8.2 Trigonometric Integrals:

The purpose is to find $\int \sin^m x \cos^n x \, dx$:

Case 1: If m is an odd number, then:

We write $m = 2k+1$, then use: $\sin^2 x = 1 - \cos^2 x$.

$$\Rightarrow \int \sin^m x \cos^n x \, dx = \int (\sin^2 x)^k \cos^n x \sin x \, dx$$

$$= \int (1 - \cos^2 x)^k \cos^n x \sin x \, dx.$$

Let $u = \cos x \Rightarrow du = -\sin x \, dx$

$$= \int -(1 - u^2)^k u^n \, du$$

Example: $\int \sin^3 x \cos^2 x \, dx$. , $m = 3$ (odd)

$$= \int \sin^2 x \sin x \cos^2 x \, dx = \int (1 - \cos^2 x) \cos^2 x \sin x \, dx$$

Let $u = \cos x \Rightarrow du = -\sin x \, dx$

$$\Rightarrow \int -(1 - u^2) u^2 \, du = -\int u^2 - u^4 \, du$$

$$= -\frac{u^3}{3} + \frac{u^5}{5} + C = -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + C$$

Case 2: $\int \sin^m x \cos^n x \, dx$

If m is even and n is odd

Then : $n = 2k+1$ & Use $\cos^2 x = 1 - \sin^2 x$.

$$\int \sin^m x \cos^n x \, dx = \int \sin^m x \left(\cos^2 x\right)^k . \cos x \, dx$$

$$= \int \sin^m x \left(1 - \sin^2 x\right)^k \cos x \, dx.$$

Let $u = \sin x$ $\Rightarrow$ $du = \cos x \, dx$

$$\Rightarrow \int u^m \left(1 - u^2\right)^k du$$

Example: $\int \cos^5 x \, dx.$

Here $m = 0$ and $n = 5$ (odd)

$$\int \cos^5 x \, dx = \int \left(\cos^2 x\right)^2 \cos x \, dx = \int \left(1 - \sin^2 x\right)^2 \cos x \, dx$$

Let $u = \sin x$ $\Rightarrow$ $du = \cos x \, dx$

$$= \int \left(1 - u^2\right)^2 du = \int 1 - 2u^2 + u^4 \, du$$

$$= u - 2\frac{u^3}{3} + \frac{u^5}{5} + C$$

$$= \sin x - 2\frac{\sin^3 x}{3} + \frac{\sin^5 x}{5} + C$$

Case 3: $\int \sin^m x \cos^n x \, dx$

If both m and n are even, we use:

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$

Example: $\int \sin^2 x \cos^4 x \, dx$.

$$\int \sin^2 x \left(\cos^2 x\right)^2 dx = \int \left(\frac{1 - \cos 2x}{2}\right)\left(\frac{1 + \cos 2x}{2}\right)^2 dx$$

$$= \frac{1}{8} \int (1 - \cos 2x)(1 + 2\cos 2x + \cos^2 2x) \, dx$$

$$= \frac{1}{8} \int (1 + 2\cos 2x + \cos^2 2x - \cos 2x - 2\cos^2 2x - \cos^3 2x) \, dx$$

$$= \frac{1}{8} \left[\int (1 + \cos 2x) \, dx - \underbrace{\int \cos^2 2x \, dx}_{\substack{\text{case 3} \\ (*)}} - \underbrace{\int \cos^3 2x \, dx}_{\substack{\text{case 2} \\ (**)}} \right.$$

$$(*) \int \cos^2 2x \, dx = \int \frac{1 + \cos 4x}{2} \, dx = \frac{1}{2}x + \frac{\sin 4x}{8}.$$

$$(**) \int \cos^3 2x \, dx = \int (\cos^2 2x) \cos 2x \, dx = \int (1 - \sin^2 2x) \cos 2x \, dx$$

Let $u = \sin 2x \Rightarrow du = 2\cos 2x \, dx$

$$\Rightarrow \int (1 - u^2) \frac{du}{2} = \frac{1}{2}u - \frac{u^3}{6} = \frac{\sin 2x}{2} - \frac{\sin^3 2x}{6}$$

Therefore: The whole Integral:

$$= \frac{1}{8}\left[x + \cancel{\frac{\sin 2x}{2}} - \frac{1}{2}x - \frac{\sin 4x}{8} - \cancel{\frac{\sin 2x}{2}} + \frac{\sin^3 2x}{6}\right] + C$$

$$= \frac{1}{8}\left[\frac{1}{2}x - \frac{\sin 4x}{8} + \frac{\sin^3 2x}{6}\right] + C.$$

Eliminating Square Roots:

