Sinusoidal Steady-State Analysis

The Sinusoidal Source

$$v_s(t) = V_m \sin \omega t$$

$V_m \equiv$ Amplitude of the Sinusoid

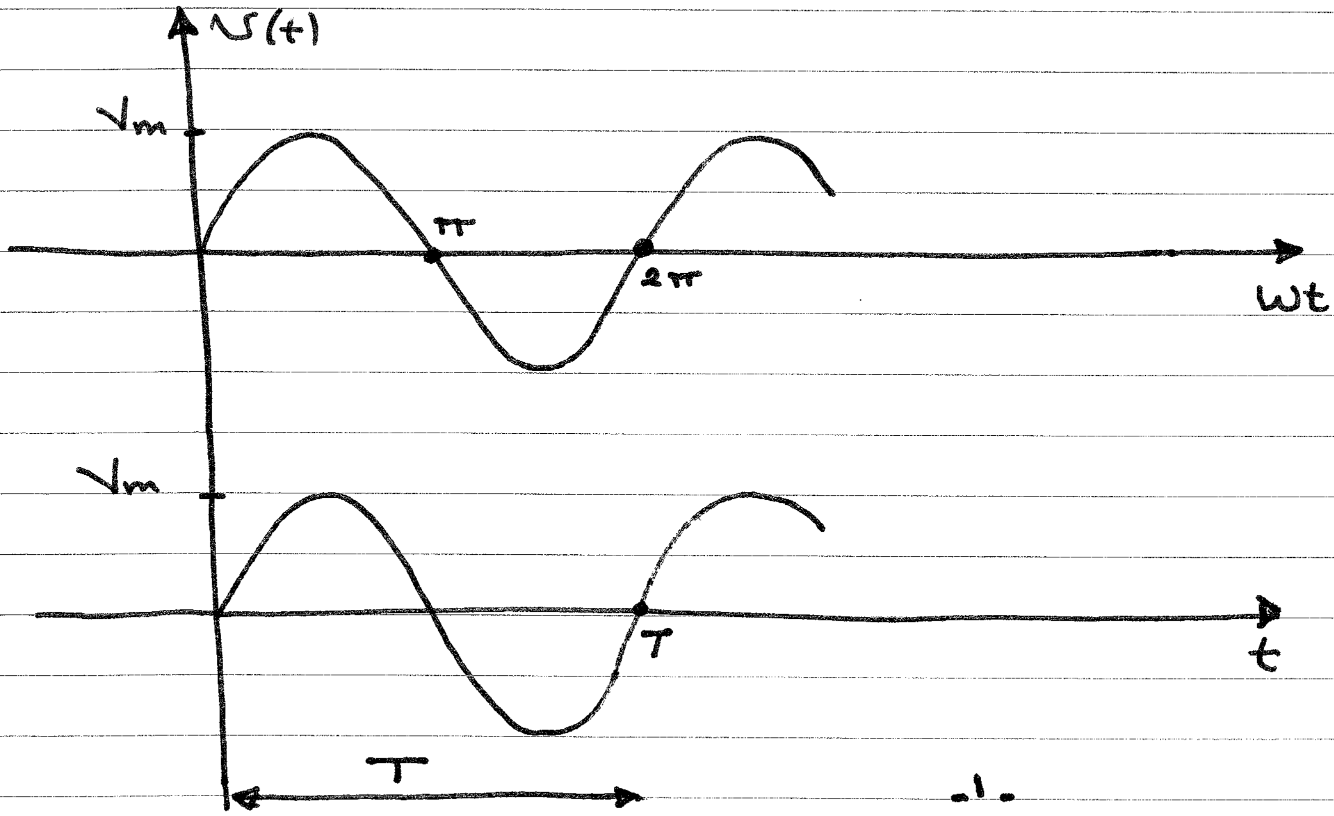
$\omega \equiv$ Angular frequency in radian/s

$$\omega = 2\pi f$$

$f \equiv$ frequency in Hertz

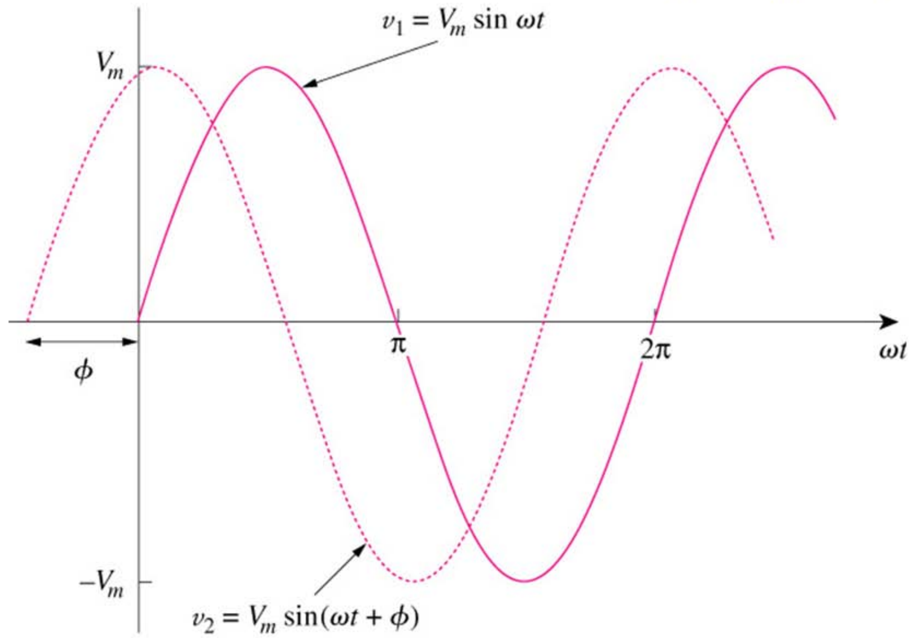
$$f = \frac{1}{T}$$

$T \equiv$ Period in seconds



Phase of Sinusoids

➢ Consider the sinusoidal voltage having phase φ, $v(t) = V_m \sin(\omega t + \phi)$



- v_2 LEADS v_1 by phase φ.
- v_1 LAGS v_2 by phase φ.
- v_1 and v_2 are out of phase.

Phase of Sinusoids

The terms Lead and Lag are used to indicate the relationship between two sinusoidal wave forms of the same frequency plotted on the same set of axes.

$v_1(t) = V_m \sin \omega t$

$v_2(t) = V_m \sin(\omega t + \theta)$

$\therefore$ $v_2(t)$ Leads $v_1(t)$ by θ

or

$v_1(t)$ Lags $v_2(t)$ by θ

Trigonometric Identities

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\sin(\omega t \pm 180^\circ) = -\sin \omega t$$

$$\cos(\omega t \pm 180^\circ) = -\cos \omega t$$

$$\sin(\omega t \pm 90^\circ) = \pm \cos \omega t$$

$$\cos(\omega t \pm 90^\circ) = \mp \sin \omega t$$

$$A \cos \omega t + B \sin \omega t = C \cos(\omega t - \theta)$$

where

$$C = \sqrt{A^2 + B^2} \quad \text{and} \quad \theta = \tan^{-1} \frac{B}{A}$$

Let $v_1(t) = 10 \sin(5t - 30^\circ)$

$v_2(t) = 15 \sin(5t + 10^\circ)$

$\therefore$ $v_2(t)$ Leads $v_1(t)$ by 40°

Let $i_1(t) = 2 \sin(377t + 45^\circ)$

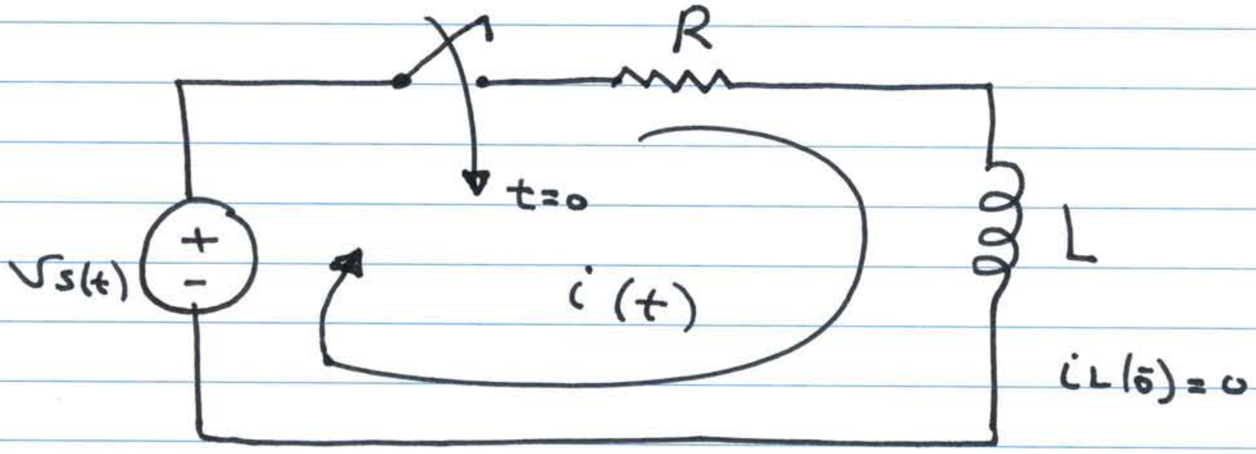
$i_2(t) = 0.5 \cos(377t + 10^\circ)$

$\cos\alpha = \sin(\alpha + 90^\circ)$

$0.5 \cos(377t + 10^\circ) = 0.5 \sin(377t + 100^\circ)$

$\therefore$ $i_2(t)$ leads $i_1(t)$ by 55°

The sinusoidal Response



Find $i(t)$ for $t > 0$

given $v_s(t) = V_m \cos \omega t$ V

KVL :

$$v_s(t) = R\,i(t) + L\frac{di(t)}{dt}$$

$$V_m \cos \omega t = R\,i(t) + L\frac{di(t)}{dt}$$

First order nonhomogenouse differential equation

$$\therefore \quad i(t) = i_n(t) + i_f(t)$$

$$i(t) = A e^{-t/\tau} + i_f(t)$$

$$i_f(t) = I_1 \cos \omega t + I_2 \sin \omega t$$

To find I_1 and I_2

$$V_m \cos\omega t = R\,i(t) + L\frac{di(t)}{dt}$$

$$V_m \cos\omega t = R\left[I_1 \cos\omega t + I_2 \sin\omega t\right]$$

$$+ L\omega\left[-I_1 \sin\omega t + I_2 \cos\omega t\right]$$

Collect the cosine and sine terms

$$0 = \left(-L I_1 \omega + R I_2\right)\sin\omega t + \left(L I_2 \omega + R I_1 - V_m\right)\cos\omega t$$

$\therefore$

$$-\omega L I_1 + R I_2 = 0$$

$$\omega L I_2 + R I_1 - V_m = 0$$

$$I_1 = \frac{R V_m}{R^2 + \omega^2 L^2}$$

$$I_2 = \frac{\omega L V_m}{R^2 + \omega^2 L^2}$$

$$\therefore i_f(t) = \frac{R V_m}{R^2 + \omega^2 L^2}\cos\omega t + \frac{\omega L V_m}{R^2 + \omega^2 L^2}\sin\omega t$$

$$i_f(t) = I_1 \cos\omega t + I_2 \sin\omega t$$

$$i_f(t) = C \cos(\omega t - \phi)$$

$$i_f(t) = C \cos\omega t \cos\phi + C \sin\omega t \sin\phi$$

$$\therefore \quad I_1 = C \cos\phi$$

$$I_2 = C \sin\phi$$

$$\frac{I_2}{I_1} = \tan\phi$$

$$\therefore \quad \phi = \tan^{-1}\frac{I_2}{I_1} = \tan^{-1}\frac{\omega L}{R} \quad \text{——— (1)}$$

$$I_1^2 + I_2^2 = C^2\cos^2\phi + C^2\sin^2\phi$$

$$I_1^2 + I_2^2 = C^2$$

$$\therefore \quad C = \sqrt{I_1^2 + I_2^2}$$

$$C = \frac{V_m}{\sqrt{R^2+\omega^2L^2}} \quad \text{——— (2)}$$

$$\therefore \quad i_f(t) = \frac{V_m}{\sqrt{R^2+\omega^2L^2}} \cos\left(\omega t - \tan^{-1}\frac{\omega L}{R}\right)$$

$$\therefore \quad i(t) = A e^{-t/\tau} + \frac{V_m}{\sqrt{R^2+\omega^2 L^2}} \cos\left(\omega t - \tan^{-1}\frac{\omega L}{R}\right)$$

$$i(0^+) = A + \frac{V_m}{\sqrt{R^2+\omega^2 L^2}} \cos\left(-\tan^{-1}\frac{\omega L}{R}\right) = 0$$

$$\therefore \quad A = -\frac{V_m}{\sqrt{R^2+\omega^2 L^2}} \cos\left(-\tan^{-1}\frac{\omega L}{R}\right)$$

$$\therefore \quad i(t) = i_n(t) + i_f(t)$$

$i(t)$ = transient Componet + Steady-state Component

* The steady-state solution is a sinusoidal function with the same frequency as the source signal.

