Chapter 12 :- Introduction to the Laplace Transform

The Laplace transform is given by the integral

$$\mathcal{L}(f(t)) = F(s) = \int_{-\infty}^{\infty} f(t)\, e^{-st}\, dt$$

wher the variable $s = \sigma + j\omega$

$f(t) \xleftrightarrow{\mathcal{L}} F(s)$ Laplace transform pairs

$\mathcal{L}(\delta(t)) \longleftrightarrow 1$

please check Table $\boxed{12.1}$ for other pairs and Table $\boxed{12.2}$ for the properties. Please also see Table $\boxed{12.3}$ for useful transform pairs.

Why Laplace :-

- Consider the steady-state and the transient response of Circuits containing more than a single node.
- Transient response of a Circuit containing complicated single Source
- To define the transfer function
- Relate time domain to s-domain and frequency domain

Inverse Laplace transform

Let $F(s)$ be Laplace transform of some function $f(t)$

we want to find $f(t)$ without using the inversion formula - we want to find $f(t)$ using the Laplace transform known table and properties.

- put $F(s)$ in a form or a sum of forms that we know it is in the Laplace transform table.

$F(s)$ in general is a ratio of two polynomials

$F(s)$ is a rational function

$$F(s) = \frac{N(s)}{D(s)} = \frac{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0}{b_m s^m + b_{m-1} s^{m-1} + \quad + b_0}$$

If $m > n$, the ratio $\frac{N(s)}{D(s)}$ is called proper rational function

If $m \leq n$, the ratio $\frac{N(s)}{D(s)}$ is called improper rational function

Only a proper rational function can be expanded as a sum of partial fraction.

1-Partial fraction expansions Nonrepeated Real Roots of D(s) (Distinct)

$$F(s) = \frac{96(s+5)(s+12)}{s(s+8)(s+6)} \equiv \frac{K_1}{s} + \frac{K_2}{s+8} + \frac{K_3}{s+6}$$

To find the value of K_1, we multiply both sides by s and then evaluate both sides at $s=0$

$$\cancel{s}\,\frac{96(s+5)(s+12)}{\cancel{s}(s+8)(s+6)}\Bigg|_{s=0} = \cancel{s}\,\frac{K_1}{\cancel{s}} + \frac{K_2 s}{s+8}\Bigg|_{s=0} + \frac{K_3 s}{s+6}\Bigg|_{s=0}$$

$$\frac{96(s+5)(s+12)}{(s+8)(s+6)}\Bigg|_{s=0} = K_1 + \frac{K_2 s}{s+8}\Bigg|_{s=0} + \frac{K_3 s}{s+6}\Bigg|_{s=0}$$

$$\frac{96(5)(12)}{(8)(6)} = K_1 + 0 + 0$$

$$\boxed{120 = K_1}$$

To find the value of K_2, we multiply both sides by $(s+8)$ and then evaluate both sides at $s=-8$

$$\frac{96(s+5)(s+12)}{s(s+6)}\Bigg|_{s=-8} = \frac{K(s+8)}{s}\Bigg|_{s=-8} + K_2 + \frac{K_3(s+8)}{(s+6)}\Bigg|_{s=-8}$$

$$\frac{96(-3)(4)}{-8(-2)} = 0 + K_2 + 0$$

$$\boxed{-72 = K_2}$$

then K_3

$$\left.\frac{96(s+5)(s+12)}{s(s+8)}\right|_{s=-6} = \left.\frac{K_1(s+6)}{s}\right|_{s=-6} + \left.\frac{K_2(s+6)}{s+8}\right|_{s=-6} + K_3$$

$$\frac{96(-1)(6)}{-6(2)} = 0+0+K_3$$

$$\boxed{48 = K_3}$$

So:-

$$\frac{96(s+5)(s+12)}{s(s+8)(s+6)} = \frac{120}{s} + \frac{48}{s+6} - \frac{72}{s+8}$$

$$\therefore \mathcal{L}^{-1}\left[\frac{96(s+5)(s+12)}{s+(s+8)(s+6)}\right] = (\underbrace{120}_{\text{steady}} + \underbrace{48e^{-6t} - 72e^{-8t}}_{\text{transient}})u(t)$$