Example: $\int_0^{\frac{\pi}{4}} \sqrt{1+\cos 4x}\, dx$

Recall: $\cos 2x = 2\cos^2 x - 1 \Rightarrow \cos 4x = 2\cos^2 2x - 1$

$$\Rightarrow \int_0^{\frac{\pi}{4}} \sqrt{1+\cos 4x}\, dx = \int_0^{\frac{\pi}{4}} \sqrt{2\cos^2 2x}\, dx = \sqrt{2}\int_0^{\frac{\pi}{4}} \cos 2x\, dx$$

$$= \sqrt{2}\left[\frac{\sin 2x}{2}\right]_0^{\frac{\pi}{4}} = \frac{\sqrt{2}}{2}$$

Integrals of Powers of $\tan x$ and $\sec x$:

Example: $\int \tan^4 x\, dx$

$$\int \tan^4 x\, dx = \int \tan^2 x\,(\sec^2 x - 1)\, dx$$

$$= \int \tan^2 x \sec^2 x - \tan^2 x\, dx$$

$$= \underbrace{\int (\tan^2 x \sec^2 x)\, dx}_{\text{substitution : } u = \tan x} - \int (\sec^2 x - 1)\, dx$$

$$= \int u^2 du - \int \sec^2 x\, dx + \int 1\, dx$$

$$= \frac{u^3}{3} - \tan x + x + C$$

$$= \frac{\tan^3 x}{3} - \tan x + x + C.$$

Example: $\int \sec^3 x \, dx$

Integration by Parts:

Let $u = \sec x$, $dv = \sec^2 x \, dx$

$du = \sec x \tan x \, dx$, $v = \tan x$.

$\Rightarrow \int \sec^3 x \, dx = \sec x \tan x - \int \tan^2 x \sec x \, dx$.

$= \sec x \tan x - \int (\sec^2 x - 1) \sec x \, dx$

$= \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx$

$\Rightarrow 2 \int \sec^3 x \, dx = \sec x \tan x + \ln|\sec x + \tan x| + C$

$\Rightarrow \int \sec^3 x \, dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln|\sec x + \tan x| + C$.

Products of Sines and Cosines:

1) $\sin mx \sin nx = \frac{1}{2}\left[\cos(m-n)x - \cos(m+n)x\right]$

2) $\sin mx \cos nx = \frac{1}{2}\left[\sin(m-n)x + \sin(m+n)x\right]$

3) $\cos mx \cos nx = \frac{1}{2}\left[\cos(m-n)x + \cos(m+n)x\right]$

Example: $\int \sin 3x \cos 5x \, dx = \int \frac{1}{2}\left(\sin(-2x) + \sin(8x)\right) dx$

$\frac{1}{4}\cos 2x - \frac{\cos 8x}{16} + C$.

Q19) $\int 16 \sin^2 x \cos^2 x \, dx$

$$= \int 16 \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right) dx$$

$$= \int 4 \left(1-\cos^2 2x\right) dx = 4\int\left(1-\left(\frac{1+\cos 4x}{2}\right)\right) dx$$

$$= 4\int\left(\frac{1}{2}-\frac{1}{2}\cos 4x\right) dx = 2\int(1-\cos 4x)\, dx$$

$$= 2x - \frac{\sin 4x}{2} + C.$$

Q34) $\int \sec x \tan^2 x \, dx = \int \sec x \tan x \tan x \, dx$

Let $u = \tan x$, $dv = \sec x \tan x \, dx$

$du = \sec^2 x \, dx$, $v = \sec x$

$$\Rightarrow \int \sec x \tan^2 x \, dx = \tan x \sec x - \underbrace{\int \sec^3 x \, dx}_{\text{done in page (1??)}}$$

$$= \tan x \sec x - \left(\frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x|\right) + C$$

$$= \frac{1}{2}\tan x \sec x - \frac{1}{2}\ln|\sec x + \tan x| + C'.$$

8.3. Trigonometric Substitutions:

The most Common substitutions are:

$$x = a\sin\theta \quad , \quad x = a\tan\theta \quad , \quad x = a\sec\theta .$$

These substitutions are effective in transforming Integrals involving:

$$\sqrt{a^2+x^2} \quad , \quad \sqrt{a^2-x^2} \quad , \quad \sqrt{x^2-a^2}$$

into Integrals we can evaluate directly.