Complex Numbers

A complex number may be written in three forms

1) Rectangular Form

$$Z = x + jy$$

$j = \sqrt{-1}$, $x = Re(Z)$, $y = Im(Z)$

2) Exponential Form

$$Z = |Z| e^{j\theta}$$

$|Z|$ = Magnitude , θ = angle

3) Polar Form

$$Z = |Z| \angle \theta$$

Euler's Law

$$e^{j\theta} = \cos\theta + j\sin\theta$$

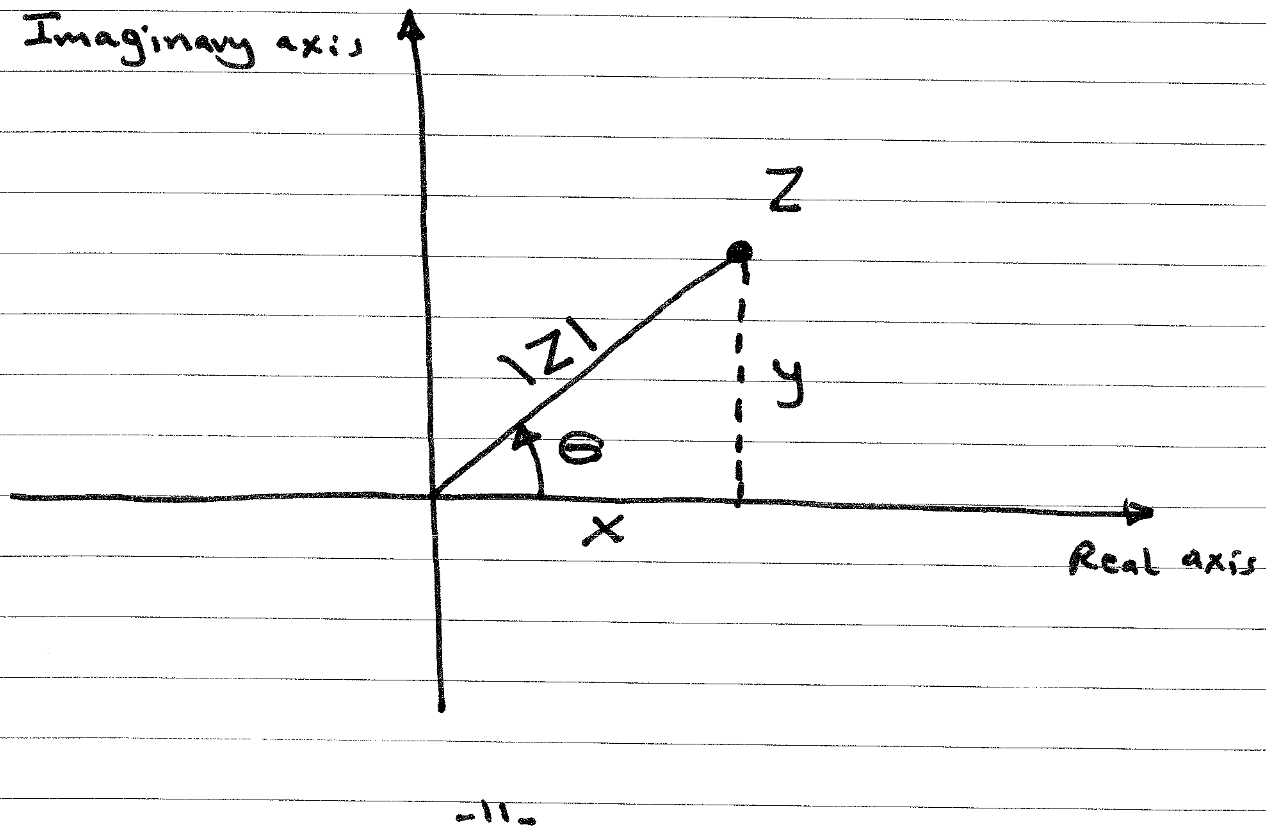
$$Z = |Z|\, e^{j\theta}$$

$$Z = |Z|\cos\theta + j|Z|\sin\theta$$

$$Z = x + jy$$

$$\therefore\ x = |Z|\cos\theta$$

$$y = |Z|\sin\theta$$



Mathematical Operations of Complex numbers

Addition : $Z_1 + Z_2 = (x_1 + x_2) + j(y_1 + y_2)$

Subtraction : $Z_1 - Z_2 = (x_1 - x_2) + j(y_1 - y_2)$

Multiplication : $Z_1 Z_2 = |Z_1||Z_2| \angle \theta_1 + \theta_2$

Division : $\dfrac{Z_1}{Z_2} = \dfrac{|Z_1|}{|Z_2|} \angle \theta_1 - \theta_2$

Complex Conjugate : $Z^* = x - jy$

$= |Z| \angle -\theta$

$$x = |Z| \cos\theta$$

$$y = |Z| \sin\theta$$

$$x^2 + y^2 = |Z|^2 \cos^2\theta + |Z| \sin^2\theta$$

$$x^2 + y^2 = |Z|^2 \left(\cos^2\theta + \sin^2\theta \right)$$

$$x^2 + y^2 = |Z|^2$$

$$\therefore |Z| = \sqrt{x^2 + y^2}$$

$$\frac{y}{x} = \frac{|Z| \sin\theta}{|Z| \cos\theta} = \tan\theta$$

$$\therefore \theta = \tan^{-1} \frac{y}{x}$$

$$Z_1 = 4 + j3 = 5\angle 36.9^\circ$$

$$Z_2 = 3 + j4 = 5\angle 53.1^\circ$$

$$Z_1 + Z_2 = 7 + j7$$

$$Z_1 - Z_2 = 1 - j1$$

$$Z_1 Z_2 = 5\angle 36.9^\circ \cdot 5\angle 53.1^\circ = 25\angle 90^\circ$$

$$\frac{Z_1}{Z_2} = \frac{5\angle 36.9^\circ}{5\angle 53.1^\circ} = 1\angle -16.2^\circ$$

or

$$Z_1 Z_2 = (4+j3)(3+j4)$$

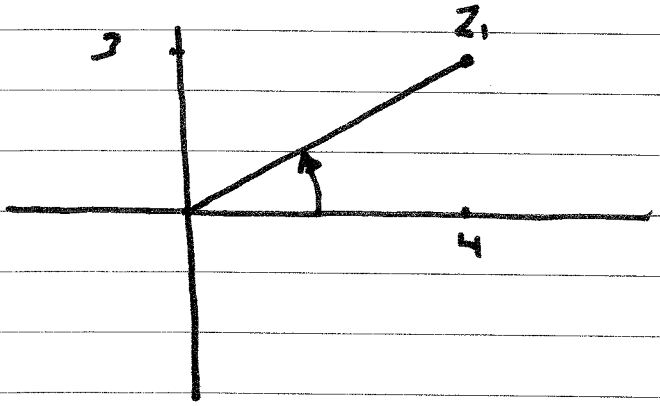
$$= 12 + j16 + j9 - 12$$

$$Z_1 Z_2 = j25$$

$$\frac{Z_1}{Z_2} = \frac{4+j3}{3+j4} \cdot \frac{3-j4}{3-j4} = \frac{12 - j16 + j9 + 12}{25}$$

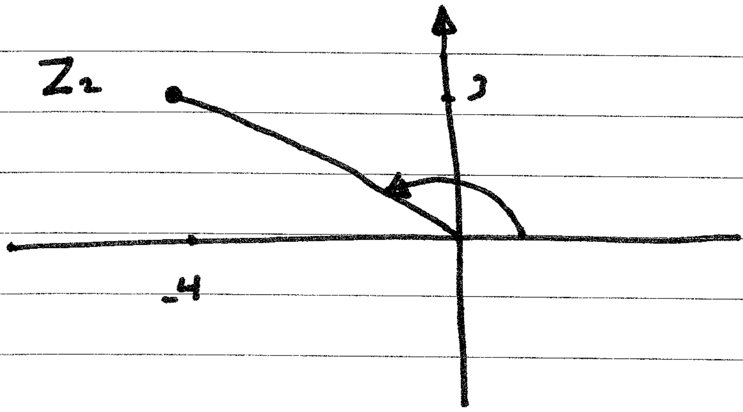
$$= \frac{24 - j7}{25} = \frac{24}{25} - j\frac{7}{25}$$

