Mustafa Jarrar: Lecture Notes in Discrete Mathematics.
Birzeit University, Palestine, 2021

Set Theory

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This lecture is based on (but not limited to) to chapter 5 in "Discrete Mathematics with Applications by Susanna S. Epp (3rd Edition)".

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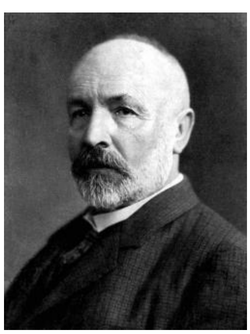
Set Theory

6.1 Basics of Sets

In this lecture:

- **Part 1: Basic Concepts and Notations**
- Part 2: Subsets, proper subsets, and Set Equalities
- Part 3: Operations on Sets
- Part 4: Empty Sets
- Part 5: Partitions of Sets
- Part 6: Power Sets & Cartesian Products

History



Georg Cantor
1845 – 1918
Born in Saint Petersburg, Russia
Moved to Germany 1856
PhD: University of Berlin 1867
Work: University of Halle

Set theory is the branch of mathematical logic that studies sets, which informally are collections of objects.

Initiated by Georg Cantor in 1870s

Basic Concepts and Notations

Cantor suggested a set as a:

"collection into a whole ***M*** of definite and separate objects of our intuition or our thought".

$\boldsymbol{M} = \{$ Ali, Adam, Sara $\}$

Each object is called an elements (or member of) of ***M***.

Ali $\in \boldsymbol{M}$ (Ali belongs to *M*)

Rami $\notin \boldsymbol{M}$ (Rami does not belong to *M*)

Basic Concepts and Notations

The order of elements is irrelevant

{Ali, Adam, Sara} = {Adam, Sara, Ali}

Redundancy is not allowed

{Ali, Adam, Adam, Sara}

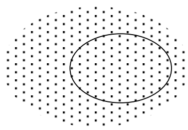
A set can be an element inside another set

{1, {1}} has two elements

Notation of elements

{Ali} ≠ Ali different elements

Defining Sets by a Property



$$A = \{x \in S \mid P(x)\}$$

The set of all "x is dummy" — Property

Examples:

The set of all integers that are more than -2 and less than 5

$\{x \in \mathbf{Z} \mid -2 < x < 5\}$

The set of all persons who born in Palestine

$\{x \in \textbf{\textit{Person}} \mid \textit{BornIn}(x, \text{Palestine})\}$

The set of all persons who born in Palestine and Love Homus

$\{x \in \textbf{\textit{Person}} \mid \textit{BornIn}(x, \text{Palestine}) \wedge \textit{Love}(x, \text{Homus})\}$

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Set Versus **Element**

In Set theory → **Set** vs. **Element** ← Mathematical Set

In JAVA → Class vs. Object

- The **extension** of a set is its elements.
- The **order** of elements is irrelevant
- In set theory: an element itself might be a set.

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- ❑ **Part 2: Subsets, proper subsets, and Set Equalities**
- ❑ Part 3: Operations on Sets
- ❑ Part 4: Empty Sets
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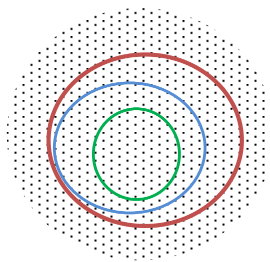
- ❑ Part 6: Power Sets & Cartesian Products

Subsets

المجموعة الجزئية

$A \subseteq B \quad \Leftrightarrow \quad \forall x, \ \text{if } x \in A \text{ then } x \in B.$

Subsets Versus JAVA SubClasses

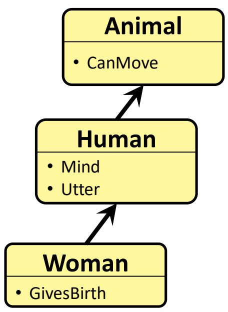


Animal = {x ∈ LivingOrganism | *CanMove*(x)}

Human = {x ∈ Animal | Has*Mind*(x) ∧ Utter (x)}

Woman = {x ∈ Human| GivesBirth(x)}

Woman ⊂ Human ⊂ Animal



Every subclass inherits the properties of its super class, thus:

- Human is a living organism that can move, has mind and utter.
- Woman is a living organism that can move, has mind and utter, and able to give birth.

Distinction between ∈ and ⊇

Which of the following are true statements?

$2 \in \{1, 2, 3\}$

$\{2\} \in \{1, 2, 3\}$

$2 \subseteq \{1, 2, 3\}$

$\{2\} \subseteq \{1, 2, 3\}$ →

$\{2\} \subseteq \{\{1\}, \{2\}\}$

$\{2\} \in \{\{1\}, \{2\}\}$

Distinction between $\in$ and $\supseteq$

Which of the following are true statements?

- ✓ $2 \in \{1, 2, 3\}$
- ✗ $\{2\} \in \{1, 2, 3\}$
- ✗ $2 \subseteq \{1, 2, 3\}$
- ✓ $\{2\} \subseteq \{1, 2, 3\}$
- ✗ $\{2\} \subseteq \{\{1\}, \{2\}\}$
- ✓ $\{2\} \in \{\{1\}, \{2\}\}$

Subsets Notations

Not Subset:

$$A \nsubseteq B \Leftrightarrow \exists x \,.\, x \in A \text{ and } x \notin B$$

Notations:

$A = B$		A equals B
$A \subset B$	$B \supset A$	A is subset of B
$A \subseteq B$	$B \supseteq A$	A is subset or equal of B
$A \not\subset B$	$B \not\supset A$	A is not a subset of B
$A \nsubseteq B$	$B \nsupseteq A$	A is not a subset or equal of B
$A \subsetneq B$	$B \supsetneq A$	A is a subset but not equal of B

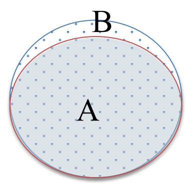
Examples: Person $\supseteq$ Man, $Z \supseteq Z^+$, $R \not\subset Z$

True True $Z \subset R$

Proper Subsets

Definition

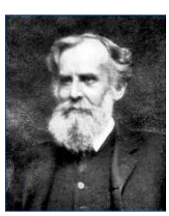
Let A and B be sets. A is a **proper subset** of B if, and only if, every element of A is in B but there is at least one element of B that is not in A.