Case 1: $\sqrt{a^2-x^2}$:

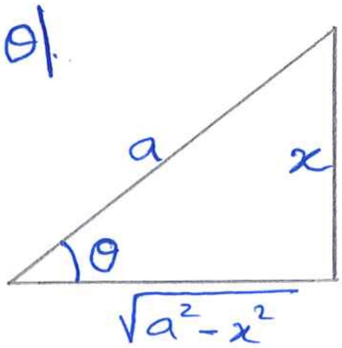
In this case, Let $x = a\sin\theta$, then

$$a^2 - x^2 = a^2 - a^2\sin^2\theta = a^2(1-\sin^2\theta)$$

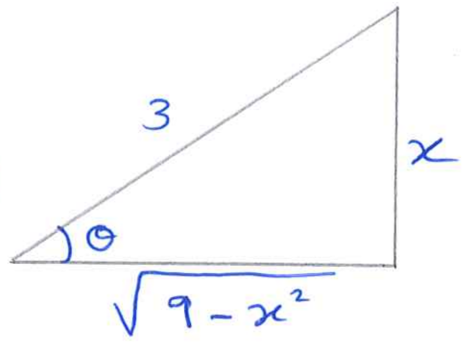
$$= a^2\cos^2\theta$$

where $\Rightarrow \sqrt{a^2-x^2} = \sqrt{a^2\cos^2\theta} = |a\cos\theta|$.

where $\theta = \sin^{-1}\left(\frac{x}{a}\right)$, $\underbrace{-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}}_{\cos\theta > 0}$.



Example: Evaluate $\displaystyle\int \frac{x^2}{\sqrt{9-x^2}}\,dx$.



Let $x = 3\sin\theta$, $\Rightarrow \theta = \sin^{-1}\left(\frac{x}{3}\right)$

$\Rightarrow dx = 3\cos\theta\, d\theta$

where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$$\Rightarrow \sqrt{9-x^2} = \sqrt{9-9\sin^2\theta} = \sqrt{9\cos^2\theta} = |3\cos\theta|$$

$$\Rightarrow \int \frac{x^2}{\sqrt{9-x^2}}\,dx = \int \frac{9\sin^2\theta}{|3\cos\theta|}\cdot(3\cos\theta)\,d\theta$$

$$= 9\int \sin^2\theta\, d\theta = 9\int\left(\frac{1-\cos 2\theta}{2}\right)d\theta$$

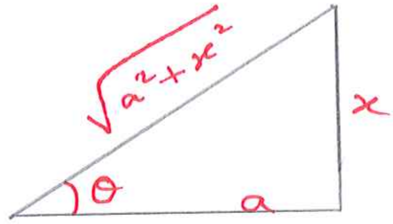
$$= 9\left(\frac{1}{2}\theta - \frac{\sin 2\theta}{2\cdot 2}\right) + C$$

$$= \frac{9}{2}\theta - \frac{9}{2}\sin\theta\cos\theta + C$$

$$= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) - \frac{9}{2}\left(\frac{x}{3}\right)\left(\frac{\sqrt{9-x^2}}{3}\right) + C$$

$$= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) - \frac{1}{2}x\sqrt{9-x^2} + C.$$

Case 2: $\sqrt{a^2 + x^2}$:



In this case, Let $x = a\tan\theta$.

$$\Rightarrow \sqrt{a^2+x^2} = \sqrt{a^2+a^2\tan^2\theta} = \sqrt{a^2(1+\tan^2\theta)}$$

$$= \sqrt{a^2\sec^2\theta} = |a\sec\theta|.$$

Notice that : $\theta = \tan^{-1}\left(\frac{x}{a}\right)$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

Example : Evaluate $\int \frac{dx}{\sqrt{4+x^2}}$

$\sec\theta > 0$
$\uparrow$

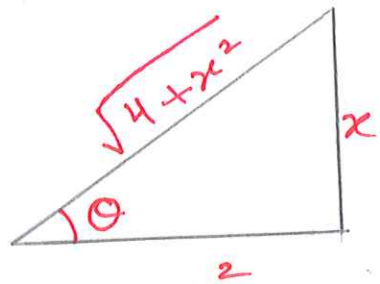
Let $x = 2\tan\theta$ $\Rightarrow \theta = \tan^{-1}\left(\frac{x}{2}\right)$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

$\Rightarrow dx = 2\sec^2\theta \, d\theta$

$$\Rightarrow \int \frac{dx}{\sqrt{4+x^2}} = \int \frac{2\sec^2\theta \, d\theta}{\sqrt{4+4\tan^2\theta}} = \int \frac{2\sec^2\theta \, d\theta}{\sqrt{4\sec^2\theta}}$$

$$= \int \frac{2\sec^2\theta \, d\theta}{2\sec\theta} = \int \sec\theta \, d\theta = \ln|\sec\theta + \tan\theta| + C$$