2- Partial Fraction Expansion:- Repeated Real Roots of D(s)

$$\frac{100(s+25)}{s(s+5)^3} = \frac{K_1}{s} + \frac{K_2}{(s+5)^3} + \frac{K_3}{(s+5)^2} + \frac{K_4}{(s+5)}$$

we find K_1 as previously described

$$\left.\frac{100(s+25)}{(s+5)^3}\right|_{s=0} = K_1 + \left.\frac{K_2 s}{(s+5)^3}\right|_{s=0} + \left.\frac{K_3 s}{(s+5)^2}\right|_{s=0} + \left.\frac{K_4 s}{(s+5)}\right|_{s=0}$$

$$\frac{100(25)}{5^3} \quad \boxed{K_1 = 20}$$

To find K_2, we multiply both sides by $(s+5)^3$ and then evaluate both sides at -5

$$\frac{100(s+25)}{s} = \left.\frac{K_1(s+5)^3}{s}\right|_{s=-5} + K_2 + \left.K_3(s+5)\right|_{s=-5} + \left.K_4(s+5)^2\right|_{s=-5}$$

$$\frac{100(20)}{-5} = 0 + K_2 + 0 + 0$$

$$\boxed{-400 = K_2}$$

To find K_3, we first multiply both sides by $(s+5)^3$. Next we differentiate both sides once with respect to s and then evaluate at $s=-5$

$$\left.\frac{d}{ds}\left[\frac{100(s+25)}{s}\right]\right|_{s=-5} = \cancelto{0}{\frac{d}{ds}\left[\frac{K_1(s+5)^3}{s}\right]_{s=-5}} + \cancelto{0}{\frac{d}{ds}[K_2]} + \frac{d}{ds}[K_3(s+5)]_{s=-5}$$
$$+ \cancelto{0}{\frac{d}{ds}\left[K_4(s+5)^2\right]_{s=-5}}$$

$$100\left[\frac{s-(s+25)}{s^2}\right]_{s=-5} = K_3$$

$$\boxed{-100 = K_3}$$

To find K_4, we first multiply both sides by $(s+5)^3$. Next we differentiate both sides with respect to s twice and then evaluate both sides at $s=-5$.

After simplifing the first derivative, the second derivative becomes:

$$100\frac{d}{ds}\left[\frac{-25}{s^2}\right]_{s=-5} = \cancelto{0}{K_1\frac{d}{ds}\left[\frac{(s+5)^2(2s-5)}{s^2}\right]_{s=-5}} + 0 + \cancelto{0}{\frac{d}{ds}[K_3]} + \frac{d}{ds}[2K_4(s+5)]_{s=-5}$$

$$100\left[\frac{25(25)}{s^4}\right]\Bigg|_{s=-5} = 2K_4$$

$$\frac{(100)(-250)}{625} = 2K_4$$

$$-40 = 2K_4$$

$$\boxed{-20 = K_4}$$

So

$$\frac{100(s+25)}{s(s+5)^3} = \frac{20}{s} - \frac{400}{(s+5)^3} - \frac{100}{(s+5)^2} - \frac{20}{s+5}$$

$$\therefore \mathcal{L}^{-1}\left[\frac{100(s+25)}{s(s+5)^3}\right] = \left[20 - 200t^2e^{-5t} - 100te^{-5t} - 20e^{-5t}\right]u(t)$$

3 - Partial Fraction Expansion: Nonrepeated Complex Roots of D(s)

$$F(s) = \frac{100(s+3)}{(s+6)(s^2+6s+25)}$$

F(s) is a proper rational function. Next we must find the roots of the quadratic term: $s^2+6s+25$ $\qquad s = \frac{-b \mp \sqrt{b^2-4ac}}{2a}$

$$s^2+6s+25 = (s+3-j4)(s+3+j4)$$

$$\frac{100(s+3)}{(s+6)(s^2+6s+25)} = \frac{K_1}{s+6} + \frac{K_2}{s+3-j4} + \frac{K_3}{s+3+j4}$$