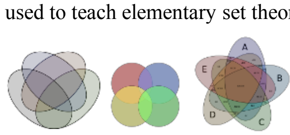
Examples of proper subsets:

{-2, -3, 4} ⊂ {-2, -2.5, -3, -3.5, -4}
{1, 2, 3, 4, …} ⊂ {0, 1, 2, 3, 4, ...}

Venn Diagrams

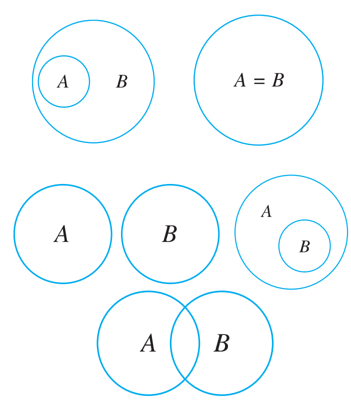


John Venn, British
(1834-1923)
Represented sets as diagrams in1881.
used to teach elementary set theory,



$A \subseteq B$

$A \nsubseteq B$



Z: integers (صحيحة)
Q: rational numbers (نسبية)
R: real numbers(حقيقية)

Proving and Disproving Subset Relations

Define sets A and B as follows:

$A = \{m \in \mathbf{Z} \mid m = 6r + 12 \text{ for some } r \in \mathbf{Z}\}$

$B = \{n \in \mathbf{Z} \mid n = 3s \text{ for some } s \in \mathbf{Z}\}$.

Prove that $A \subseteq B$.

Suppose x is a particular but arbitrarily chosen element of A.

Show that $x \in B$, means show that $x = 3 \cdot (integer)$.

$$\begin{aligned} x &= 6r + 12 \\ &= 3 \cdot (2r + 4). \end{aligned}$$

Let $s = 2r + 4$.

Also, $3s = 3(2r + 4)$
$= 6r + 12$
$= x$

Therefore, x is an element of B.

Set Equality

Definition

Given sets A and B, A **equals** B, written $\boldsymbol{A = B}$, if, and only if, every element of A is in B and every element of B is in A.

Symbolically: $A = B \iff A \subseteq B$ and $B \subseteq A$.

Example: Define sets A and B as follows:

$A = \{m \in \mathbf{Z} \mid m = 2a \text{ for some integer } a\}$

$B = \{n \in \mathbf{Z} \mid n = 2b - 2 \text{ for some integer } b\}$

Is $A = B$?

Yes. To prove this, both subset relations $\boldsymbol{A \subseteq B}$ and $\boldsymbol{B \subseteq A}$ must be proved.

Part 1, Proof That $A \subseteq B$:

…..

Part 2, Proof That $B \subseteq A$:

…..

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6.1 Basics of Sets

In this lecture:

- Part 1: Basic Concepts and Notations
- Part 1: Subsets, proper subsets, and Set Equalities
- Part 3: **Set Operations (Union, Intersection, Difference, Complement)**
- Part 4: Empty Sets
- Part 5: Partitions of Sets
- Part 6: Power Sets & Cartesian Products

Operations on Sets

• Definition

Let A and B be subsets of a universal set U.

1. The **union** of A and B, denoted $\boldsymbol{A \cup B}$, is the set of all elements that are in at least one of A or B.

2. The **intersection** of A and B, denoted $\boldsymbol{A \cap B}$, is the set of all elements that are common to both A and B.

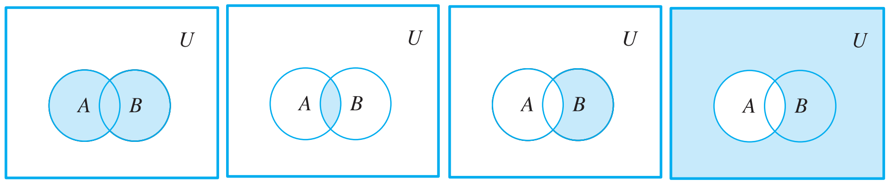
3. The **difference** of B minus A (or **relative complement** of A in B), denoted $\boldsymbol{B - A}$, is the set of all elements that are in B and not A.

4. The **complement** of A, denoted $\boldsymbol{A^c}$, is the set of all elements in U that are not in A. (U – A)

Symbolically:

$$A \cup B = \{x \in U \mid x \in A \text{ or } x \in B\},$$
$$A \cap B = \{x \in U \mid x \in A \text{ and } x \in B\},$$
$$B - A = \{x \in U \mid x \in B \text{ and } x \notin A\},$$
$$A^c = \{x \in U \mid x \notin A\}.$$

Operations on Sets



Shaded region represents $A \cup B$. | Shaded region represents $A \cap B$. | Shaded region represents $B - A$. | Shaded region represents A^c.

