where $a_{1n}, a_{2n} \dots a_{in}$ are the elements of the row that it zero.
then, any thing multiple by zero gonna be zero.

11. Let A and B be 2×2 matrices.

(a) Does $\det(A + B) = \det(A) + \det(B)$?

(b) Does $\det(AB) = \det(A)\det(B)$?

(c) Does $\det(AB) = \det(BA)$?

Justify your answers.

(a) No, by counter example :-

$A = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$, $B = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}$

$\det(A) = 1$, $\det(B) = 1$

① $\det(A+B) = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$

② $\det(A) + \det(B) = 1 + 1 = 2$

they are not equivilant.

(b) $\det(AB) = \det(A)\det(B)$.

Case 1 :- if B is singular, then $\det(B) = 0$

$AB = 0$, by taking det for both sides.

① $\det(AB) = 0$

② $\det(A) . \det(B)$
$= \det(A) . 0 = 0$

so they are equal.

Case 2 :- if B is non singular, then $\det(B) \neq 0$.

and since B is nonsingular, then B is row equivilant to I.

$B = E_k E_{k-1} \dots E_2 E_1 I = E_k E_{k-1} \dots E_2 E_1$

$\det(AB) = \det(A E_k E_{k-1} \dots E_2 E_1 I)$
$= \det(A E_k E_{k-1} \dots E_2 E_1) \det(I)$
$= \det(A E_k E_{k-1} \dots E_2) \det(E_1)$
$= \det(A E_k E_{k-1} \dots E_3) \det(E_2) \det(E_1)$
$\vdots$
$= \det(A) \det(E_k) \det(E_{k-1}) \dots \det(E_1)$.

$= \det(A) . \det(\underbrace{E_k E_{k-1} \dots E_2 E_1}_{B})$.

$= \det(A) . \det(B)$

(c) $\det(AB) = \det(A) \cdot \det(B)$, $\det(A) = k$, $\det(B) = d \longrightarrow$ they are constants

$= k \cdot d$

$= d \cdot k$

$= \det(B) \cdot \det(A)$

$= \det(BA)$ #

ch.2.3

2. Let

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \\ -2 & -2 & 3 & 3 \\ 1 & 2 & -2 & -3 \end{bmatrix}$$

(a) Use the elimination method to evaluate $\det(A)$.

(b) Use the value of $\det(A)$ to evaluate

$$\begin{vmatrix} 0 & 1 & 2 & 3 \\ -2 & -2 & 3 & 3 \\ 1 & 2 & -2 & -3 \\ 1 & 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \\ -1 & -1 & 4 & 4 \\ 2 & 3 & -1 & -2 \end{vmatrix}$$

(2) $A = \begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \\ -2 & -2 & 3 & 3 \\ 1 & 2 & -2 & -3 \end{vmatrix} = - \begin{vmatrix} 1 & 2 & -2 & -3 \\ 1 & 1 & 1 & 1 \\ -2 & -2 & 3 & 3 \\ 0 & 1 & 2 & 3 \end{vmatrix} \underset{\substack{-R_1+R_2 \\ 2R_1+R_3}}{=} - \begin{vmatrix} 1 & 2 & -2 & -3 \\ 0 & -1 & 3 & 4 \\ 0 & 2 & -1 & -3 \\ 0 & 1 & 2 & 3 \end{vmatrix}$

$\underset{\substack{2R_2+R_3 \\ R_2+R_4}}{=} - \begin{vmatrix} 1 & 2 & -2 & -3 \\ 0 & -1 & 3 & 4 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & 5 & 7 \end{vmatrix} \underset{-R_3+R_4}{=} - \begin{vmatrix} 1 & 2 & -2 & -3 \\ 0 & -1 & 3 & 4 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & 0 & 2 \end{vmatrix} = -(1)(-1)(5)(2)$

$= \underline{\underline{10}}$

~~**3.**~~ For each of the following, compute the determinant and state whether the matrix is singular or nonsingular:

(e) $\begin{bmatrix} 2 & -1 & 3 \\ -1 & 2 & -2 \\ 1 & 4 & 0 \end{bmatrix}$

(f) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & -1 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 7 & 3 \end{bmatrix}$