The graphical Representation



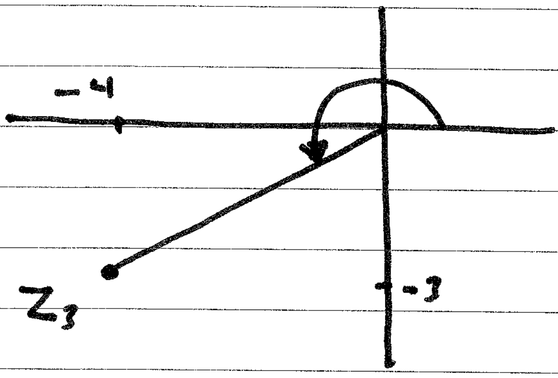
$Z_1 = 4 + j3$

$Z_1 = 5\angle 36.9^\circ$



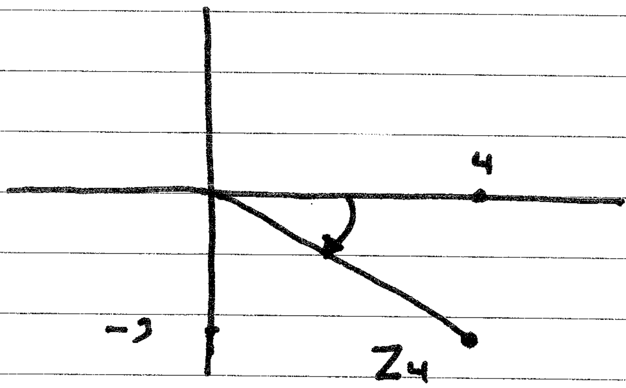
$Z_2 = -4 + j3$

$Z_2 = 5\angle 143.1$



$Z_3 = -4 - j3$

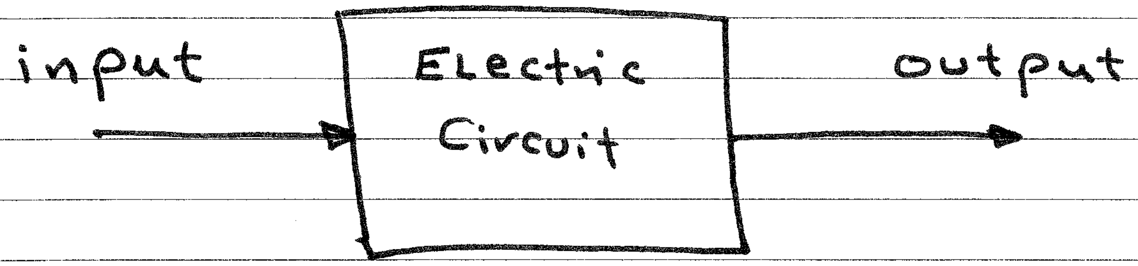
$Z_3 = 5\angle 216.9$



$Z_4 = 4 - j3$

$Z_4 = 5\angle -36.9^\circ$

The phasor Concept



$$V_m \cos(\omega t + \theta) \longrightarrow I_m \cos(\omega t + \phi)$$

$$V_m \sin(\omega t + \theta) \longrightarrow I_m \sin(\omega t + \phi)$$

$$j V_m \sin(\omega t + \theta) \longrightarrow j I_m \sin(\omega t + \phi)$$

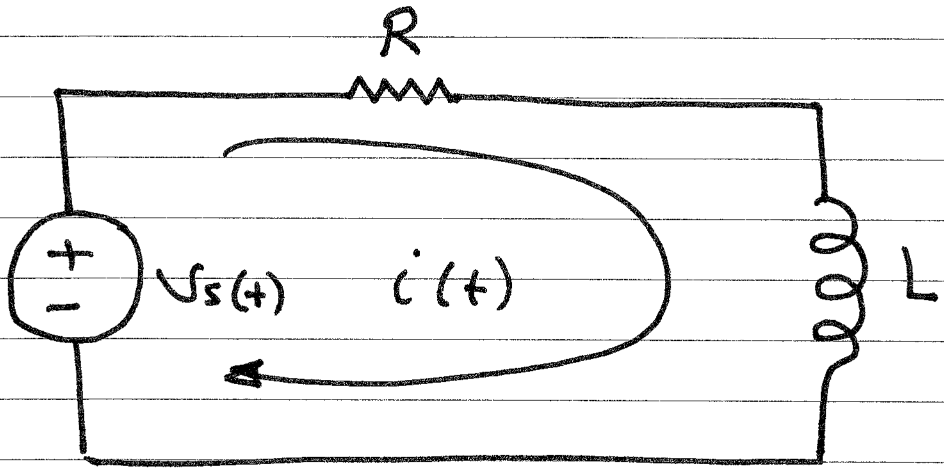
$$\begin{matrix} V_m \cos(\omega t + \theta) \\ + \\ j V_m \sin(\omega t + \theta) \end{matrix} \Longrightarrow \begin{matrix} I_m \cos(\omega t + \phi) \\ + \\ j I_m \sin(\omega t + \phi) \end{matrix}$$

$$V_m e^{j(\omega t + \theta)} \longrightarrow I_m e^{j(\omega t + \phi)}$$

Instead of Applying a real forcing function to obtain the desired real response, we apply a Complex forcing function whose real part is the given real forcing function.

We obtain a Complex response whose real part is the desired real response.

Sinusoidal and Complex forcing function



$$V_s(t) = V_m \cos \omega t$$

$$\therefore \quad i(t) = I_m \cos(\omega t + \phi)$$

$$V_s(t) \longrightarrow V_m e^{j\omega t}$$

$$i(t) \longrightarrow I_m e^{j(\omega t + \phi)}$$

KVL :

$$V_s(t) = R\, i(t) + L \frac{d\, i(t)}{dt}$$

$$V_m e^{j\omega t} = R I_m e^{j(\omega t + \phi)} + j\omega L I_m e^{j(\omega t + \phi)}$$

a Complex algebraic equation

To find I_m and ϕ ; devide by $e^{j\omega t}$

$$V_m = R I_m e^{j\phi} + j\omega L I_m e^{j\phi}$$

$$V_m = I_m e^{j\phi}(R + j\omega L)$$

$$\therefore I_m e^{j\phi} = \frac{V_m}{R + j\omega L}$$

$$I_m e^{j\phi} = \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}\; e^{j\tan^{-1}\frac{\omega L}{R}}}$$

$$I_m e^{j\phi} = \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}}\, e^{-j\tan^{-1}\frac{\omega L}{R}}$$

$$\therefore I_m = \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}}$$

$$\phi = -\tan^{-1}\frac{\omega L}{R}$$

$$\therefore i(t) = I_m \cos(\omega t + \phi)$$

$$i(t) = \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos(\omega t + \phi)$$

Phasors

Given the sinusoids $i(t) = I_m \cos(\omega t + \phi_i)$

and $v(t) = V_m \cos(\omega t + \theta_v)$

We can obtain the phasor forms as:

$i(t) = I_m \cos(\omega t + \phi_i)$, then $\vec{I} = I_m \angle \phi_i$

$v(t) = V_m \cos(\omega t + \theta_v)$, then $\vec{V} = V_m \angle \theta_v$

$$i(t) = 6 \cos(50t - 40^\circ) \text{ A}$$

$$\therefore \vec{I} = 6 \angle -40^\circ \text{ A}$$

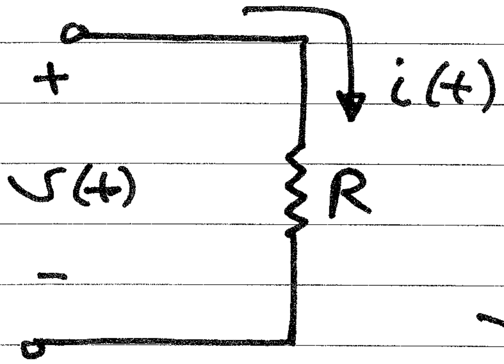
$$v(t) = -4 \sin(30t + 50^\circ) \text{ V}$$

$$v(t) = 4 \cos(30t + 140^\circ) \text{ V}$$

$$\therefore \vec{V} = 4 \angle 140^\circ \text{ V}$$

Phasor Relationships for Circuit Elements

Resistor :



$$v(t) = R\,i(t)$$

$$V_m e^{j(\omega t + \theta_v)} = R\,I_m e^{j(\omega t + \phi)}$$

$$V_m e^{j\theta_v} = R\,I_m e^{j\phi_i}$$

$$V_m \underline{\angle \theta_v} = R\,I_m \underline{\angle \phi_i}$$

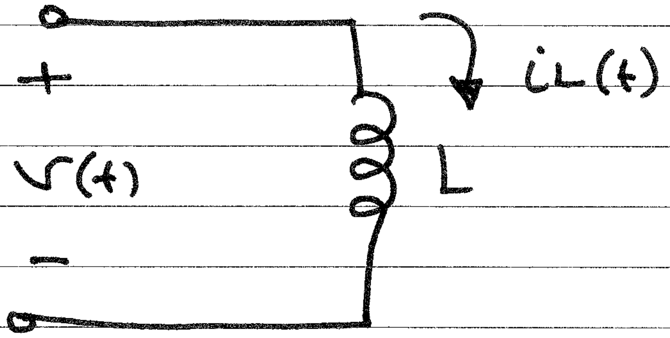
$$\boxed{\vec{V} = R\,\vec{I}}$$

$$V_m = R I_m$$

$$\theta_v = \phi_i$$

* Voltage and Current of a resistor are in phase.

Inductor :



$$v(t) = L\,\frac{di(t)}{dt}$$

$$V_m e^{j(\omega t+\theta_v)} = L\,\frac{d}{dt}\left(I_m e^{j(\omega t+\phi_i)}\right)$$

$$V_m e^{j(\omega t+\theta_v)} = j\omega L\, I_m e^{j(\omega t+\phi_i)}$$

$$V_m e^{j\theta_v} = j\omega L\, I_m e^{j\phi_i}$$

$$V_m \underline{\angle \theta_v} = j\omega L\, I_m \underline{\angle \phi_i}$$

$$\boxed{\vec{V} = j\omega L\,\vec{I}}$$

$*$ $\quad V_m = \omega L I$

$$V_m \underline{\angle \theta_v} = \omega L \underline{\angle 90^\circ} \cdot I_m \underline{\angle \phi_i}$$

$$V_m \underline{\angle \theta_v} = \omega L I_m \underline{\angle \phi_i + 90^\circ}$$

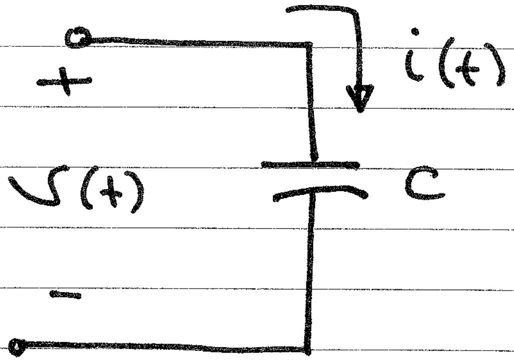
$$V_m \angle \theta_v = \omega L I_m \angle \phi_i + 90^\circ$$

$$\therefore \quad V_m = \omega L I_m$$

$$\theta_v = \phi_i + 90^\circ$$

The voltage Leads the Current by 90°

Capacitor :



$$i(t) = C\,\frac{dv(t)}{dt}$$

$$I_m\, e^{j(\omega t+\phi_i)} = C\,\frac{d}{dt}\left(V_m\, e^{j(\omega t+\theta_v)}\right)$$

$$I_m\, e^{j(\omega t+\phi_i)} = j\omega C\, V_m\, e^{j(\omega t+\theta_v)}$$

$$I_m\, e^{j\phi_i} = j\omega C\, V_m\, e^{j\theta_v}$$

$$I_m \angle \phi_i = j\omega C\, V_m \angle \theta_v$$

$$\boxed{\vec{I} = j\omega C\, \vec{V}}$$

$$I_m \angle \phi_i = \omega C \angle 90^\circ\ V_m \angle \theta_v$$

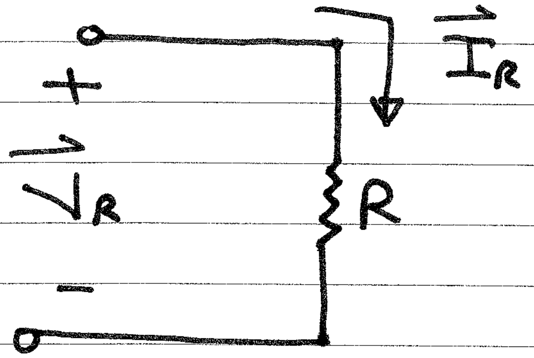
$$I_m \angle \phi_i = \omega C V_m \angle \theta_v + 90^\circ$$

$\therefore \quad I_m = \omega C V_m$

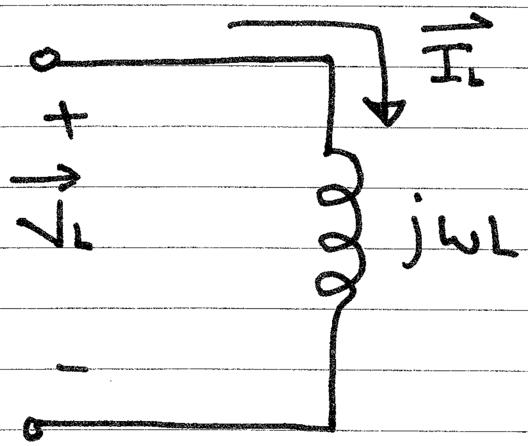
$\phi_i = \theta_v + 90^\circ$

The Current Leads the voltage by 90°.