Ex. Draw the following:

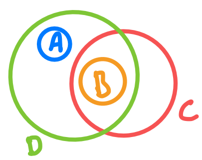
$A \cap B = \emptyset$

$B \subseteq C$

$C \cap D \neq \emptyset$

$A \subseteq D$

$D \cap B = B$



Distinction between $\cap$ and $\wedge$

Between sets → $\cap$ and $\wedge$ ← Between predicate and propositions

$\cup$ and $\vee$

Indexed Collection of Sets

• **Definition**

Unions and Intersections of an Indexed Collection of Sets

Given sets $A_0, A_1, A_2, \ldots$ that are subsets of a universal set U and given a nonnegative integer n,

$$\bigcup_{i=0}^{n} A_i = \{x \in U \mid x \in A_i \text{ for at least one } i = 0, 1, 2, \ldots, n\}$$

$$\bigcup_{i=0}^{\infty} A_i = \{x \in U \mid x \in A_i \text{ for at least one nonnegative integer } i\}$$

$$\bigcap_{i=0}^{n} A_i = \{x \in U \mid x \in A_i \text{ for all } i = 0, 1, 2, \ldots, n\}$$

$$\bigcap_{i=0}^{\infty} A_i = \{x \in U \mid x \in A_i \text{ for all nonnegative integers } i\}.$$

Finding Unions and Intersections of More than Two Sets

For each positive integer *i*, let $A_i = \left\{x \in \mathbf{R} \mid -\frac{1}{i} < x < \frac{1}{i}\right\} = A_i = \left(-\frac{1}{i}, \frac{1}{i}\right)$

A_1: set of all real numbers between -1 and 1
A_2: set of all real numbers between -1/2 and 1/2
A_3: set of all real numbers between - 1/3 and 1/3

Find $A_1 \cup A_2 \cup A_3$ $= (-1,1)$, because $\left(-\frac{1}{2}, \frac{1}{2}\right)\left(-\frac{1}{3}, \frac{1}{3}\right)$ included

Find $A_1 \cap A_2 \cap A_3$ $= \left(-\frac{1}{3}, \frac{1}{3}\right)$, because (-1,1)$\left(-\frac{1}{2}, \frac{1}{2}\right)$ are included

Find $\bigcup_{i=1}^{\infty} A_i$ $= (-1,1)$ Find $\bigcap_{i=1}^{\infty} A_i$ $= 0$

Formalizing Statements in Set Theory

- **All smart students**

  Smart ∩ student
- **Students who are not Smart**

  student − smart / student ∩ smart^c
- **There are no smart students from Palestine**

  Smart ∩ student ∩ Palestinian = ∅
- **There are no smart students from Palestine among the winners**

  Smart ∩ student ∩ Palestinian ∩ winners = ∅

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- ❑ Part 1: Subsets, proper subsets, and Set Equalities
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- ❑ Part 4: **Empty Sets**
- ❑ Part 5: Partitions of Sets
- ❑ Part 6: Power Sets & Cartesian Products

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The Empty Set

The empty set is not the same thing as nothing; rather, it is a set with nothing inside it and a set is always something. This issue can be overcome by **viewing a set as a bag—an empty bag undoubtedly still exists.**

Describe the set $D = \{x \in \mathbf{R} \mid 3 < x < 2\}$.

$\forall A: \emptyset \subseteq A$

$\forall A: A \cup \emptyset \subseteq A$

$\forall A: A \cap \emptyset \subseteq \emptyset$

$\forall A: A \times \emptyset = \emptyset$

$\forall A: A \times \emptyset \Rightarrow A = \emptyset$

While the empty set is a standard and widely accepted mathematical concept, it remains an ontological curiosity, **whose meaning and usefulness are debated by philosophers and logicians**.

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- ❑ Part 6: Power Sets & Cartesian Products

Disjoint Sets

• Definition

Two sets are called **disjoint** if, and only if, they have no elements in common. Symbolically:

$$A \text{ and } B \text{ are disjoint} \quad \Leftrightarrow \quad A \cap B = \emptyset.$$

Man ∩ Woman = ∅

• Definition

Sets $A_1, A_2, A_3 \ldots$ are **mutually disjoint** (or **pairwise disjoint** or **nonoverlapping**) if, and only if, no two sets A_i and A_j with distinct subscripts have any elements in common. More precisely, for all $i, j = 1, 2, 3, \ldots$

$$A_i \cap A_j = \emptyset \quad \text{whenever } i \neq j.$$

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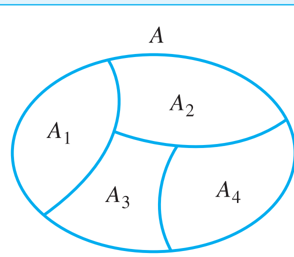
Partitions of Sets

• Definition

A finite or infinite collection of nonempty sets $\{A_1, A_2, A_3 \ldots\}$ is a **partition** of a set A if, and only if,

1. A is the union of all the A_i
2. The sets $A_1, A_2, A_3, \ldots$ are mutually disjoint.



Man ∩ Woman = ∅

Person = Man ∪ Woman

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Example

Let **Z** be the set of all integers and let

$T_0 = \{n \in \mathbf{Z} \mid n = 3k, \quad \text{for some integer } k\}$,
$T_1 = \{n \in \mathbf{Z} \mid n = 3k + 1, \text{ for some integer } k\}$,
$T_2 = \{n \in \mathbf{Z} \mid n = 3k + 2, \text{ for some integer } k\}$.

Is $\{T_0, T_1, T_2\}$ a partition of Z?

Yes. By the quotient-remainder theorem, every integer n can be represented in exactly one of the three forms

$$n=3k \quad \text{or} \quad n=3k+1 \quad \text{or} \quad n=3k+2$$

It also implies that every integer is in one of the sets T_0, T_1, or T_2.
So $\mathbf{Z} = T_0 \cup T_1 \cup T_2$.

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- ❑ Part 4: Empty Sets
- ❑ Part 5: Partitions of Sets
- ❑ Part 6: **Power Sets & Cartesian Products**

Power Sets

• Definition

Given a set A, the **power set** of A, denoted $\mathscr{P}(A)$**,** is the set of all subsets of A.