to find K_1, K_2 and K_3 we use the same process as before:-

$$\left.\frac{100(s+3)}{s^2+6s+25}\right|_{s=-6} = K_1 + \left.\frac{K_2\cancel{(s+6)}}{s+3-j4}\right|_{s=-6} + \left.\frac{K_3\cancel{(s+6)}}{s+3+j4}\right|_{s=-6}$$

$$\frac{100(-3)}{25} = K_1$$

$$\boxed{-12 = K_1}$$

to find K_2, we multiply both sides by $s+3-j4$, next evaluate at $s=-3+j4$

$$\left.\frac{100(s+3)}{(s+6)(s+3+j4)}\right|_{s=-3+j4} = K_2$$

$$\frac{100(-3+j4+3)}{(-3+4j+6)(-3+j4+3+j4)} = K_2$$

$$\frac{100(j4)}{(3+4j)(+j8)} = K_2$$

$$\boxed{\begin{aligned} 6-8j &= K_2 \\ 10e^{-j53.13} &= K_2 \end{aligned}}$$

to find K_3, we multiply both sides by $s+3+j4$ and evaluate at $s=-3-j4$

$$\left.\frac{100(s+3)}{(s+6)(s+3-j4)}\right|_{s=-3-j4} = K_3$$

$$\frac{100(-j4)}{(3-j4)(-j8)} = K_3$$

$$\boxed{\begin{aligned} 6+j8 &= K_3 \\ 10e^{+j53.13} &= K_3 \end{aligned}} = K_2^*$$

$$\frac{100(s+3)}{(s+6)(s^2+6s+25)} = \frac{-12}{s+6} + \frac{10\angle -53.13}{s+3-j4} + \frac{10\angle 53.13}{s+3+j4}$$

$$\mathcal{L}^{-1}\left[\frac{100(s+3)}{(s+6)(s^2+6s+25)}\right] = \left[-12e^{-6t} + 10e^{-j53.13}e^{-(3-j4)t} + 10e^{j53.13}e^{-(3+j4)t}\right]u(t)$$

$$10e^{-j53.13}e^{-(3-j4)t} + 10e^{j53.13}e^{-(3+j4)t}$$

$$= 10e^{-3t}\left(e^{j(4t-53.13)} + e^{-j(4t-53.13)}\right)$$

$$= 20e^{-3t}\cos(4t-53.13)$$

$$\therefore \mathcal{L}^{-1}\left[\frac{K}{s+\alpha-j\beta} + \frac{K^*}{s+\alpha+j\beta}\right] = 2|K|e^{-\alpha t}\cos(\beta t+\theta)$$

4- Partial Fraction Expansion: Repeated Complex Roots of $D(s)$

$$F(s) = \frac{768}{(s^2+6s+25)^2} = \frac{768}{(s+3-j4)^2(s+3+j4)^2} \equiv$$

$$\equiv \frac{K_1}{(s+3-j4)^2} + \frac{K_2}{(s+3-j4)} + \frac{K_3}{(s+3+j4)^2} + \frac{K_4}{(s+3+j4)}$$

$$K_3 = K_1^*, \quad K_4 = K_2^*$$

to find K_1, multiply both sides by $(s+3-j4)^2$ and then evaluate at $s=-3+j4$

$$\left.\frac{768}{(s+3+j4)^2}\right|_{s=-3+j4} = K_1$$

$$\frac{768}{(j8)^2} = K_1$$

$$\boxed{-12 = K_1} \Rightarrow K_3 = K_1^* = -12$$

to find K_2

$$\frac{d}{ds}\left[\frac{768}{(s+3+j4)^2}\right]_{s=-3+j4} = K_2$$

$$= \frac{-2(768)(s+\cancel{3}+j4)}{(s+3+j4)^{\cancel{4}\,3}} = K_2$$

$$-\frac{2(768)}{(j8)^3} = K_2$$

$$-j3 = K_2$$

$$\boxed{3\angle -90 = K_2} \Rightarrow K_4 = K_2^* = 3\angle 90$$

$$F(s) = \left[\frac{-12}{(s+3-j4)^2} - \frac{12}{(s+3+j4)^2}\right] + \left(\frac{3\angle -90}{s+3-j4} + \frac{3\angle 90}{s+3+j4}\right)$$