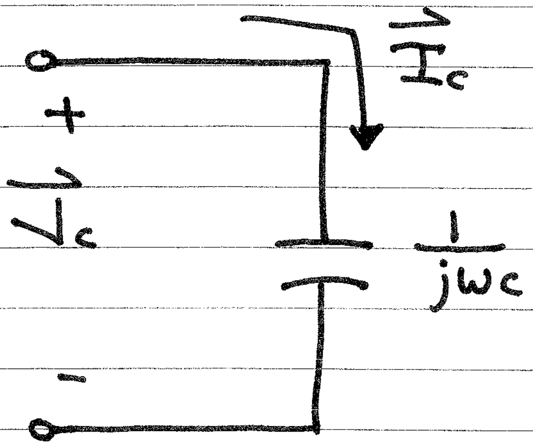
Phasor Relationships for Circuit Elements



$$\vec{V}_R = R\,\vec{I}_R$$



$$\vec{V}_L = j\omega L\,\vec{I}_L$$



$$\vec{V}_c = \frac{1}{j\omega c}\,\vec{I}_c$$

$$\vec{V} = Z(j\omega)\,\vec{I}$$

Impedance and Admittance

$$Z(j\omega) = \frac{\vec{V}}{\vec{I}}$$ Impedance, Ω

or $\vec{V} = Z(j\omega)\,\vec{I}$

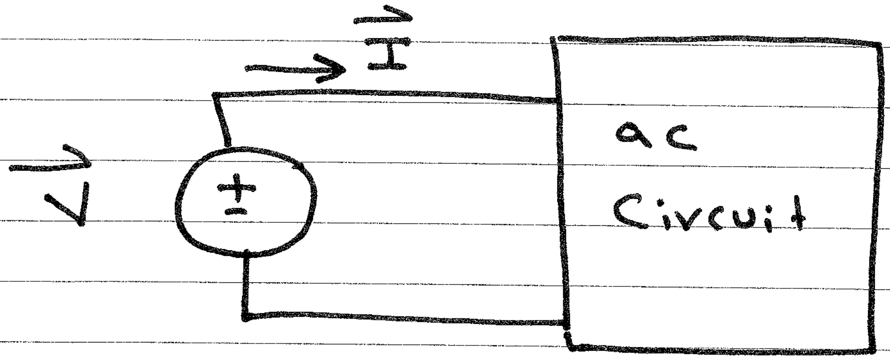
$$Y(j\omega) = \frac{\vec{I}}{\vec{V}}$$ Admittance, $\mho$

or $\vec{I} = Y(j\omega)\,\vec{V}$

$$\therefore \quad Z(j\omega) = \frac{1}{Y(j\omega)}$$

Element	Impedance	Admittance
R	$Z(j\omega) = R$	$Y(j\omega) = \frac{1}{R}$
C	$Z(j\omega) = \frac{1}{j\omega C}$	$Y(j\omega) = j\omega C$
L	$Z(j\omega) = j\omega L$	$Y(j\omega) = \frac{1}{j\omega L}$

Impedance : $Z(j\omega)$



$$Z(j\omega) = \frac{\vec{V}}{\vec{I}} = \frac{V_m \angle \theta_v}{I_m \angle \phi_i}$$

$$Z(j\omega) = \frac{V_m}{I_m} \angle \theta_v - \phi_i$$

$$Z(j\omega) = |Z| \angle \theta_z$$

The unit of impedance is Ohm

Impedance is not aphasor but a Complex number that Can be Written in polar or Cartesian forms

$$\vec{Z} = R + jX$$

$R \equiv$ Resistive part

$X \equiv$ Reactive part

$$Z = |Z| \angle \theta_z$$

$$Z = R + jX$$

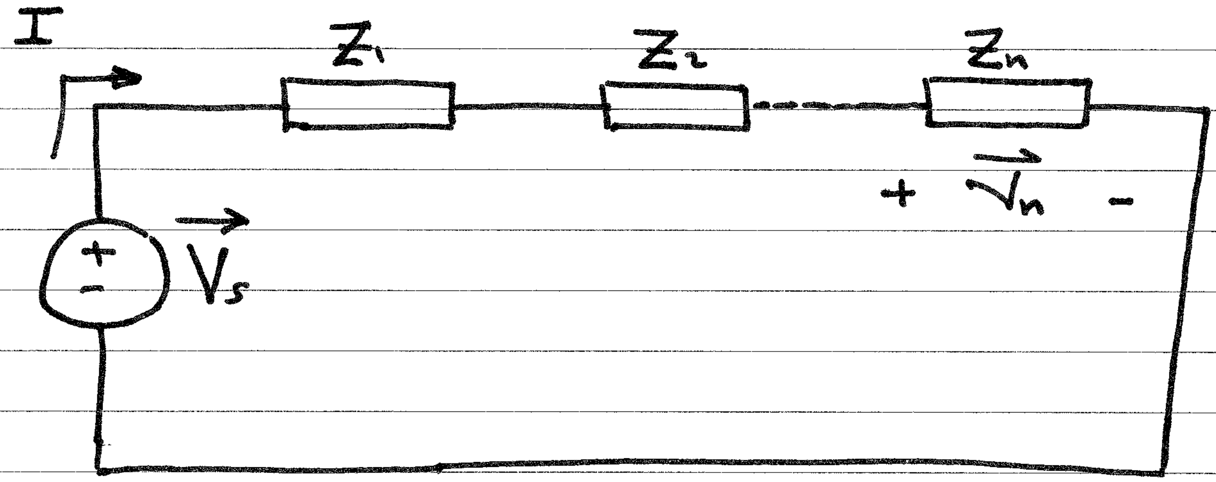
$$|Z| = \sqrt{R^2 + X^2}$$

$$\theta_z = \tan^{-1} \frac{X}{R}$$

$$X = |Z| \sin \theta_z$$

$$R = |Z| \cos \theta_z$$

Application of KVL for phasors



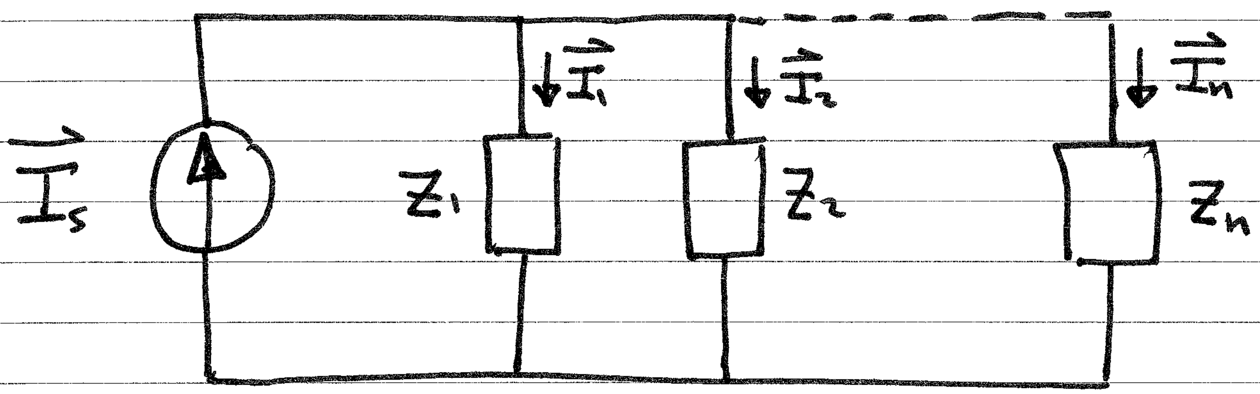
KVL : $v_s(t) = v_1(t) + v_2(t) + \dots\dots v_n(t)$

$$\vec{V}_s = \vec{V}_1 + \vec{V}_2 + \dots\dots \vec{V}_n$$

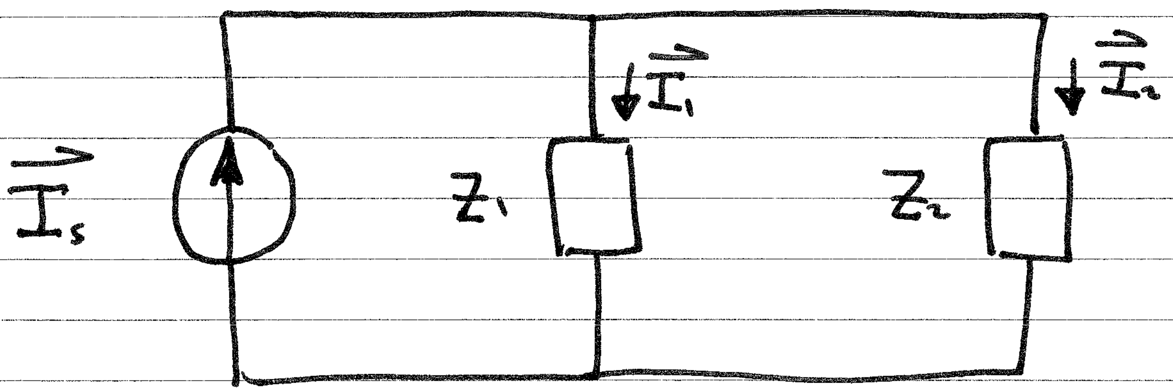
$$Z_{eq} = Z_1 + Z_2 + Z_3 + \dots\dots Z_n$$

$$\vec{V}_n = \frac{Z_n}{Z_1 + Z_2 + \dots\dots Z_n} \cdot \vec{V}_s$$

Application of KCL for phasors



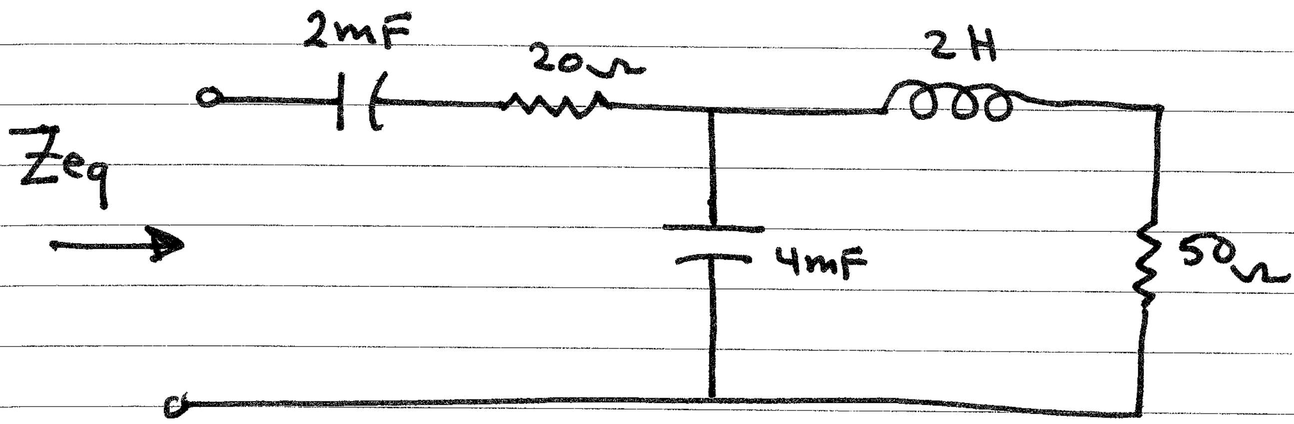
KCL : $\vec{I}_s = \vec{I}_1 + \vec{I}_2 + \cdots\cdots \vec{I}_n$