$$= \ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + C.$$

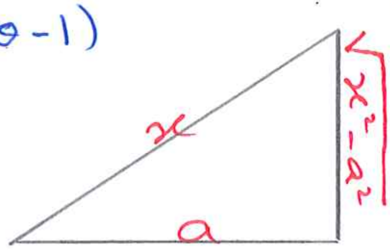


Case 3: $\sqrt{x^2 - a^2}$

In this case, Let $x = a\sec\theta$

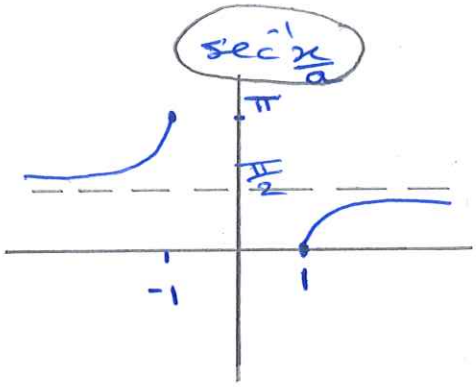
$$\Rightarrow \sqrt{x^2 - a^2} = \sqrt{a^2\sec^2\theta - a^2} = \sqrt{a^2(\sec^2\theta - 1)}$$

$$= \sqrt{a^2\tan^2\theta} = |a\tan\theta|$$



Notice that: $\theta = \sec^{-1}\left(\frac{x}{a}\right)$

where $\begin{cases} 0 \le \theta < \frac{\pi}{2}, & \frac{x}{a} \ge 1 \\ \frac{\pi}{2} < \theta \le \pi, & \frac{x}{a} \le -1 \end{cases}$



Example: Evaluate $\displaystyle\int \frac{dx}{\sqrt{25x^2 - 4}}$, $x > \frac{2}{5}$.

$$\sqrt{25x^2 - 4} = \sqrt{25\left(x^2 - \frac{4}{25}\right)} = 5\sqrt{x^2 - \left(\frac{2}{5}\right)^2}$$

$\Rightarrow$ Let $x = \frac{2}{5}\sec\theta$, $0 \le \theta < \frac{\pi}{2}$

$dx = \frac{2}{5}\sec\theta\tan\theta\, d\theta$

$$\left(\frac{x}{a} \ge 1 \Leftrightarrow \frac{x}{\frac{2}{5}} \ge 1 \Leftrightarrow x \ge \frac{2}{5} \right)$$

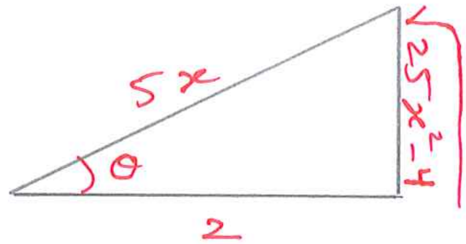
Now, $5\sqrt{\left(\frac{2}{5}\right)^2\sec^2\theta - \left(\frac{2}{5}\right)^2} = 5\sqrt{\left(\frac{2}{5}\right)^2(\sec^2\theta - 1)}$

$$= 5\cdot\left|\frac{2}{5}\tan\theta\right| = 2\tan\theta$$

$$\Rightarrow \int \frac{dx}{\sqrt{25x^2-4}} = \int \frac{\frac{2}{5}\sec\theta\tan\theta\, d\theta}{2\tan\theta}$$

$$= \frac{1}{5}\int \sec\theta\, d\theta = \frac{1}{5}\ln\left|\sec\theta + \tan\theta\right| + C.$$

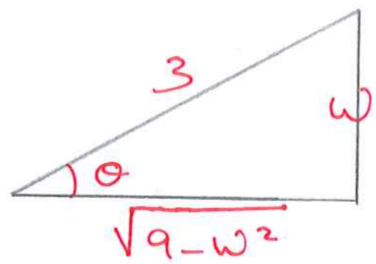
$$= \frac{1}{5}\ln\left|\frac{5x}{2} + \frac{\sqrt{25x^2-4}}{2}\right| + C'$$



Q20) Evaluate $\int \frac{\sqrt{9-w^2}}{w^2}\, dw$

Let $w = 3\sin\theta \quad , \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}.$

$dw = 3\cos\theta\, d\theta$



$$\Rightarrow \sqrt{9-w^2} = \sqrt{9-9\sin^2\theta} = \sqrt{9\cos^2\theta} = 3\cos\theta$$

$$\Rightarrow \int \frac{\sqrt{9-w^2}}{w^2}\, dw = \int \frac{3\cos\theta\,(3\cos\theta)\, d\theta}{9\sin^2\theta}$$

$$= \int \frac{\cos^2\theta}{\sin^2\theta}\, d\theta = \int \frac{1-\sin^2\theta}{\sin^2\theta}\, d\theta = \int (\csc^2\theta - 1)\, d\theta$$

$$= -\cot\theta - \theta + C = -\frac{\sqrt{9-w^2}}{w} - \sin^{-1}\left(\frac{w}{3}\right) + C'$$