Find the power set of the set $\{x, y\}$. That is, find $\mathscr{P}(\{x, y\})$

$$= \{\emptyset, \{x\}, \{y\}, \{x, y\}\}.$$

n-tuples

• Definition

Let n be a positive integer and let $x_1, x_2, \ldots, x_n$ be (not necessarily distinct) elements. The **ordered *n*-tuple, $(x_1, x_2, \ldots, x_n)$,** consists of $x_1, x_2, \ldots, x_n$ together with the ordering: first x_1, then x_2, and so forth up to x_n. An ordered 2-tuple is called an **ordered pair,** and an ordered 3-tuple is called an **ordered triple.**

Two ordered n-tuples $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ are **equal** if, and only if, $x_1 = y_1, x_2 = y_2, \ldots, x_n = y_n$.

Symbolically:

$$(x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_n) \quad \Leftrightarrow \quad x_1 = y_1, x_2 = y_2, \ldots, x_n = y_n.$$

In particular,

$$(a, b) = (c, d) \quad \Leftrightarrow \quad a = c \text{ and } b = d.$$

Order *n*-tuples:

Is (1,2) =(2,1)? No $1 \neq 2$ and $2 \neq 1$

Is $(3, (-2)^2, 1/3) = (\sqrt{9}, 4, \frac{3}{9})$? Yes $3 = \sqrt{9}$. $(-2)^2 = 4$, $\frac{1}{3} = \frac{3}{9}$

Cartesian Products

• Definition

Given sets $A_1, A_2, \ldots, A_n$, the **Cartesian product** of $A_1, A_2, \ldots, A_n$ denoted $\mathbf{A_1 \times A_2 \times \ldots \times A_n}$, is the set of all ordered n-tuples $(a_1, a_2, \ldots, a_n)$ where $a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n$.

Symbolically:

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\}.$$

In particular,

$$A_1 \times A_2 = \{(a_1, a_2) \mid a_1 \in A_1 \text{ and } a_2 \in A_2\}$$

is the Cartesian product of A_1 and A_2.

Example: Let $A_1 = \{x, y\}$, $A_2 = \{1,2,3\}$, and $A_3 = \{a,b\}$.

$A_1 \times A_2 =$
$= \{(x,1),(x,2),(x,3),(y,1),(y,2),(y,3)\}$

Example

Let A = {Ali, Ahmad},
B = {AI, Dmath, DB},
C = {Pass, Fail}

Find (A × B) × C = {(Ali, AI), (Ali, Dmath), (Ali, DB), (Ahmad, AI), (Ahmad, Dmath), (Ahmad, DB)} × C
= {((Ali, AI), Pass), ((Ali, AI), Fail), ((Ali, Dmath), Pass), ((Ali, Dmath), Fail), … }

Find A × B × C =
{(Ali, AI, Pass), (Ali, AI, Fail), (Ali, Dmath, Pass), (Ali, Dmath, Fail), … }

Set Theory

Set Theory

6.2 Properties of Sets

In this lecture:

- ❑ **Part 1: Set Relations and Identities**
- ❑ Part 2: Proving Set Identities (Element Argument)
- ❑ Part 3: Examples of proving Set Identities

Set Relations

Theorem 6.2.1 Some Subset Relations

1. *Inclusion of Intersection:* For all sets A and B,

$$\text{(a) } A \cap B \subseteq A \quad \text{and} \quad \text{(b) } A \cap B \subseteq B.$$

2. *Inclusion in Union:* For all sets A and B,

$$\text{(a) } A \subseteq A \cup B \quad \text{and} \quad \text{(b) } B \subseteq A \cup B.$$

3. *Transitive Property of Subsets:* For all sets A, B, and C,

$$\text{if } A \subseteq B \text{ and } B \subseteq C, \text{ then } A \subseteq C.$$

Procedural Versions of Set Definitions

Let X and Y be subsets of a universal set U and suppose x and y are elements of U.

1. $x \in X \cup Y \iff x \in X$ or $x \in Y$
2. $x \in X \cap Y \iff x \in X$ and $x \in Y$
3. $x \in X - Y \iff x \in X$ and $x \notin Y$
4. $x \in X^c \iff x \notin X$
5. $(x,y) \in X \times Y \iff x \in X$ and $y \in Y$

Theorem 6.2.2 Set Identities

Let all sets referred to below be subsets of a universal set U.

1. *Commutative Laws:* For all sets A and B,

$$\text{(a) } A \cup B = B \cup A \quad \text{and} \quad \text{(b) } A \cap B = B \cap A.$$

2. *Associative Laws:* For all sets A, B, and C,

$$\text{(a) } (A \cup B) \cup C = A \cup (B \cup C) \quad \text{and}$$
$$\text{(b) } (A \cap B) \cap C = A \cap (B \cap C).$$

3. *Distributive Laws:* For all sets, A, B, and C,

$$\text{(a) } A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \quad \text{and}$$
$$\text{(b) } A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

4. *Identity Laws:* For all sets A,

$$\text{(a) } A \cup \emptyset = A \quad \text{and} \quad \text{(b) } A \cap U = A.$$

5. *Complement Laws:*

$$\text{(a) } A \cup A^c = U \quad \text{and} \quad \text{(b) } A \cap A^c = \emptyset.$$

6. *Double Complement Law:* For all sets A,

$$(A^c)^c = A.$$

Theorem 6.2.2 Set Identities

Let all sets referred to below be subsets of a universal set U .