(e) $\underset{\substack{\frac{1}{2}R_1+R_2 \\ -\frac{1}{2}R_1+R_3}}{=} \begin{vmatrix} 2 & -1 & 3 \\ 0 & \frac{3}{2} & \frac{-1}{2} \\ 0 & \frac{9}{2} & \frac{-3}{2} \end{vmatrix} \underset{-3R_2+R_3}{=} \begin{vmatrix} 2 & -1 & 3 \\ 0 & \frac{3}{2} & \frac{-1}{2} \\ 0 & 0 & 0 \end{vmatrix}$

row of zeros then the $\det(A) = 0$

So A is singular.

f) $-2R_1+R_2 \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -3 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 7 & 3 \end{vmatrix} \underset{\frac{1}{3}R_2+R_3}{=} \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & \frac{7}{3} & 1 \\ 0 & 0 & 7 & 3 \end{vmatrix}$

$-3R_3+R_4$

$$= \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & \frac{7}{3} & 1 \\ 0 & 0 & 0 & 0 \end{vmatrix}$$

$\det(A) = 0$ So its singular

4. Find all possible choices of c that would make the following matrix singular:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 9 & c \\ 1 & c & 3 \end{bmatrix}$$

$-R_1+R_2$
$-R_1+R_3$

$$\begin{vmatrix} 1 & 1 & 1 \\ 0 & 8 & -1+c \\ 0 & -1+c & 2 \end{vmatrix} = (1) \times (-1)^2 \times \begin{vmatrix} 8 & -1+c \\ -1+c & 2 \end{vmatrix}$$

$$= 16 - (-1+c)^2$$
$$= 16 - (1 - 2c + c^2)$$
$$0 = -c^2 + 2c + 15$$
$$0 = c^2 - 2c - 15$$
$$(c+3)(c-5) = 0$$
$$c = 5, \quad c = -3$$

5. Let A be an $n \times n$ matrix and α a scalar. Show that

$$\det(\alpha A) = \alpha^n \det(A)$$

αI is a matrix wich all the diagonals equals α. So

$$|\alpha I| = \alpha_{11} \cdot \alpha_{22} \cdot \alpha_{33} \cdots \alpha_{nn} = \alpha^n$$

$$\alpha I A = (\alpha I)(A)$$

$$\det(\alpha A) = \det(\alpha I A)$$
$$= \det(\alpha I) \cdot \det(A)$$
$$= \alpha^n \cdot \det(A)$$

\#

6. Let A be a nonsingular matrix. Show that

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

$$A \cdot A^{-1} = I$$

$$\det(A \cdot A^{-1}) = \det(I)$$
$$\det(A) \cdot \det(A^{-1}) = 1, \quad \det(A) \neq 0, \text{ Since it's non Singular}$$

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

7. Let A and B be 3×3 matrices with $\det(A) = 4$ and $\det(B) = 5$. Find the value of

(a) $\det(AB)$ **(b)** $\det(3A)$

(c) $\det(2AB)$ **(d)** $\det(A^{-1}B)$

(a) $= \det(A)\det(B$
$= 4 \times 5 = 20$

(b) $= 3^3 \times \det(A)$
$= 27 \times 4$
$= 108$

(c) $= 2^3 \times \det(A) \times \det(B)$
$= 8 \times 4 \times 5$
$= 160$

(d) $= \det(A^{-1})\det(B)$
$= \frac{1}{\det(A)} \cdot \det(B)$
$= \frac{1}{4} \times 5 = \frac{5}{4}$

8. Show that if E is an elementary matrix, then E^T is an elementary matrix of the same type as E.

if E is from type I or II then it's Symmetric So $E^T = E$ of the same type.
and if E is an elemantary matrix from III, the formal form of the identity matrix by adding a row to a row times by number then E^T will be an elementary matrix of type III from the identity.