$$\vec{I}_1 = \frac{Z_2}{Z_1 + Z_2} \vec{I}_s$$

$$\vec{I}_2 = \frac{Z_1}{Z_1 + Z_2} \vec{I}_s$$

Find Z_{eq}



$\omega = 10\ r/s$

$$Z_1 = 20 + \frac{1}{j(10)(2)(10^{-3})} = 20 - j50$$

$$Z_2 = 50 + j(10)(2) = 50 + j20$$

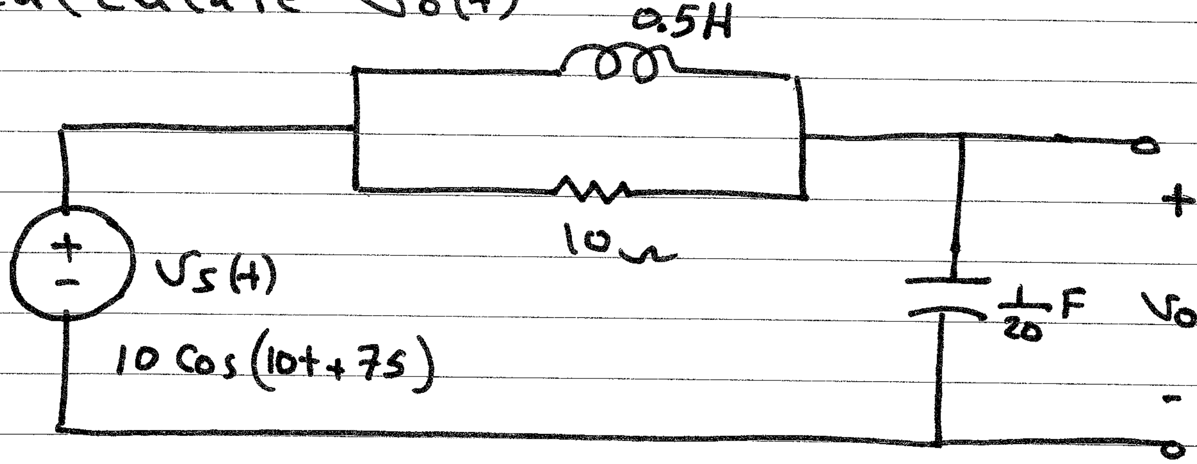
$$Z_3 = (50+j20) \parallel \frac{1}{j(10)(4)(10^{-3})}$$

$$Z_3 = (50+j20) \parallel -j25$$

$$Z_3 = \frac{(50+j20)(-j25)}{50+j20-j25} = 12.38 - j23.76$$

$$Z_{eq} = Z_1 + Z_3 = 32.38 - j73.76\ \Omega$$

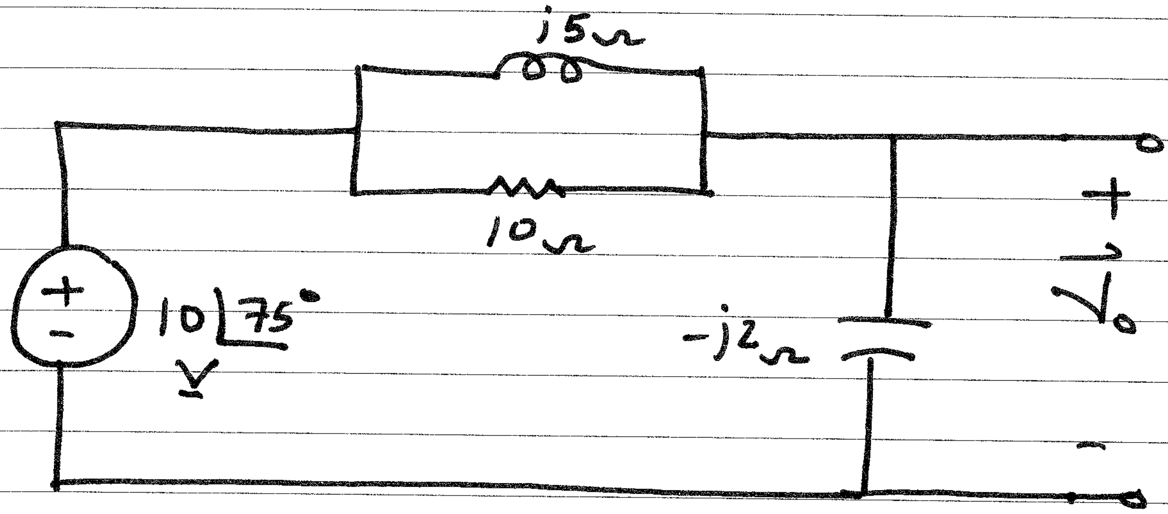
Calculate $v_o(t)$



$Z_L(j\omega) = j\omega L = j5\ \Omega$

$Z_c(j\omega) = -j\frac{1}{\omega c} = -j2\ \Omega$

$\vec{V}_s = 10\angle 75^\circ\ V$

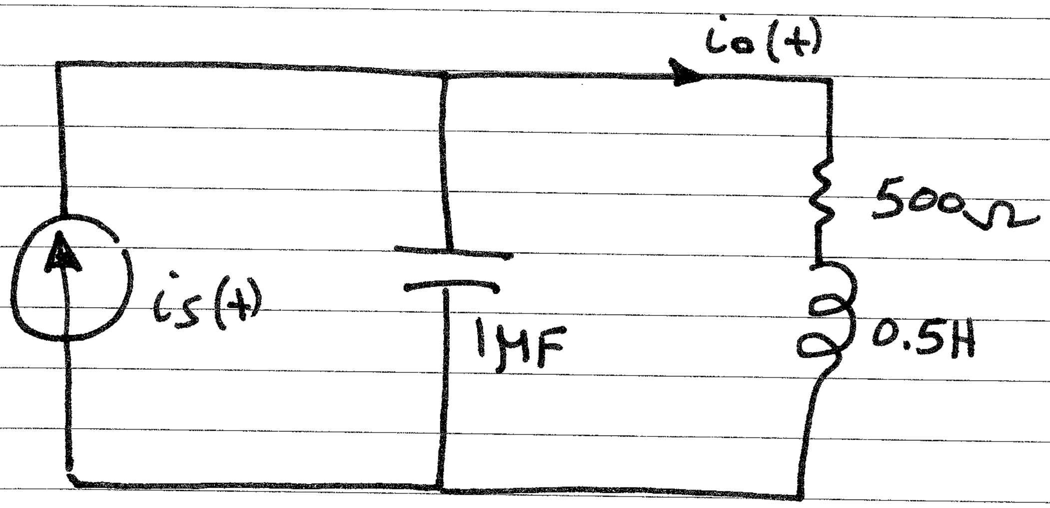


$$\vec{V}_o = \frac{-j2}{-j2 + 10 \| j5} \cdot 10\angle 75^\circ$$

$$\vec{V}_o = 7.071\angle -60^\circ\ V$$

$\therefore v_o(t) = 7.071 \cos(10t - 60^\circ)\ V$

Calculate $i_o(t)$

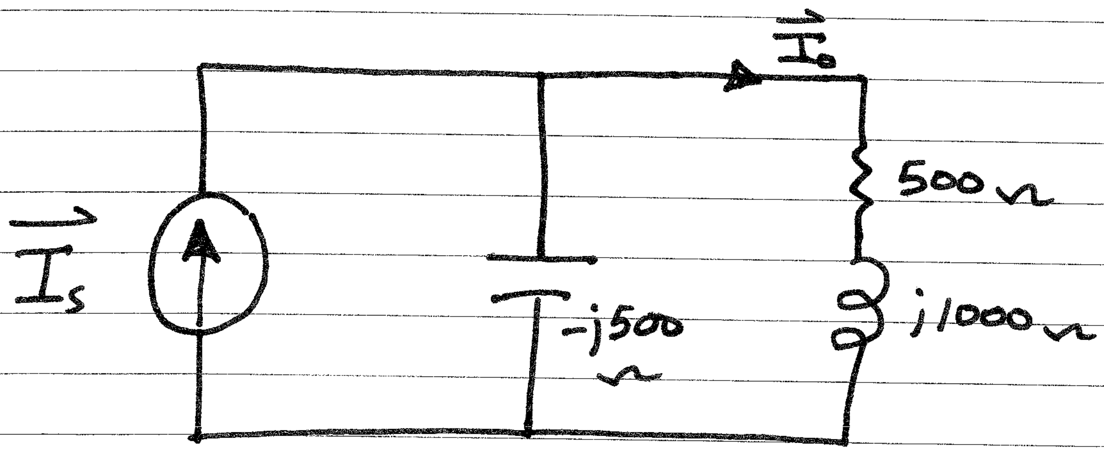


$$i_s(t) = 0.05 \cos 2000t \quad A$$

$$Z_C(j\omega) = -j\frac{1}{\omega C} = -j500\ \Omega$$

$$Z_L(j\omega) = j\omega L = j1000\ \Omega$$

$$\vec{I_s} = 0.05\angle 0^\circ \quad A$$

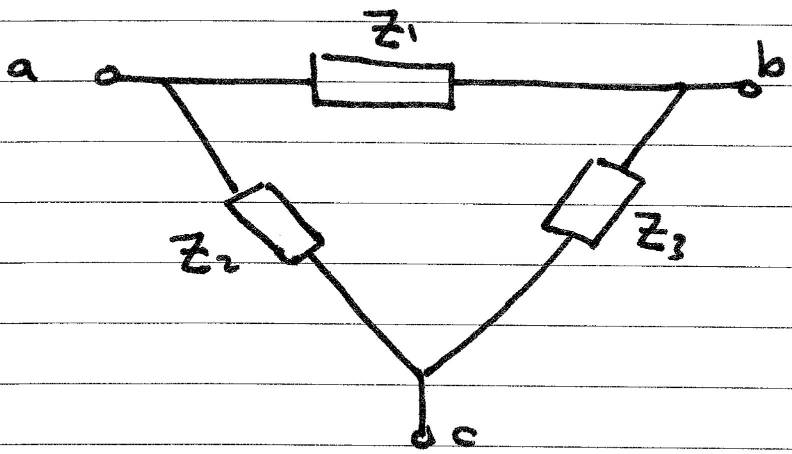


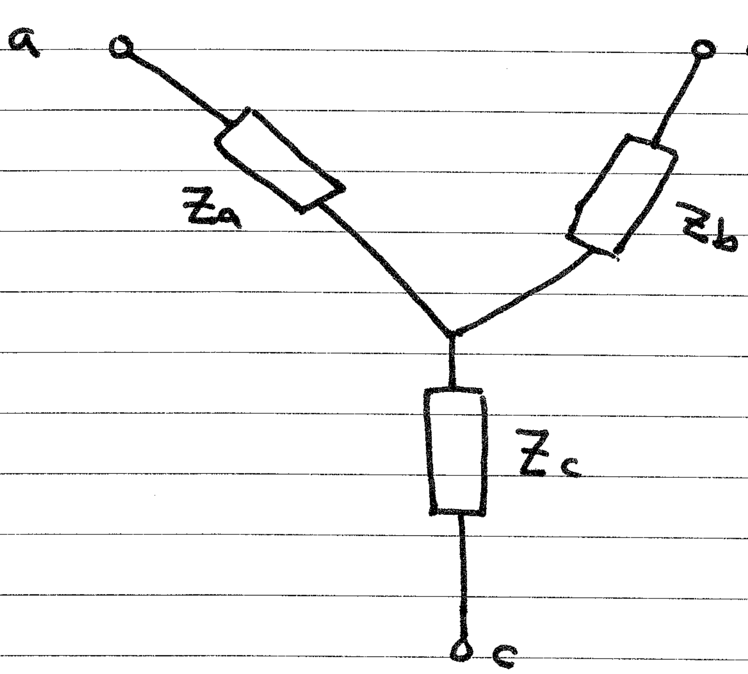
$$\vec{I_o} = \frac{-j500}{-j500+500+j1000}\left(0.05\angle 0^\circ\right)$$