7. *Idempotent Laws:* For all sets A,

$$\text{(a) } A \cup A = A \quad \text{and} \quad \text{(b) } A \cap A = A.$$

8. *Universal Bound Laws:* For all sets A,

$$\text{(a) } A \cup U = U \quad \text{and} \quad \text{(b) } A \cap \emptyset = \emptyset.$$

9. *De Morgan's Laws:* For all sets A and B,

$$\text{(a) } (A \cup B)^c = A^c \cap B^c \quad \text{and} \quad \text{(b) } (A \cap B)^c = A^c \cup B^c.$$

10. *Absorption Laws:* For all sets A and B,

$$\text{(a) } A \cup (A \cap B) = A \quad \text{and} \quad \text{(b) } A \cap (A \cup B) = A.$$

11. *Complements of U and $\emptyset$:*

$$\text{(a) } U^c = \emptyset \quad \text{and} \quad \text{(b) } \emptyset^c = U.$$

12. *Set Difference Law:* For all sets A and B,

$$A - B = A \cap B^c.$$

➔ We will prove some of these theories in the lecture, please prove others at home

Set Theory

6.2 Properties of Sets

In this lecture:

- ❑ Part 1: Set Relations and Identities
- ❑ Part 2: **Proving Set Identities (Element Argument)**
- ❑ Part 3: Examples of proving Set Identities

Proving Set Identities

Proving That Sets Are Equal

e.g., prove: *HumanMale* = *Man*

Basic Method for Proving That Sets Are Equal

Let sets X and Y be given. To prove that $X = Y$:

1. Prove that $X \subseteq Y$.
2. Prove that $Y \subseteq X$.

But: How to prove that $X \subseteq Y$?

The Element Argument Method

For Proving a set is a subset of another

e.g., prove: *HumanMale* $\subseteq$ *Man*

i.e., Prove that every element in *HumanMale* is an element in *Man*

The Element Argument Method:

Let sets X and Y be Given, To Prove that $\boldsymbol{X \subseteq Y}$**:**

Step 1. Suppose that x is a particular but arbitrarily chosen element in X.

Step 2. Show that x is an element of Y.

The Element Argument Method

In details

Example: Prove that: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

That is:

Prove: $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$

That is, show $\forall x$, if $x \in A \cup (B \cap C)$ then $x \in (A \cup B) \cap (A \cup C)$

Suppose $x \in A \cup (B \cap C)$. [Show $x \in (A \cup B) \cap (A \cup C)$.]

…

Thus $x \in (A \cup B) \cap (A \cup C)$.

Hence $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$.

Prove: $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$

That is, show $\forall x$, if $x \in (A \cup B) \cap (A \cup C)$ then $x \in A \cup (B \cap C)$.

Suppose $x \in (A \cup B) \cap (A \cup C)$. [Show $x \in A \cup (B \cap C)$.]

…

Thus $x \in A \cup (B \cap C)$.

Hence $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$.

Thus $(A \cup B) \cap (A \cup C) = A \cup (B \cap C)$.

Set Theory

6.2 Properties of Sets

In this lecture:

- ❑ Part 1: Set Relations and Identities
- ❑ Part 2: Proving Set Identities (Element Argument)
- ❑ Part 3: **Examples of proving Set Identities**

Proving: **A Distributive Law for Sets**

Theorem 6.2.2(3)(a) A Distributive Law for Sets
For all sets A, B, and C,

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

$A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$:

Suppose $x \in A \cup (B \cap C)$.
$x \in A$ or $x \in B \cap C$. (by def. of union)
Case 1 ($x \in A$): then
$x \in A \cup B$ (by def. of union) and
$x \in A \cup C$ (by def. of union)
$\therefore x \in (A \cup B) \cap (A \cup C)$ (def. of intersection)
Case 2 ($x \in B \cap C$): then
$x \in B$ and $x \in C$ (def. of intersection)
As $x \in B$, $x \in A \cup B$ (by def. of union)
As $x \in C$, $x \in A \cup C$, (by def. of union)
$\therefore x \in (A \cup B) \cap (A \cup C)$ (def. of intersection)

In both cases, $x \in (A \cup B) \cap (A \cup C)$.
Thus: $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$
by definition of subset

$(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$:

Suppose $x \in (A \cup B) \cap (A \cup C)$.
$x \in A \cup B$ and $x \in A \cup C$. (def. of intersection)
Case 1 ($x \in A$): then
$x \in A \cup (B \cap C)$ (by def. of union)
Case 2 ($x \notin A$): then
$x \in B$ and $x \in C$, (def. of intersection)
Then, $x \in B \cap C$ (def. of intersection)
$\therefore x \in A \cup (B \cap C)$

In both cases $x \in A \cup (B \cap C)$.
Thus: $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$
by definition of subset,

Conclusion: Since both subset relations have been proved, it follows by definition of set equality that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Proving: **A De Morgan's Law for Sets**

Theorem 6.2.2(9)(a) A De Morgan's Law for Sets

For all sets A and B, $(A \cup B)^c = A^c \cap B^c$

Same As: proving whether: the people who are not students or employees is the same as the people who are either students nor employees.

$(A \cup B)^c \subseteq A^c \cap B^c$

Suppose $x \in (A \cup B)^c$. *[We must show that $x \in A^c \cap B^c$.]* By definition of complement,

$$x \notin A \cup B.$$

But to say that $x \notin A \cup B$ means that

it is false that (x is in A or x is in B).

By De Morgan's laws of logic, this implies that

x is not in A and x is not in B,

which can be written $x \notin A$ and $x \notin B$.

Hence $x \in A^c$ and $x \in B^c$ by definition of complement. It follows, by definition of intersection, that $x \in A^c \cap B^c$ *[as was to be shown]*. So $(A \cup B)^c \subseteq A^c \cap B^c$ by definition of subset.

$A^c \cap B^c \subseteq (A \cup B)^c$

Suppose $x \in A^c \cap B^c$. *[We must show that $x \in (A \cup B)^c$.]* By definition of intersection, $x \in A^c$ and $x \in B^c$, and by definition of complement,

$$x \notin A \quad \text{and} \quad x \notin B.$$

In other words, x is not in A and x is not in B.