9. Let E_1, E_2, and E_3 be 3×3 elementary matrices of types I, II, and III, respectively, and let A be a 3×3 matrix with $\det(A) = 6$. Assume, additionally, that E_2 was formed from I by multiplying its second row by 3. Find the values of each of the following:

(a) $\det(E_1A)$ **(b)** $\det(E_2A)$

(c) $\det(E_3A)$ **(d)** $\det(AE_1)$

(e) $\det(E_1^2)$ **(f)** $\det(E_1E_2E_3)$

(a) $= \det(E_1)\det(A)$
$= -6.$

(b) $= \det(E_2).\det(A)$
$= 3.6 = 18$

(c) $= \det(E_3)\det(A)$
$= 1.6 = 6$

(d) $= \det(A).\det(E_1)$
$= 6.-1 = -6$

(e) $= \det(E_1).\det(E_1)$
$= (-1).(-1) = 1$

(f) $= \det(E_1).\det(E_2).\det(E_3)$
$= -1.3.1 = -3$

14. Let A and B be $n \times n$ matrices. Prove that the product AB is nonsingular if and only if A and B are both nonsingular.

if $\det(AB) = 0$, then $\det(A) \cdot \det(B) = 0$

either $\det(A) = 0$, or $\det(B) = 0$ or $\det(A)$ and $\det(B)$ are both $\underline{0}$

Which is Contradiction.

A or B or both are Singular.

16. A matrix A is said to be *skew symmetric* if $A^T = -A$. For example,

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

is skew symmetric, since

$$A^T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -A$$

If A is an $n \times n$ skew-symmetric matrix and n is odd, show that A must be singular.

$A^T = -A$

$\det(A^T) = \det(-A)$

$\det(A^T) = (-1)^n \det(A)$, since n is odd and $\det(A^T) = \det(A)$.

$\det(A) = -\det(A)$

$2\det(A) = 0$

$\det(A) = 0 \rightarrow$ So A is singular.

1. For each of the following, compute (i) $\det(A)$, (ii) adj A, and (iii) A^{-1}:

(a) $A = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}$ **(b)** $A = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}$

(c) $A = \begin{array}{c} \begin{pmatrix} 1 & 3 & 1 \\ 2 & 1 & 1 \\ -2 & 2 & -1 \end{pmatrix} \end{array} \begin{array}{c} 13 \\ 23 \\ 33 \end{array}$

(d) $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

(C) $|A| = a_{13}A_{13} + a_{23}A_{23} + a_{33}A_{33}$

$= 6 + (-1)^5 8 + 5$

$= 3$

$\text{Adj}(A) =$

(b) * $|A| = 12 - 2 = 10$

* $\text{adj}\, A = \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}$

* $A^{-1} = \frac{1}{10} \begin{bmatrix} 4 & -1 \\ -2 & 3 \end{bmatrix}$

2. Use Cramer's rule to solve each of the following systems:

(a) $x_1 + 2x_2 = 3$
$3x_1 - x_2 = 1$

(b) $2x_1 + 3x_2 = 2$
$3x_1 + 2x_2 = 5$

(c) $2x_1 + x_2 - 3x_3 = 0$
$4x_1 + 5x_2 + x_3 = 8$
$-2x_1 - x_2 + 4x_3 = 2$

(d) $x_1 + 3x_2 + x_3 = 1$
$2x_1 + x_2 + x_3 = 5$
$-2x_1 + 2x_2 - x_3 = -8$

(b) $A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$, $b = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$, $|A| = -5$

$x_1 = \dfrac{\begin{vmatrix} 2 & 3 \\ 5 & 2 \end{vmatrix}}{-5} = \dfrac{-11}{-5} = \dfrac{11}{5}$

$x_2 = \dfrac{\begin{vmatrix} 2 & 2 \\ 3 & 5 \end{vmatrix}}{-5} = \dfrac{-4}{5}$

$A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 1 \\ -2 & 2 & -1 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 5 \\ -8 \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$,

$x_1 = \dfrac{\begin{vmatrix} 1 & 3 & 1 \\ 5 & 1 & 1 \\ -8 & 2 & -1 \end{vmatrix}}{10} = \dfrac{-6}{-3} = 2$

$|A| = \begin{bmatrix} 1 & 3 & 1 \\ 0 & -5 & -1 \\ 0 & \underline{8} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 1 \\ 0 & -5 & -1 \\ 0 & 0 & \frac{-3}{5} \end{bmatrix} = -3$

$X_2 = \dfrac{\begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 1 \\ -2 & -8 & -1 \end{bmatrix}}{-3} = \dfrac{-3}{-3} = 1$

$X_3 = \dfrac{\begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 5 \\ -2 & 2 & -8 \end{bmatrix}}{-3} = \dfrac{6}{-3} = -2$

3. Given

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 4 & 3 \\ 1 & 2 & 2 \end{bmatrix}$$

determine the $(2, 3)$ entry of A^{-1} by computing a quotient of two determinants.