$$\vec{I_o} = 0.03535 \angle -45^\circ \ A$$

$$\therefore \ i_o(t) = 0.03535 \cos(2000t - 45^\circ) \ A$$

Y–Δ Transformation





$$Z_a = \frac{Z_1 Z_2}{Z_1 + Z_2 + Z_3}$$

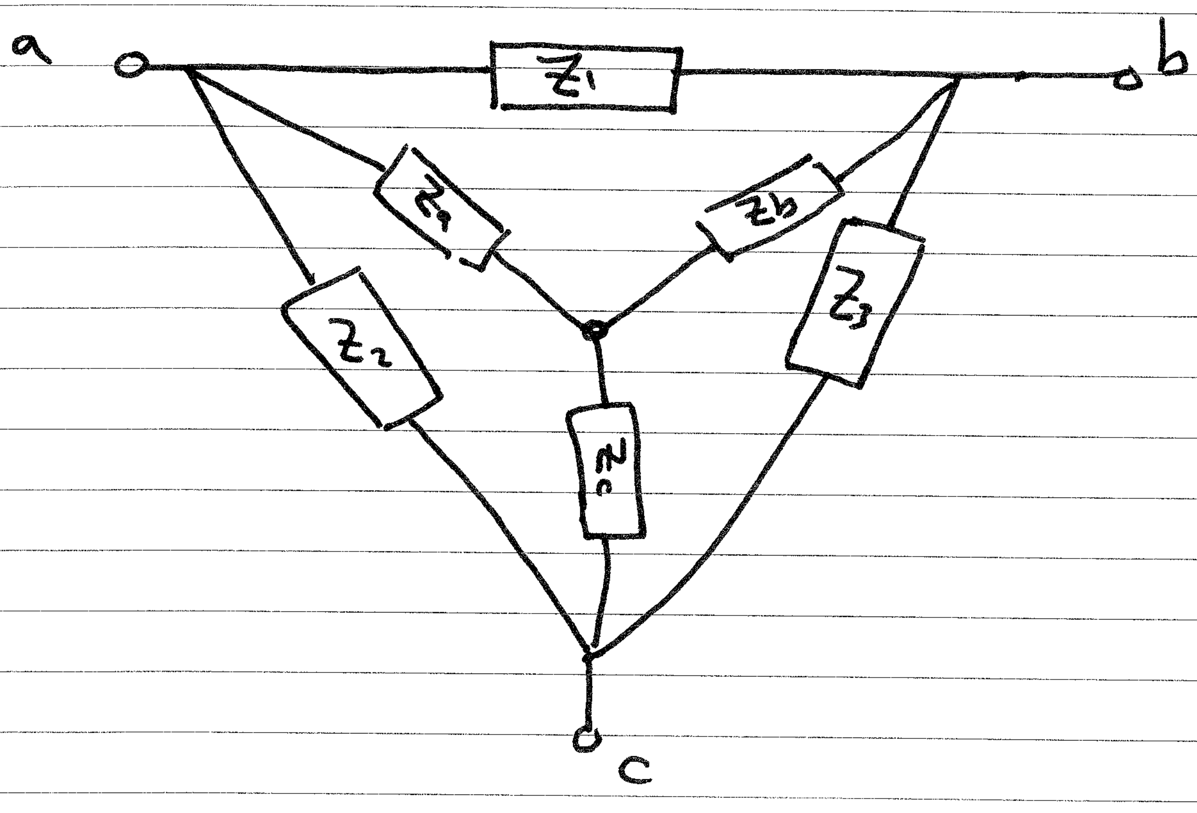
$$Z_b = \frac{Z_1 Z_3}{Z_1 + Z_2 + Z_3}$$

$$Z_c = \frac{Z_2 Z_3}{Z_1 + Z_2 + Z_3}$$

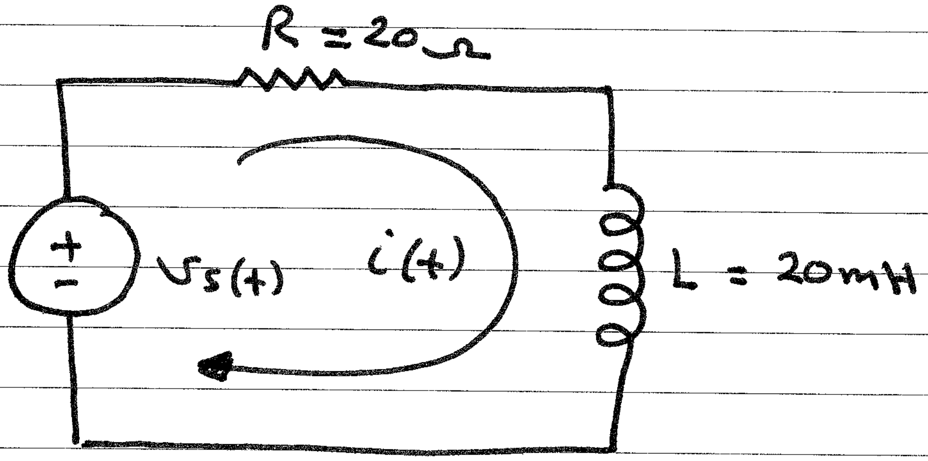
$$Z_1 = \frac{Z_a Z_b + Z_b Z_c + Z_c Z_a}{Z_c}$$

$$Z_2 = \frac{Z_a Z_b + Z_b Z_c + Z_c Z_a}{Z_b}$$

$$Z_3 = \frac{Z_a Z_b + Z_b Z_c + Z_c Z_a}{Z_a}$$



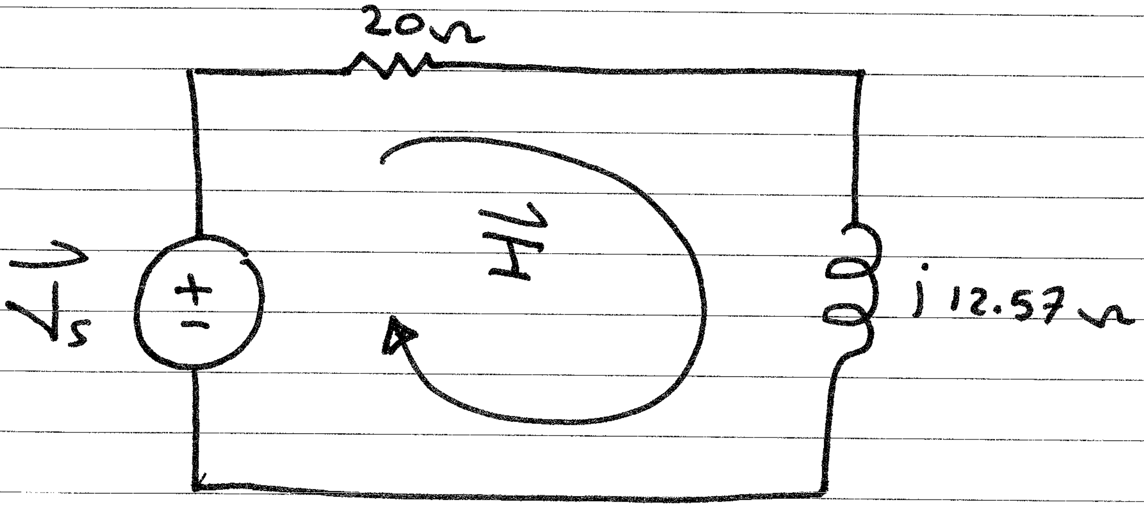
Series RL circuit



$$v_s(t) = 60 \cos(200\pi t) \ V$$

Find $i(t)$

$$Z_R(j\omega) = 20 \ \Omega$$
$$Z_L(j\omega) = j12.57 \ \Omega$$
$$\vec{V}_s = 60\angle 0^\circ \ V$$



KVL : $\vec{V}_s = \vec{V}_R + \vec{V}_L$

$$60\angle 0^\circ = 20\vec{I} + j12.57\vec{I}$$

$$\vec{I} = \frac{60\angle 0^\circ}{20+j12.57} = \frac{60\angle 0^\circ}{23.6\angle 32.1^\circ}$$

$$\therefore \quad \vec{I} = 2.54\angle -32.1^\circ \quad A$$

$$\vec{V_R} = 20\vec{I} = 50.8\angle -32.1^\circ \quad V$$

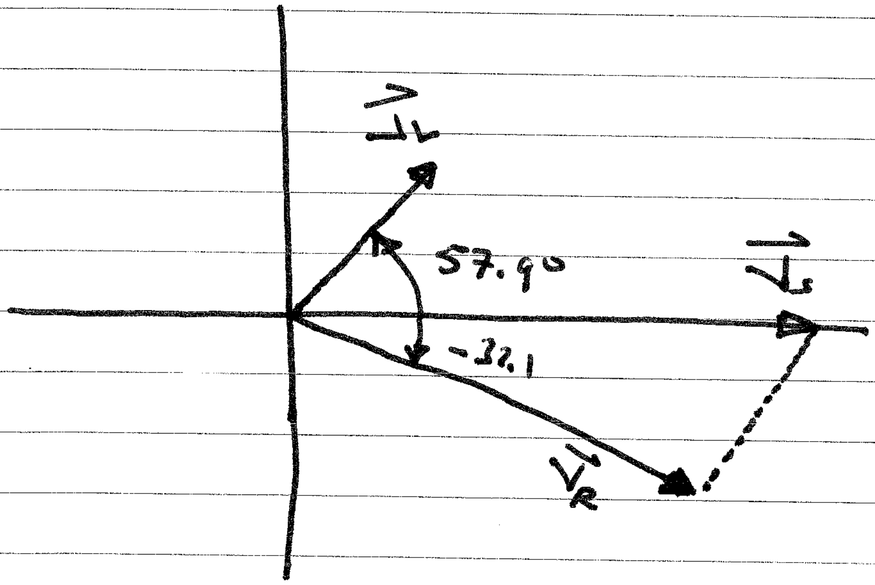
$$\vec{V_L} = j12.57\vec{I} = 31.9\angle +57.9^\circ \quad V$$

$\vec{V_L}$ Leads $\vec{V_R}$ by 90°

$\vec{I_L}$ Lags $\vec{V_s}$ by 32.1°

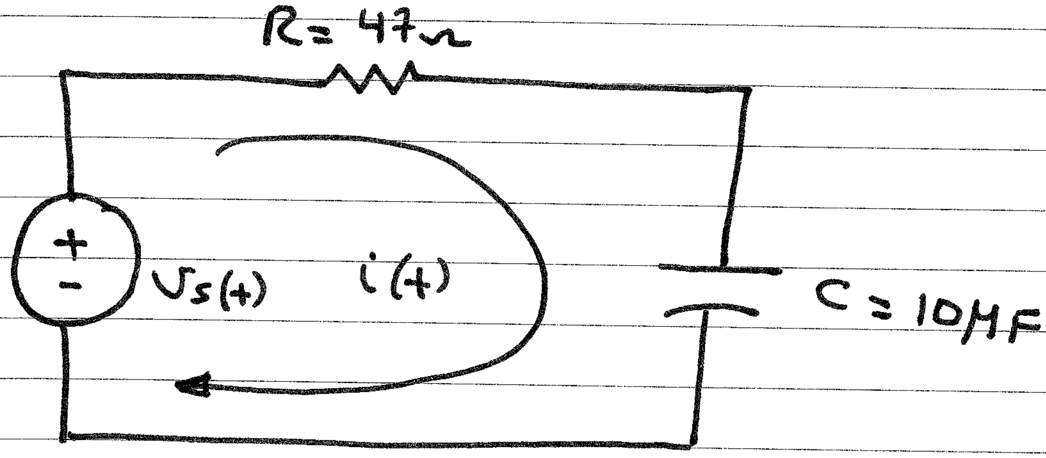
$$Z_{eq} = 20+j12.57 \ \Omega \quad \text{inductive}$$

