$$V_1 = Z_{11} I_1 + Z_{12} I_2 \quad \text{①}$$
$$V_2 = Z_{21} I_1 + Z_{22} I_2 \quad \text{②}$$

$$V_1 = V_g - I_1 Z_g \quad \text{③}$$
$$V_2 = -Z_L I_2 \quad \text{④}$$

1 $\quad Z_{in} = \dfrac{V_1}{I_1}$

eq ② $\quad V_2 = Z_{21} I_1 + Z_{22} I_2$

④ $\quad -Z_L I_2 = Z_{21} I_1 + Z_{22} I_2$

$$-(Z_L + Z_{22}) I_2 = Z_{21} I_1$$

$$\boxed{I_2 = \dfrac{-Z_{21}}{Z_L + Z_{22}} I_1} \quad \text{ⓐ}$$

eq ① $\quad V_1 = Z_{11} I_1 + Z_{12} I_2 \qquad$ sub ⓐ in ①

$$= Z_{11} I_1 - \dfrac{Z_{12} Z_{21}}{Z_L + Z_{22}} I_1$$

$$\therefore \boxed{Z_{in} = \dfrac{V_1}{I_1} = Z_{11} - \dfrac{Z_{12} Z_{21}}{Z_{22} + Z_L}}$$

2 $\quad I_2 = ??$

from ⓐ $\quad \boxed{I_1 = -\dfrac{Z_L + Z_{22}}{Z_{21}} I_2} \quad \text{Ⓑ}$

eq ① $\quad V_1 = Z_{11} I_1 + Z_{12} I_2$

eq ③ in ①

$$V_g - I_1 Z_g = Z_{11} I_1 + Z_{12} I_2$$

$$Z_{12} I_2 = V_g - (Z_g + Z_{11}) I_1 \qquad \text{sub Ⓑ}$$

$$\dfrac{Z_{21} Z_{12} I_2}{Z_{21}} = V_g + \dfrac{(Z_g + Z_{11})(Z_L + Z_{22})}{Z_{21}} I_2$$

$$\left(\dfrac{Z_{12} Z_{21} - (V_g + Z_{11})(Z_L + Z_{22})}{Z_{21}} \right) I_2 = V_g$$

$$\boxed{I_2 = \dfrac{Z_{21} V_g}{Z_{12} Z_{21} - (V_g + Z_{11})(Z_L + Z_{22})}}$$

$\boxed{3}$ $V_{Th} = ??$

$$V_{Th} = V_2 \Big|_{I_2 = 0}$$

$\therefore$ eq ① $\quad V_1 = Z_{11} I_1 \quad \text{——} Ⓒ$

eq ② $\quad V_2 \Big|_{I_2 = 0} = Z_{21} I_1 \quad \text{——} Ⓓ$

Sub Ⓒ in Ⓓ

$\longrightarrow V_{Th} = \dfrac{Z_{21}}{Z_{11}} V_1 \longleftarrow$

from ③ $\quad V_1 = V_g - I_1 Z_g \quad$ Sub in V_{Th}

$V_{Th} = \dfrac{Z_{21}}{Z_{11}} \left(V_g - \boxed{I_1 Z_g} \right) \longleftarrow$

$\qquad \downarrow \text{from D}$

$= \dfrac{Z_{21}}{Z_{11}} \left(V_g - \dfrac{V_{Th}}{Z_{21}} Z_g \right)$

$\therefore V_{Th} \left(1 + \dfrac{Z_{21}}{Z_{11}} \dfrac{Z_g}{Z_{21}} \right) = \dfrac{Z_{21}}{Z_{11}} V_g$

$V_{Th} \left(\dfrac{Z_{11} + Z_g}{Z_{11}} \right) = \dfrac{Z_{21}}{Z_{11}} V_g$

$$\boxed{V_{Th} = \dfrac{Z_{21}}{Z_{11} + Z_g} V_g}$$

OR

$V_1 = Z_{11} I_1$

$V_g - I_1 Z_g = Z_{11} I_1$

$\therefore \boxed{I_1 = \dfrac{V_g}{Z_{11} + Z_g}}$

$\longrightarrow V_{Th} = \dfrac{Z_{21}}{Z_{11}} \left(V_g - \dfrac{Z_g V_g}{Z_{11} + Z_g} \right)$

$= V_g \left(\dfrac{Z_{21}}{Z_{11}} - \dfrac{Z_{21} Z_g}{Z_{11}(Z_{11} + Z_g)} \right)$

$= \dfrac{Z_{21}}{Z_{11}} V_g \left(1 - \dfrac{Z_g}{Z_{11} + Z_g} \right)$

$= \dfrac{Z_{21}}{Z_{11}} V_g \left(\dfrac{Z_{11}}{Z_{11} + Z_g} \right)$

$$\boxed{V_{Th} = \dfrac{Z_{21}}{Z_{11} + Z_g} V_g}$$

[4] Thev. impedance = output impedance

$$Z_{Th} = \left.\frac{V_2}{I_2}\right|_{V_g=0} \qquad V_g \text{ is S.C.}$$

when $V_g = 0$, eq ③ becomes

$$V_1 = -I_1 Z_g$$

sub in ①

$$-I_1 Z_g = Z_{11} I_1 + Z_{12} I_2$$

$$I_1 (Z_{11} + Z_g) = -Z_{12} I_2$$

$$\boxed{I_1 = -\frac{Z_{12}}{Z_{11}+Z_g} I_2} \quad — \text{ⓔ}$$

sub ⓔ in ②

$$\left.V_2\right|_{V_g=0} = Z_{21} I_1 + Z_{22} I_2$$

$$= \left(\frac{-Z_{21} Z_{12}}{Z_{11}+Z_g} + Z_{22}\right) I_2$$

$$\therefore \boxed{\left.\frac{V_2}{I_2}\right|_{V_g=0} = Z_{Th} = Z_{22} - \frac{Z_{12} Z_{21}}{Z_{11}+Z_g}}$$