$$f(t) = \left[-12te^{-(3+j4)t} - 12t\,e^{-(3-j4)t} + 3e^{-j90}e^{-(3+j4)t} + 3e^{j90}e^{-(3-j4)t}\right]u(t)$$

$$= \left[-12t\,e^{-3t}e^{-j4t} - 12t\,e^{-3t}e^{+j4t} + 3e^{-j90}e^{-3t}e^{-j4t} + 3e^{j90}e^{-3t}e^{j4t}\right]u(t)$$

$$= \left[-12t\,e^{-3t}\left(e^{j4t} + e^{-j4t}\right) + 3e^{-3t}\left(e^{j(4t+90)} + e^{-j(4t+90)}\right)\right]u(t)$$

$$= \left[-24t\,e^{-3t}\cos(4t) + 6e^{-3t}\cos(4t+90)\right]u(t)$$

$$\mathcal{L}^{-1}\left(\frac{K}{(s+a)^r}\right) = \frac{K\, t^{r-1} e^{-at}}{(r-1)!}\, u(t)$$

$$\mathcal{L}^{-1}\left(\frac{K}{(s+\alpha-j\beta)^r} + \frac{K^*}{(s+a+j\beta)^r}\right) = \left[\frac{2|K|\, t^{r-1}}{(r-1)!}\, e^{-at}\cos(\beta t+\theta)\right]u(t)$$

5- Partial Fraction Expansion: Improper Rational Function

$$F(s) = \frac{s^4 + 13s^3 + 66s^2 + 200s + 300}{s^2 + 9s + 20}$$

$$
\begin{array}{r|l}
 & s^2 + 4s + 10 \\
\hline
s^2 + 9s + 20 & s^4 + 13s^3 + 66s^2 + 200s + 300 \\
 & -\;(s^4 + 9s^3 + 20s^2) \\
\hline
 & 0 \quad 4s^3 + 46s^2 + 200s + 300 \\
 & -\;(4s^3 + 36s^2 + 80s) \\
\hline
 & 0 + 10s^2 + 120s + 300 \\
 & 10s^2 + 90s + 200 \\
\hline
 & 0 + 30s + 100
\end{array}
$$

$$\therefore F(s) = s^2 + 4s + 10 + \frac{30s+10}{s^2+9s+20}$$

where $\frac{30s+10}{s^2+9s+20}$ is a proper rational function

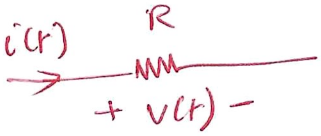
$$\frac{30s+10}{s^2+9s+20} = \frac{30s+10}{(s+4)(s+5)} = \frac{-20}{s+4} + \frac{50}{s+5}$$

$$F(s) = s^2 + 4s + 10 - \frac{20}{s+4} + \frac{50}{s+5}$$

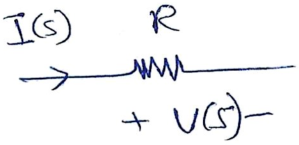
$$f(t) = \frac{d^2}{dt^2}\delta(t) + 4\frac{d\,\delta(t)}{dt} + 10\,\delta(t) - \left(20e^{-4t} - 50e^{-5t}\right)u(t)$$

Circuit element models — Chapter 13:–

* Resistor:–



$\Longleftrightarrow$

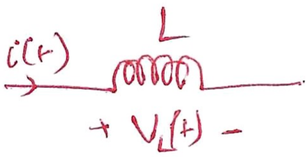


$v(t) = R\,i(t)$

$V(s) = R\,I(s)$

$Z_R = R \ \Omega$

* Inductor:–



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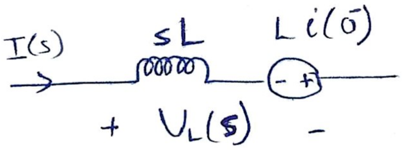
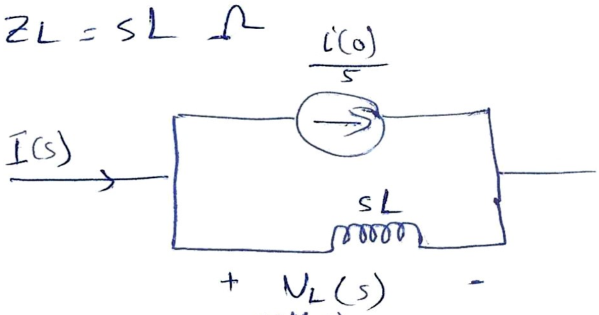