$$= 23.6\angle 32.1^\circ \ \Omega \quad \text{inductive}$$



phasor diagram

Series RC Circuit

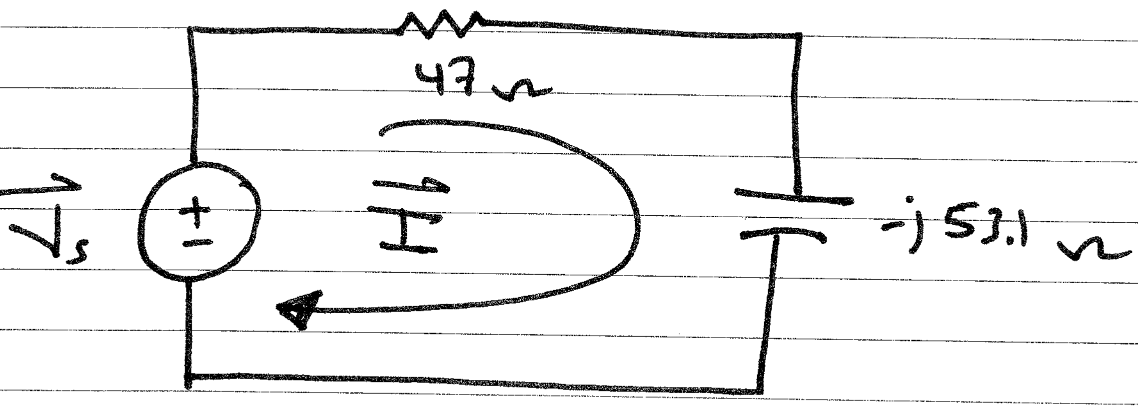


$$v_s(t) = 100 \cos 600\pi t \ \text{V}$$

$$Z_R(j\omega) = 47 \ \Omega$$

$$Z_C(j\omega) = -j\frac{1}{\omega C} = -j\,53.1 \ \Omega$$

$$\vec{V}_s = 100\angle 0^\circ \ \text{V}$$



KVL :

$$\vec{V}_s = 47\vec{I} - j\,53.1\,\vec{I}$$

$$\vec{I} = \frac{\vec{V}_s}{47 - j\,53.1} = \frac{100\angle 0^\circ}{47 - j\,53.1}$$

$$\vec{I} = \frac{100\angle 0^\circ}{70.9\angle -48.5^\circ}$$

$$\vec{I} = 1.41\angle 48.5^\circ \quad A$$

$\vec{I}$ Leads $\vec{V}_s$ by 48.5°

$\longrightarrow$ Capacitive Circuit

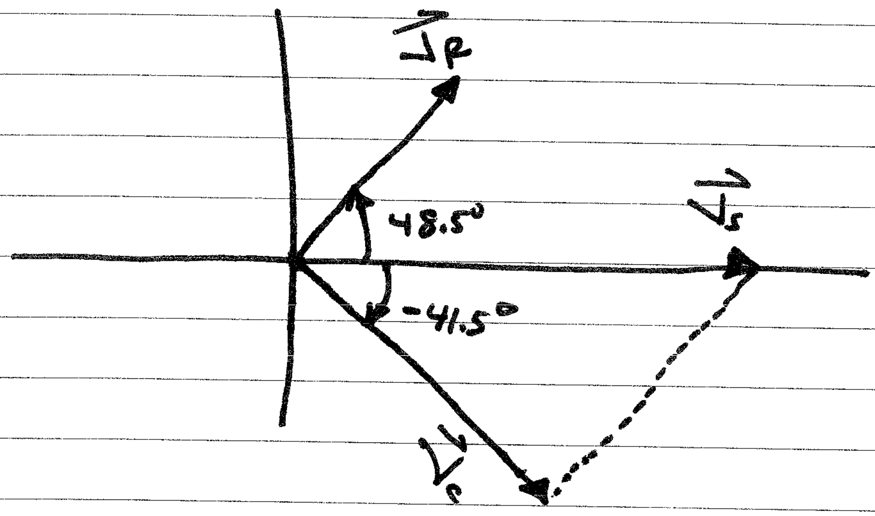
$Z(jw) = 47 - j\,53.1^\circ$ Capacitive

$Z(jw) = 70.9\angle -48.5^\circ$ Capacitive

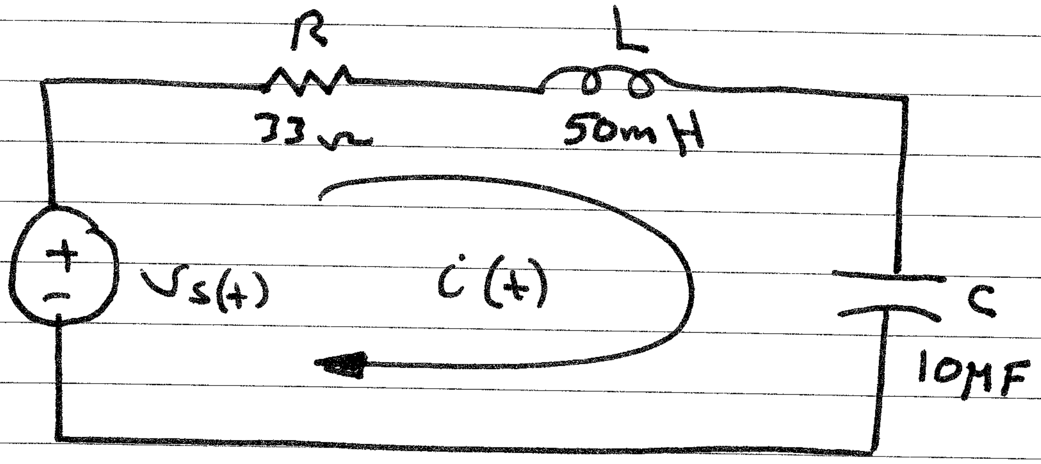
$$\vec{V}_R = 47\vec{I} = 66.3\angle 48.5^\circ \quad V$$

$$\vec{V}_c = -j53.1\vec{I} = 74.9\angle -41.5^\circ \quad V$$

$\vec{V}_c$ Lags $\vec{I}$ by 90°



Series RLC

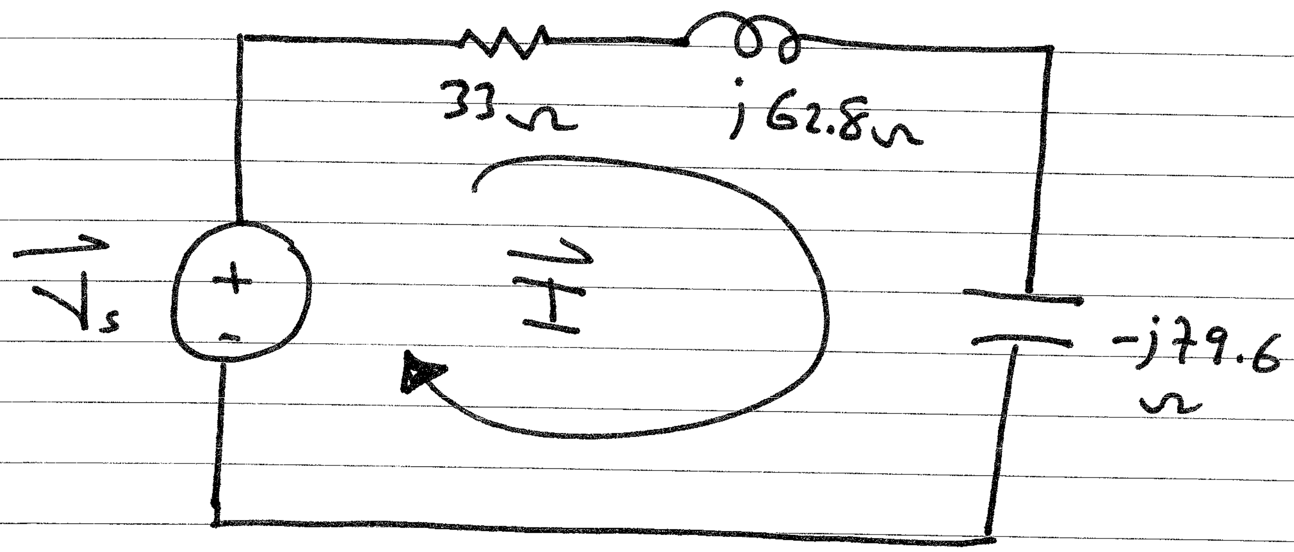


$$v_s(t) = 75 \cos 400\pi t \ \underline{v}$$

$$Z_R(j\omega) = 33\ \Omega$$

$$Z_L(j\omega) = j\omega L = j\,62.8\ \Omega$$

$$Z_C(j\omega) = -j\frac{1}{\omega C} = -j\,79.6\ \Omega$$



$$\text{KVL}: \quad \vec{V}_s = \vec{V}_R + \vec{V}_L + \vec{V}_C$$

$$\vec{V}_s = 33\vec{I} + j62.8\vec{I} - j79.6\vec{I}$$

$$\vec{I} = \frac{\vec{V}_s}{33 - j16.8} = \frac{75\angle 0^\circ}{37\angle -27^\circ}$$

$$\vec{I} = 2.03\angle 27^\circ \quad A$$

$\vec{I}$ Leads $\vec{V}_s$ by 27°

$\therefore$ Capacitive Circuit

$$Z_{eq} = R + j\omega L - j\frac{1}{\omega C}$$

$$Z_{eq} = 33 + j62.8 - j79.6$$

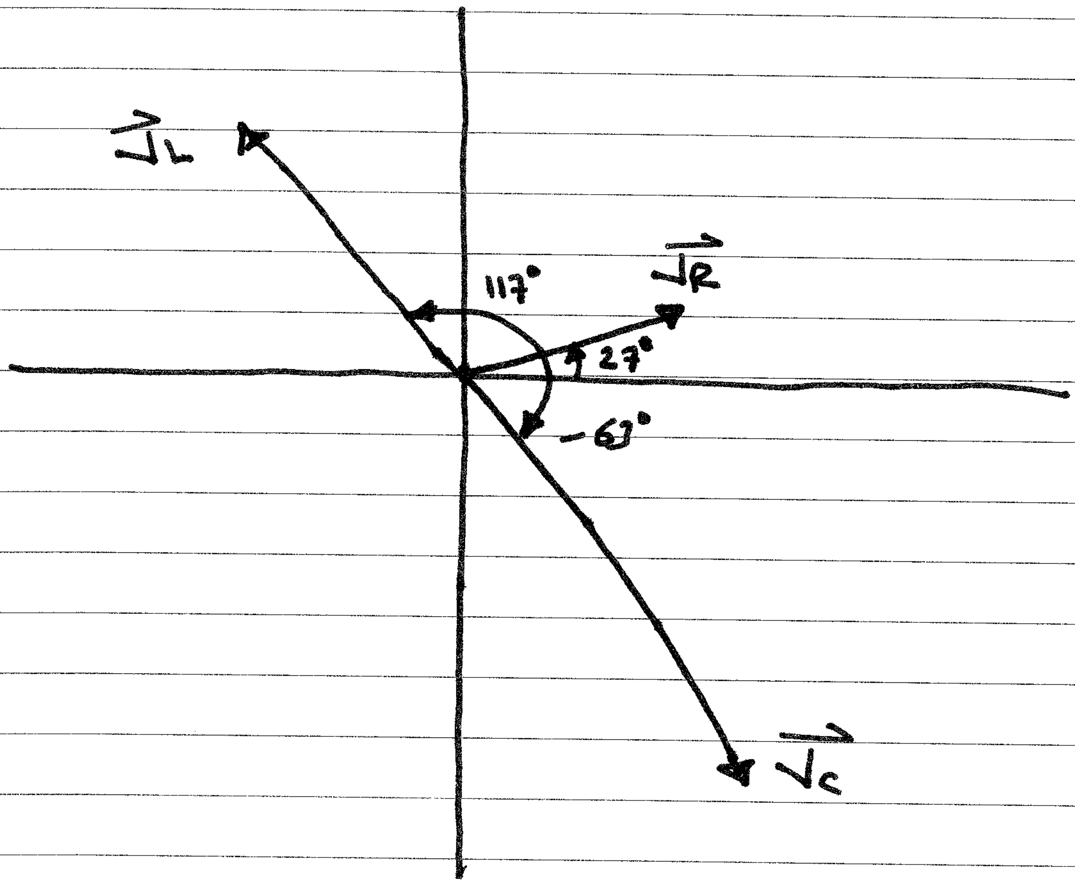
$Z_{eq} = 33 - j16.8 \ \Omega$ Capacitive