By De Morgan's laws of logic this implies that

it is false that (x is in A or x is in B),

which can be written $x \notin A \cup B$

by definition of union. Hence, by definition of complement, $x \in (A \cup B)^c$ *[as was to be shown]*. It follows that $A^c \cap B^c \subseteq (A \cup B)^c$ by definition of subset.

Theorem 6.2.3 Intersection and Union with a Subset

For any sets A and B, if $A \subseteq B$, then

$$\text{(a) } A \cap B = A \quad \text{and} \quad \text{(b) } A \cup B = B.$$

Prove at home

Proof: If every person is a student, then the set of persons and students are students

Part (a): Suppose A and B are sets with $A \subseteq B$. To show part (a) we must show both that $A \cap B \subseteq A$ and that $A \subseteq A \cap B$. We already know that $A \cap B \subseteq A$ by the inclusion of intersection property. To show that $A \subseteq A \cap B$, let $x \in A$. *[We must show that $x \in A \cap B$.]* Since $A \subseteq B$, then $x \in B$ also. Hence

$$x \in A \quad \text{and} \quad x \in B,$$

and thus

$$x \in A \cap B$$

by definition of intersection *[as was to be shown]*.

Theorem 6.2.4 A Set with No Elements Is a Subset of Every Set

If E is a set with no elements and A is any set, then $E \subseteq A$.

Proof by Contradiction:

Suppose not. [We take the negation of the theorem and suppose it to be true.]
That is, Suppose: E with no elements, and E $\not\subseteq$ A.
assuming (E $\not\subseteq$ A) means there $x \in$E and this x $\notin$ A [by definition of subset].

But there can be no such element since E has no elements. **This is a contradiction.**

Hence the supposition that there are sets E and A, where E has no elements and E $\not\subseteq$ A, is false, and so the theorem is true.

Proving: **Uniqueness of the Empty Set**

Corollary 6.2.5 Uniqueness of the Empty Set

There is only one set with no elements.

Proof:

Suppose E_1 and E_2 are both sets with no elements.

By Theorem 6.2.4, $E_1 \subseteq E_2$ since E_1 has no elements.

Also $E_2 \subseteq E_1$ since E_2 has no elements.

Thus $E_1 = E_2$ by definition of set equality.

Proving: **a Conditional Statement**

Example: If every student is smart and every smart is not-foolish, then there are no foolish students

Proposition 6.2.6

For all sets A, B, and C, if $A \subseteq B$ and $B \subseteq C^c$, then $A \cap C = \emptyset$.

Proof:

Suppose not, Suppose there is an element x in $A \cap C$.

Then $x \in A$ and $x \in C$ (By definition of intersection).

As $A \subseteq B$ then $x \in B$ (by definition of subset).

Also, as $B \subseteq C^c$, then $x \in C^c$ (by definition of subset).

So, $x \notin C$ (by definition of complement)

Thus, $x \in C$ and $x \notin C$, which is a contradiction.

So the supposition that there is an element x in $A \cap C$ is false,
and thus $A \cap C = \emptyset$ [as was to be shown].

Set Theory

6.1. Basics of Set Theory

6.2 Properties of Sets and Element Argument

6.3 Algebraic Proofs

6.4 Boolean Algebras

Set Theory

6.3 Algebraic Proofs

In this lecture:

- ❑ **Part 1: Disapproving and Problem-Solving**
- ❑ Part 2: Algebraic Proofs of Sets

(Dis)proving

Prove that: For all sets A, B, and C, **(A - B) U (B - C) ≠ *A - C* ?**

Example: All people except who are Palestinians with the set of Palestinians except who are female, are the same set as all people except who are female?

Counterexample 1: Let $A = \{1, 2, 4, 5\}$, $B = \{2, 3, 5, 6\}$, and $C = \{4, 5, 6, 7\}$. Then

$$A - B = \{1, 4\}, \quad B - C = \{2, 3\}, \quad \text{and} \quad A - C = \{1, 2\}.$$

Hence

$$(A - B) \cup (B - C) = \{1, 4\} \cup \{2, 3\} = \{1, 2, 3, 4\}, \quad \text{whereas} \quad A - C = \{1, 2\}.$$

Since $\{1, 2, 3, 4\} \neq \{1, 2\}$, we have that $(A - B) \cup (B - C) \neq A - C$.

Counterexample 2: Let $A = \emptyset$, $B = \{3\}$, and $C = \emptyset$. Then

$$A - B = \emptyset, \quad B - C = \{3\}, \quad \text{and} \quad A - C = \emptyset.$$

Hence $(A - B) \cup (B - C) = \emptyset \cup \{3\} = \{3\}$, whereas $A - C = \emptyset$.

Since $\{3\} \neq \emptyset$, we have that $(A - B) \cup (B - C) \neq A - C$.

Problem-Solving Strategy

How can you discover whether a given universal statement about sets is true or false?

حاول قليلا ان تثبت الصحة،
وان احسست عدم الصحة
حاول ايجاد مثال داحض،
ولكن ان احسست الصحة حاول الاثبات،
وهكذا ...