[5] current gain $\dfrac{I_2}{I_1}$

Directly from Ⓐ or Ⓑ

$$\boxed{\frac{I_2}{I_1} = \frac{-Z_{21}}{Z_L + Z_{22}}}$$

5 Voltage current gain $\left(\dfrac{V_2}{V_1}\right)$

from ② $\quad V_2 = Z_{21} I_1 + Z_{22} I_2$

from ④

$$\boxed{I_2 = \dfrac{-V_2}{Z_L}} \quad - \quad \text{Ⓧ}$$

sub X in ②

$$\boxed{V_2 = Z_{21} I_1 - \dfrac{Z_{22}}{Z_L} V_2} \quad - \quad \text{Ⓕ}$$

from ①

$$V_1 = Z_{11} I_1 + Z_{12} I_2$$

Sub X in ①

$$V_1 = Z_{11} I_1 - \dfrac{Z_{12}}{Z_L} V_2$$

$$\boxed{I_1 = \dfrac{V_1}{Z_{11}} + \dfrac{Z_{12}}{Z_{11} Z_L} V_2} \quad - \quad \text{Ⓖ}$$

Sub Ⓖ in Ⓕ

$$V_2 = Z_{21}\left(\dfrac{V_1}{Z_{11}} + \dfrac{Z_{12}}{Z_{11} Z_L} V_2\right) - \dfrac{Z_{22}}{Z_L} V_2$$

$$= \dfrac{Z_{21}}{Z_{11}} V_1 + \dfrac{Z_{12} Z_{21}}{Z_{11} Z_L} V_2 - \dfrac{Z_{22}}{Z_L} V_2$$

$$V_2\left(1 - \dfrac{Z_{12} Z_{21}}{Z_{11} Z_L} + \dfrac{Z_{22}}{Z_L}\right) = \dfrac{Z_{21}}{Z_{11}} V_1$$

$$V_2\left(\dfrac{Z_{11} Z_L}{Z_{11} Z_L} - \dfrac{Z_{12} Z_{21}}{Z_{11} Z_L} + \dfrac{Z_{11} Z_{22}}{Z_{11} Z_L}\right) = \dfrac{Z_{21}}{Z_{11}} V_1$$

$$V_2\left(\dfrac{Z_{11} Z_L + Z_{11} Z_{22} - Z_{12} Z_{21}}{Z_{11} Z_L}\right) = \dfrac{Z_{21}}{Z_{11}} V_1$$

$$\boxed{\dfrac{V_2}{V_1} = \dfrac{Z_{21} Z_L}{\underbrace{Z_{11} Z_L + Z_{11} Z_{22} - Z_{12} Z_{21}}_{\Delta Z}}}$$

$$= \dfrac{Z_{21} Z_L}{Z_{11} Z_L + \Delta Z}$$

7) The Voltage ratio $\frac{V_2}{V_g}$

Combine ① , ③ , ④ to find I_1

$$V_1 = Z_{11} I_1 + Z_{12} I_2$$

$$V_g - I_1 Z_g = Z_{11} I_1 + Z_{12} \left(\frac{-V_2}{Z_L} \right)$$

$$I_1 (Z_g + Z_{11}) = V_g + \frac{Z_{12}}{Z_L} V_2$$

$$\boxed{ I_1 = \frac{V_g}{Z_g + Z_{11}} + \frac{Z_{12}}{Z_L (Z_g + Z_{11})} V_2 } - Ⓗ$$

Combine ④ & Ⓗ with ②

eq② $\qquad V_2 = Z_{21} I_1 + Z_{22} I_2$

$$V_2 = \frac{Z_{21}}{Z_g + Z_{11}} V_g + \frac{Z_{12} Z_{21}}{Z_L (Z_g + Z_{11})} V_2 - \frac{Z_{22}}{Z_L} V_2$$

$$V_2 \left(1 + \frac{Z_{22}}{Z_L} \frac{(Z_g + Z_{11})}{(Z_g + Z_{11})} - \frac{Z_{12} Z_{21}}{Z_L (Z_g + Z_{11})} \right) = \frac{Z_{21}}{Z_g + Z_{11}} V_g$$

$$V_2 \left(\frac{Z_L (Z_g + Z_{11}) + Z_{22} (Z_g + Z_{11}) - Z_{12} Z_{21}}{Z_L (Z_g + Z_{11})} \right) = \frac{Z_{21}}{(Z_g + Z_{11})} V_g$$

$$\boxed{ \frac{V_2}{V_g} = \frac{Z_{21} Z_L}{Z_L (Z_g + Z_{11}) + Z_{22} (Z_g + Z_{11}) - Z_{12} Z_{21}} }$$

OR

$$\boxed{ \frac{V_2}{V_g} = \frac{Z_{21} Z_L}{(Z_{11} + Z_g)(Z_{22} + Z_L) - Z_{12} Z_{21}} }$$