$V_L(t) = L\,\frac{di}{dt}$

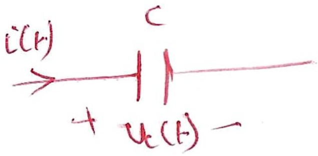
$V_L(s) = L\left[sI(s) - i(0^-)\right]$

$= sL\,I(s) - L\,i(0^-)$

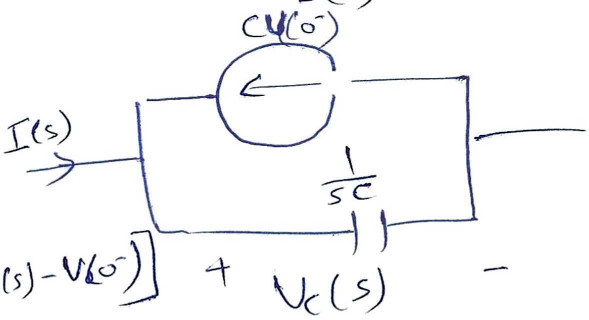
$Z_L = sL \ \Omega$



* Capacitor:–



$\Longleftrightarrow$

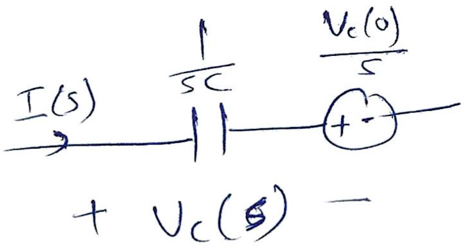


$i(t) = C\,\frac{dV_C(t)}{dt}$

$I(s) = C\left[sV_C(s) - V(0^-)\right]$

$= sC\,V_C(s) - C\,V(0)$

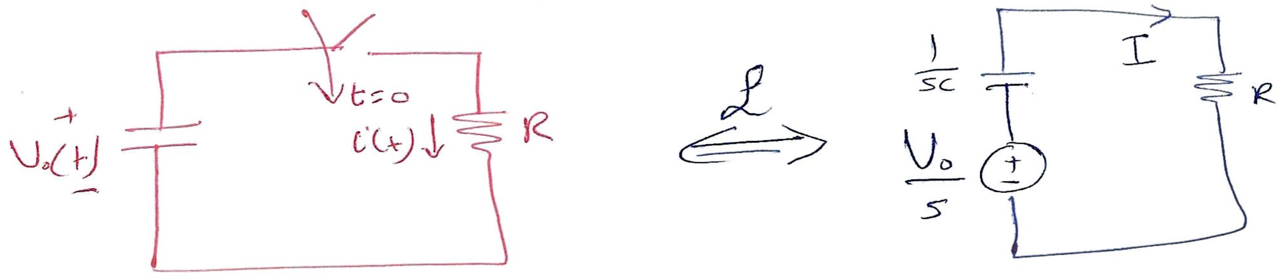
$Z_C = \frac{1}{sC} \ \Omega$



Natural response of an RC Circuit

Natural response :- No active source in the Circuit

find $i(t)$ and $v(t)$



KVL :-

$$-\frac{V_o}{s} + I\left(\frac{1}{sC}\right) + IR = 0$$

$$I\left(\frac{1}{sC} + R\right) = \frac{V_o}{s} \Rightarrow I = \frac{\frac{V_o}{s}}{\frac{1}{sC} + R} = \frac{V_o}{\frac{1}{C} + Rs}$$