Set Theory

6.3 Algebraic Proofs

In this lecture:

- ❑ Part 1: Disapproving and problem-Solving
- ❑ **Part 2: Algebraic Proofs of Sets**

Remember the following

$$\underbrace{A_1}_{A} \cap (\underbrace{A_2}_{(B} \cup \underbrace{A_3}_{C)}) = (\underbrace{A_1}_{(A} \cap \underbrace{A_2}_{B)}) \cup (\underbrace{A_1}_{(A} \cap \underbrace{A_3}_{C)}),$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$(\underbrace{W \cap X}) \cap (Y \cup Z) = ((\underbrace{W \cap X}) \cap Y) \cup ((\underbrace{W \cap X}) \cap Z),$$

$$\updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Algebraic Proofs

Deriving a Set Difference Property

Construct an algebraic proof that for all sets A, B, and C,

$(A \cup B) - C = (A - C) \cup (B - C)$.

$$\begin{aligned}
(A \cup B) - C &= (A \cup B) \cap C^c && \text{by the set difference law} \\
&= C^c \cap (A \cup B) && \text{by the commutative law for } \cap \\
&= (C^c \cap A) \cup (C^c \cap B) && \text{by the distributive law} \\
&= (A \cap C^c) \cup (B \cap C^c) && \text{by the commutative law for } \cap \\
&= (A - C) \cup (B - C) && \text{by the set difference law.}
\end{aligned}$$

Cite a property from Theorem 6.2.2 for every step of the proof.

Algebraic Proofs

Deriving a Set Identity Using Properties of ∅

Construct an algebraic proof that for all sets A and B,

$$\boldsymbol{A - (A \cap B) = A - B.}$$

$A - (A \cap B)$	$= A \cap (A \cap B)^c$	by the set difference law
	$= A \cap (A^c \cup B^c)$	by De Morgan's laws
	$= (A \cap A^c) \cup (A \cap B^c)$	by the distributive law
	$= \emptyset \cup (A \cap B^c)$	by the complement law
	$= (A \cap B^c) \cup \emptyset$	by the commutative law for ∪
	$= A \cap B^c$	by the identity law for ∪
	$= A - B$	by the set difference law.

Set Theory

6.1. Basics of Set Theory

6.2 Properties of Sets and Element Argument

6.3 Algebraic Proofs

6.4 **Boolean Algebra**

Set Theory

6.4 Boolean Algebra

In this lecture:

- Part 1: **History of Algebra**
- Part 2: What is Boolean Algebra
- Part 3: Proving Boolean Algebra Properties

What is Algebra?

Al-Khwarizmi 850 – 780 (Baghdad)

الكتاب المختصر في حساب الجبر والمقابلة

The Compendious Book on Calculation by Completion and Balancing

Developed an advanced arithmetical system with which they were able to do calculations in an algorithmic fashion.

Statements to describe relationships between things

Symbols and the rules for manipulating these symbols

Do you know any algebra (جبر)?

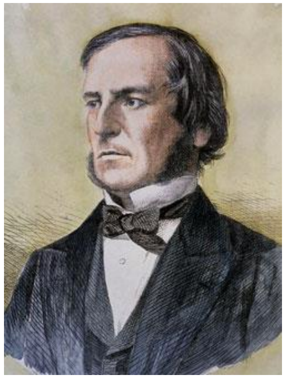
Set Theory

6.4 Boolean Algebra

In this lecture:

- ❑ Part 1: History of Algebra
- ❑ Part 2: **What is Boolean Algebra**
- ❑ Part 3: Proving Boolean Algebra Properties

Boolean Algebra



George Boole
1815-1864,
England

Introduced by George Boole in his first book The Mathematical Analysis of Logic (1847),

A structure abstracting the computation with the truth values false and true.

Instead of elementary algebra where the values of the variables are numbers, and the main operations are addition and multiplication, the main operations of Boolean algebra are the conjunction (∧) the disjunction (∨) and the negation not (¬).

Used extensively in the simplification of logic Circuits

Compare

Logical Equivalences	Set Properties
For all statement variables p, q, and r:	For all sets A, B, and C:
a. $p \vee q \equiv q \vee p$ b. $p \wedge q \equiv q \wedge p$	a. $A \cup B = B \cup A$ b. $A \cap B = B \cap A$
a. $p \wedge (q \wedge r) \equiv p \wedge (q \wedge r)$ b. $p \vee (q \vee r) \equiv p \vee (q \vee r)$	a. $A \cup (B \cup C) \equiv A \cup (B \cup C)$ b. $A \cap (B \cap C) \equiv A \cap (B \cap C)$
a. $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ b. $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	a. $A \cap (B \cup C) \equiv (A \cap B) \cup (A \cap C)$ b. $A \cup (B \cap C) \equiv (A \cup B) \cap (A \cup C)$
a. $p \vee \mathbf{c} \equiv p$ b. $p \wedge \mathbf{t} \equiv p$	a. $A \cup \emptyset = A$ b. $A \cap U = A$
a. $p \vee \sim p \equiv \mathbf{t}$ b. $p \wedge \sim p \equiv \mathbf{c}$	a. $A \cup A^c = U$ b. $A \cap A^c = \emptyset$
$\sim(\sim p) \equiv p$	$(A^c)^c = A$
a. $p \vee p \equiv p$ b. $p \wedge p \equiv p$	a. $A \cup A = A$ b. $A \cap A = A$
a. $p \vee \mathbf{t} \equiv \mathbf{t}$	a. $A \cup U = U$

Compare

Logical Equivalences	Set Properties
For all statement variables p, q, and r:	For all sets A, B, and C:
a. $p \vee q \equiv q \vee p$ b. $p \wedge q \equiv q \wedge p$	a. $A \cup B = B \cup A$ b. $A \cap B = B \cap A$
a. $p \wedge (q \wedge r) \equiv p \wedge (q \wedge r)$ b. $p \vee (q \vee r) \equiv p \vee (q \vee r)$	a. $A \cup (B \cup C) \equiv A \cup (B \cup C)$ b. $A \cap (B \cap C) \equiv A \cap (B \cap C)$
a. $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ b. $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	a. $A \cap (B \cup C) \equiv (A \cap B) \cup (A \cap C)$ b. $A \cup (B \cap C) \equiv (A \cup B) \cap (A \cup C)$
a. $p \vee \mathbf{c} \equiv p$ b. $p \wedge \mathbf{t} \equiv p$	a. $A \cup \emptyset = A$ b. $A \cap U = A$
a. $p \vee \sim p \equiv \mathbf{t}$ b. $p \wedge$	a. $A \cup A^c = U$
$\sim(\sim$	
a. $p \vee p \equiv p$ b. $p \wedge p \equiv p$	a. $A \cup A = A$ b. $A \cap A = A$
a. $p \vee \mathbf{t} \equiv \mathbf{t}$	a. $A \cup U = U$

Both are special cases of the same general structure, known as a *Boolean Algebra*.