$|A| = (1).(+4) = +4.$

$A_{32} = (-1)^5 . 3 = -3$

$A^{-1}_{23} = \dfrac{1}{|A|} A_{32}$

$= \dfrac{3}{4}$

4. Let A be the matrix in Exercise 3. Compute the third column of $\underline{A^{-1}}$ by using Cramer's rule to solve $A\mathbf{x} = \mathbf{e}_3$.

$a^{-1}_{13} \longrightarrow A^{-1}_{13} = \dfrac{1}{|A|} A_{31} = \dfrac{1}{4} . 2 = \dfrac{1}{2}$

a^{-1}_{23}

a^{-1}_{33}

$A^{-1}_{23} = \dfrac{1}{|A|} . A_{32} = \dfrac{1}{4} . (-1)^5 . 3 = \dfrac{-3}{4}$

$A^{-1}_{33} = \dfrac{1}{|A|} . A_{33} = \dfrac{1}{4} . (-1)^6 . (4) = 1$

6. If A is singular, what can you say about the product $A \operatorname{adj} A$?

if A is non singular, $A.\operatorname{adj}(A) = \det(A).I$
and then $A . \frac{\operatorname{adj}(A)}{\det(A)} = I$

if A is Singular then $A . \frac{\operatorname{adj}(A)}{\det(A)} \neq I$, Since $\det(A) = 0$

So $\operatorname{adj}^{-}(A) = 0$

8. Let A be a nonsingular $n \times n$ matrix with $n > 1$. Show that

$$\det(\operatorname{adj} A) = (\det(A))^{n-1}$$

Since $A.\operatorname{Adj}(A) = \det(A).I$

$\det(A. \operatorname{Adj}(A)) = \det(\det(A)\, I)$.

$\det(A). \det(\operatorname{adj}(A) = \det(A)^n. \det(I)$

$\det(\operatorname{adj}(A)) = \frac{\det(A)^n}{\det(A)} . 1$

$= \det(A)^{n-1}$

10. Show that if A is nonsingular, then $\operatorname{adj} A$ is nonsingular and

$$(\operatorname{adj} A)^{-1} = \det(A^{-1})A = \operatorname{adj} A^{-1}$$

11. Show that if A is singular, then $\operatorname{adj} A$ is also singular.

12. Show that if $\det(A) = 1$, then

$$\operatorname{adj}(\operatorname{adj} A) = A$$

$A. \operatorname{adj}(A) = \det(A) . I$

Since,

$A^{-1} = \frac{\operatorname{adj}(A)}{\det(A)}$, $\det(A^{-1}) = \frac{1}{\det(A)}$

$A^{-1} = \operatorname{adj}(A). \det(A^{-1})$

$\times A$ $\times A$

$\frac{I}{\operatorname{adj}(A)} = \frac{\operatorname{adj}(A)}{\operatorname{adj}(A)} . \det(A^{-1}) A.$

$(\operatorname{adj}(A))^{-1} = \det(A^{-1}) A$

$[\operatorname{adj}(A) = |A| . A^{-1}]^{-1}$

$\operatorname{adj}^{-1}(A) = A. |A|^{-1}$

$\operatorname{adj}^{-1}(A) = \frac{A}{|A|}$

$\operatorname{adj}(A) = |A|.A^{-1}$

$\operatorname{adj}(A^{-1}) = |A^{-1}|.A$

11 $adj(A) = |A| \cdot A^{-1}$, if A is singular, then $|A| = 0$

$adj(A) = 0$

$\det(adj(A)) = \det(0)$

$\det(adj(A)) = 0 \rightarrow$ so $adj(A)$ is also singular, since the determinate equals zero.

12 if $|A| = 1$, $(adj(A) = |A| \cdot A^{-1}) \; adj$

$= A^{-1}$

$$adj(adj(A)) = adj(A^{-1})$$
$$= |A^{-1}| \cdot A$$
$$= \frac{1}{|A|} \cdot A = 1 \cdot A = A \; /\!/$$