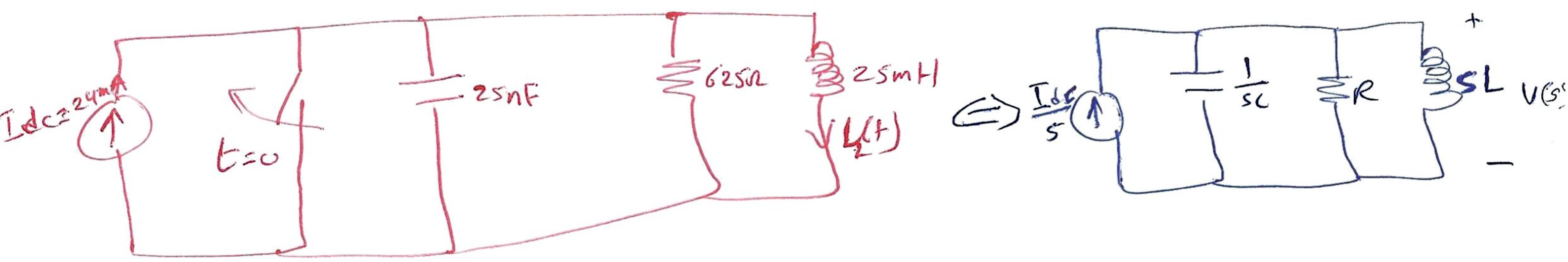
$$I(s) = \frac{V_o/R}{s + \frac{1}{RC}}$$

$$\therefore i(t) = \frac{V_o}{R}\, e^{-\frac{1}{RC}t}\, u(t)$$

$$v(t) = V_o\, e^{-\frac{1}{RC}t}\, u(t)$$

step response of a parallel RLC Circuit

step response:- active source is applied to the Circuit



KCL

$$\frac{I_{dc}}{s} = \frac{V(s)}{\frac{1}{sC}} + \frac{V(s)}{R} + \frac{V(s)}{sL} = V\left(sC + \frac{1}{R} + \frac{1}{sL}\right)$$

$$V = \frac{\frac{I_{dc}}{s}}{sC + \frac{1}{R} + \frac{1}{sL}} = \frac{I_{dc}}{s^2C + \frac{s}{R} + \frac{1}{L}} = \frac{I_{dc}/C}{s^2 + \frac{s}{RC} + \frac{1}{LC}}$$

$$I_L = \frac{V}{sL} = \frac{I_{dc}/C}{s^2 + \frac{s}{RC} + \frac{1}{LC}} \cdot \frac{1}{sL}$$

$$= \frac{I_{dc}/CL}{s\left(s^2 + \frac{s}{RC} + \frac{1}{LC}\right)}$$

$$= \frac{3.84 \times 10^7}{s\left(s^2 + (6.4 \times 10^4)s + (1.6 \times 10^9)\right)}$$

$$= \frac{24}{s} + \frac{20\angle 127}{s - (-32k + j24k)} + \frac{20\angle -127}{s - (-32k - j24k)} \text{ mA.s}$$

$$i_L(t) = 24\,u(t) + 20 e^{j127} e^{-(32k)t} e^{j(24k)t} u(t) + 20 e^{-j127} e^{-(32k)t} e^{-j24kt}$$

$$= \left[24 + 40 e^{-32kt} \cos(24kt + 127)\right] u(t) \text{ mA}$$

In the previous example if $i(t) = I_m \cos(\omega t)\, u(t)$

$$\therefore I(s) = I_m \frac{s}{s^2 + \omega^2}$$

$$\Rightarrow V(s) = \frac{(I_m/C)\, s^2}{(s^2 + \omega^2)\left(s^2 + \frac{s}{RC} + \frac{1}{LC}\right)}$$

$$I(s) = \frac{V(s)}{sL} = \frac{(I_m/CL)\, s}{(s^2 + \omega^2)\left(s^2 + \frac{s}{RC} + \frac{1}{LC}\right)}$$

$$= \frac{K_1}{s - j\omega} + \frac{K_1^*}{s + j\omega} + \frac{K_2}{s - (-\alpha + j\beta)} + \frac{K_2^*}{s - (-\alpha - j\beta)}$$

$$i(t) = \underbrace{2|K_1| \cos(\omega t + \angle K_1)}_{\text{steady}} + \underbrace{2|K_2| e^{-\alpha t} \cos(\beta t + \angle K_2)}_{\text{transient}}$$