Boolean Algebra

• Definition: Boolean Algebra

A **Boolean algebra** is a set B together with two operations, generally denoted $+$ and $\cdot$, such that for all a and b in B both $a + b$ and $a \cdot b$ are in B and the following properties hold:

1. *Commutative Laws:* For all a and b in B,

$$\text{(a) } a + b = b + a \quad \text{and} \quad \text{(b) } a \cdot b = b \cdot a.$$

2. *Associative Laws:* For all a, b, and c in B,

$$\text{(a) } (a + b) + c = a + (b + c) \quad \text{and} \quad \text{(b) } (a \cdot b) \cdot c = a \cdot (b \cdot c).$$

3. *Distributive Laws:* For all a, b, and c in B,

$$\text{(a) } a + (b \cdot c) = (a + b) \cdot (a + c) \quad \text{and} \quad \text{(b) } a \cdot (b + c) = (a \cdot b) + (a \cdot c).$$

4. *Identity Laws:* There exist distinct elements 0 and 1 in B such that for all a in B,

$$\text{(a) } a + 0 = a \quad \text{and} \quad \text{(b) } a \cdot 1 = a.$$

5. *Complement Laws:* For each a in B, there exists an element in B, denoted $\overline{a}$ and called the **complement** or **negation** of a, such that

$$\text{(a) } a + \overline{a} = 1 \quad \text{and} \quad \text{(b) } a \cdot \overline{a} = 0.$$

 ,

Properties of a Boolean Algebra

Theorem 6.4.1 Properties of a Boolean Algebra

Let B be any Boolean algebra.

1. *Uniqueness of the Complement Law:* For all a and x in B, if $a + x = 1$ and $a \cdot x = 0$ then $x = \overline{a}$.
2. *Uniqueness of* 0 *and* 1: If there exists x in B such that $a + x = a$ for all a in B, then $x = 0$, and if there exists y in B such that $a \cdot y = a$ for all a in B, then $y = 1$.
3. *Double Complement Law:* For all $a \in B$, $\overline{(\overline{a})} = a$.
4. *Idempotent Law:* For all $a \in B$,

   (a) $a + a = a$ and (b) $a \cdot a = a$.
5. *Universal Bound Law:* For all $a \in B$,

   (a) $a + 1 = 1$ and (b) $a \cdot 0 = 0$.
6. *De Morgan's Laws:* For all a and $b \in B$,

   (a) $\overline{a + b} = \overline{a} \cdot \overline{b}$ and (b) $\overline{a \cdot b} = \overline{a} + \overline{b}$.
7. *Absorption Laws:* For all a and $b \in B$,

   (a) $(a + b) \cdot a = a$ and (b) $(a \cdot b) + a = a$.
8. *Complements of* 0 *and* 1:

   (a) $\overline{0} = 1$ and (b) $\overline{1} = 0$.

Set Theory

6.4 Boolean Algebra

In this lecture:

- ❑ Part 1: History of Algebra
- ❑ Part 2: What is Boolean Algebra
- ❑ Part 3: **Proving Boolean Algebra Properties**

Proving of Boolean Algebra Properties

Uniqueness of the Complement Law: For all a and x in B, if $a + x = 1$ and $a \cdot x = 0$ then $x = \overline{a}$.

Proof:

Suppose a and x are particular, but arbitrarily chosen, elements of B that satisfy the following hypothesis: $a + x = 1$ and $a \cdot x = 0$. Then

$x = x \cdot 1$	because 1 is an identity for $\cdot$
$= x \cdot (a + \overline{a})$	by the complement law for $+$
$= x \cdot a + x \cdot \overline{a}$	by the distributive law for $\cdot$ over $+$
$= a \cdot x + x \cdot \overline{a}$	by the commutative law for $\cdot$
$= 0 + x \cdot \overline{a}$	by hypothesis
$= a \cdot \overline{a} + x \cdot \overline{a}$	by the complement law for $\cdot$
$= (\overline{a} \cdot a) + (\overline{a} \cdot x)$	by the commutative law for $\cdot$
$= \overline{a} \cdot (a + x)$	by the distributive law for $\cdot$ over $+$
$= \overline{a} \cdot 1$	by hypothesis
$= \overline{a}$	because 1 is an identity for $\cdot$.

Proving of Boolean Algebra Properties

Theorem 6.4.1(3) Double Complement Law

For all elements a in a Boolean algebra B, $\overline{(\overline{a})} = a$.

Proof:

Suppose B is a Boolean algebra and a is any element of B. Then

$$\begin{aligned} \overline{a} + a &= a + \overline{a} && \text{by the commutative law} \\ &= 1 && \text{by the complement law for 1} \end{aligned}$$

and

$$\begin{aligned} \overline{a} \cdot a &= a \cdot \overline{a} && \text{by the commutative law} \\ &= 0 && \text{by the complement law for 0.} \end{aligned}$$

Thus a satisfies the two equations with respect to $\overline{a}$ that are satisfied by the complement of $\overline{a}$. From the fact that the complement of a is unique, we conclude that $\overline{(\overline{a})} = a$.