$Z_{eq} = 37\angle -27^\circ \ \Omega$ Capacitive

$$\vec{V}_R = R\vec{I} = 67\angle 27^\circ \quad V$$

$$\vec{V}_L = j\omega L\vec{I} = 127\angle 117^\circ \quad V$$

$$\vec{V}_C = -j\frac{1}{\omega C}\vec{I} = 162\angle -63^\circ \quad V$$

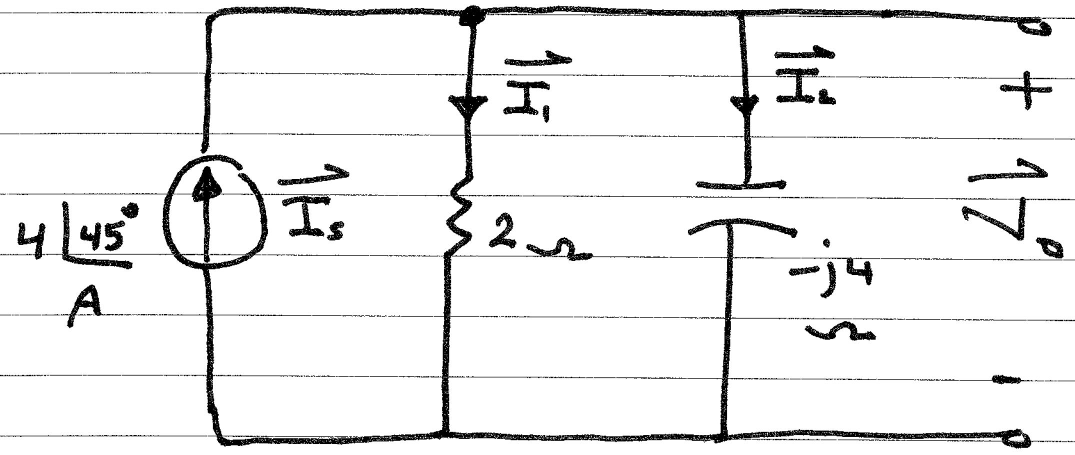


If $\quad j\omega L - j\dfrac{1}{\omega C} = 0$

$$\omega = \frac{1}{\sqrt{LC}}$$

resonant frequency

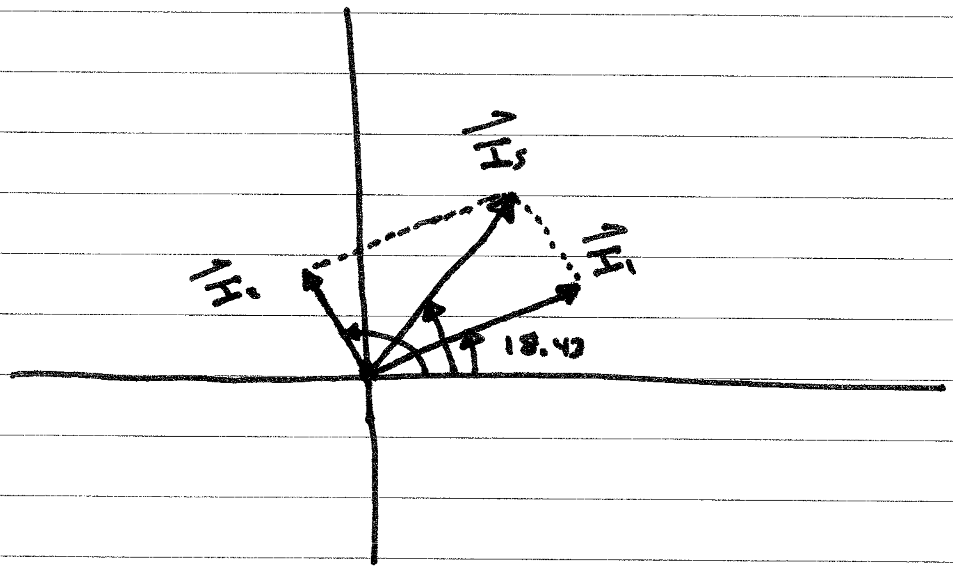
$Z_{eq} = R \qquad$ resistive



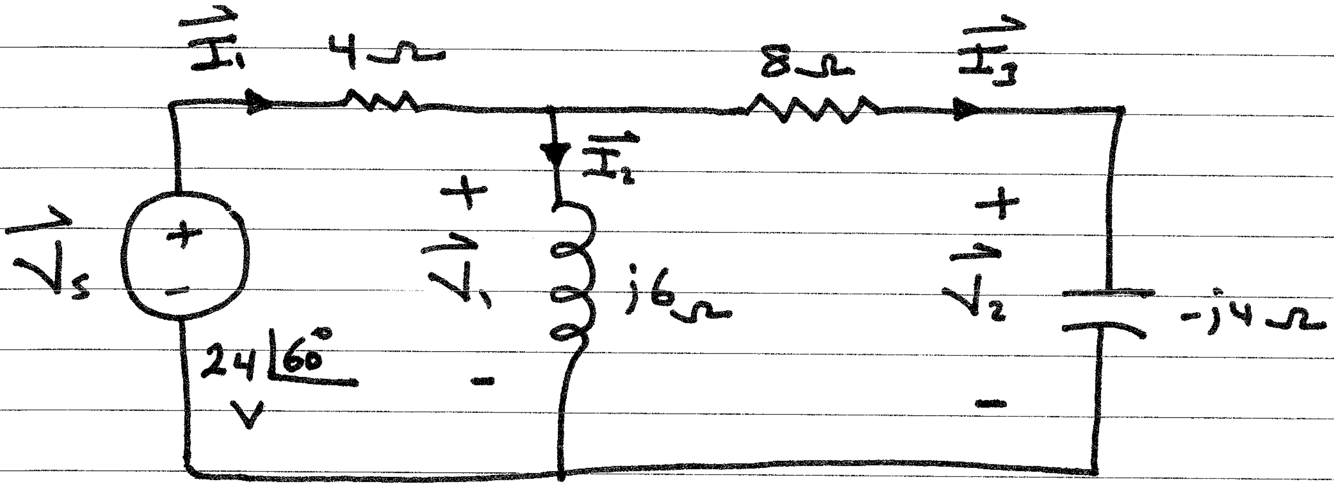
$$\vec{I}_1 = \frac{-j4}{-j4+2}\vec{I}_s = 3.578\angle 18.435^\circ \text{ A}$$

$$\vec{I}_2 = \frac{2}{2-j4}\vec{I}_s = 1.789\angle 108.435^\circ \text{ A}$$

$$\vec{V}_o = 2\vec{I}_1 = 7.156\angle 18.435^\circ \text{ V}$$



phasor diagram



Calculate all the voltages and currents

$$Z_{eq} = 4 + j6 \parallel (8 - j4)$$

$$Z_{eq} = 9.604 \angle 30.964^\circ \ \Omega$$

$$\vec{I}_1 = \frac{\vec{V}_s}{Z_{eq}} = \frac{24\angle 60^\circ}{9.604\angle 30.964^\circ} = 2.498 \angle 29.036^\circ \ A$$

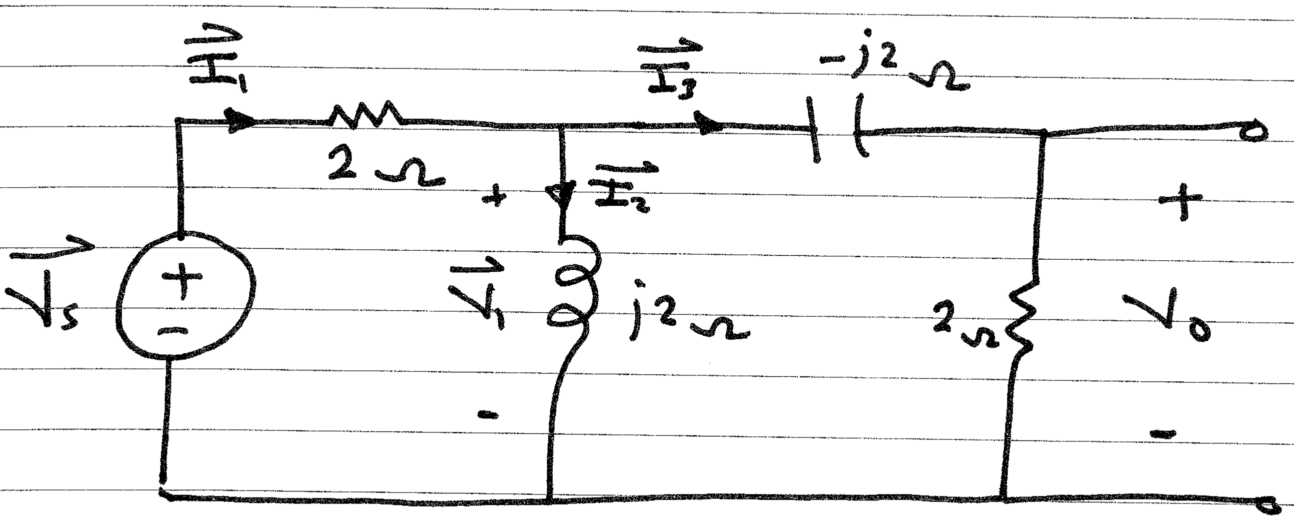
$$\vec{I}_3 = \frac{j6}{j6+8-j4}\,\vec{I}_1 = 1.82\angle 105^\circ \ A$$

$$\vec{I}_2 = \frac{8-j4}{8-j4+j6}\,\vec{I}_1 = 2.71\angle -11.58 \ A$$

$$\vec{V}_1 = j6\,\vec{I}_2 = 16.26\angle 78.42^\circ \ V$$

$$\vec{V}_2 = -j4\,I_3 = 7.28\angle 15^\circ \ V$$

If $\vec{V_o} = 8\angle 45^\circ$ V, find $\vec{V_s}$



$$I_3 = \frac{\vec{V_o}}{2} = 4\angle 45^\circ \text{ A}$$

$$\vec{V_1} = (2-j2)\vec{I_3} = 11.314\angle 0^\circ$$

$$\vec{I_2} = \frac{\vec{V_1}}{j2} = 5.657\angle -90^\circ \text{ A}$$

$$\vec{I_1} = \vec{I_2} + \vec{I_3} = (2.828 - j2.829) \text{ A}$$

$$\vec{V_s} = 2\vec{I_1} + \vec{V_1}$$

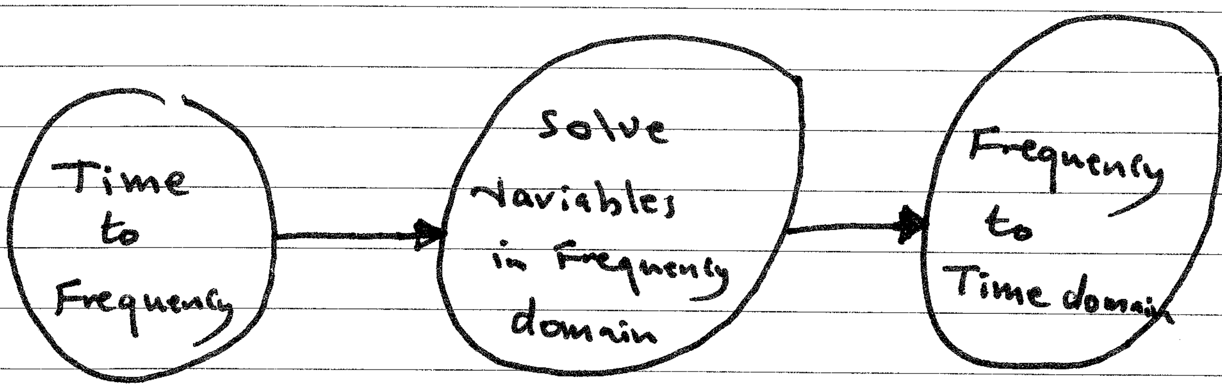
$$\vec{V_s} = 17.888\angle -18.439^\circ \text{ V}$$

Steps to Analyze Ac Circuits

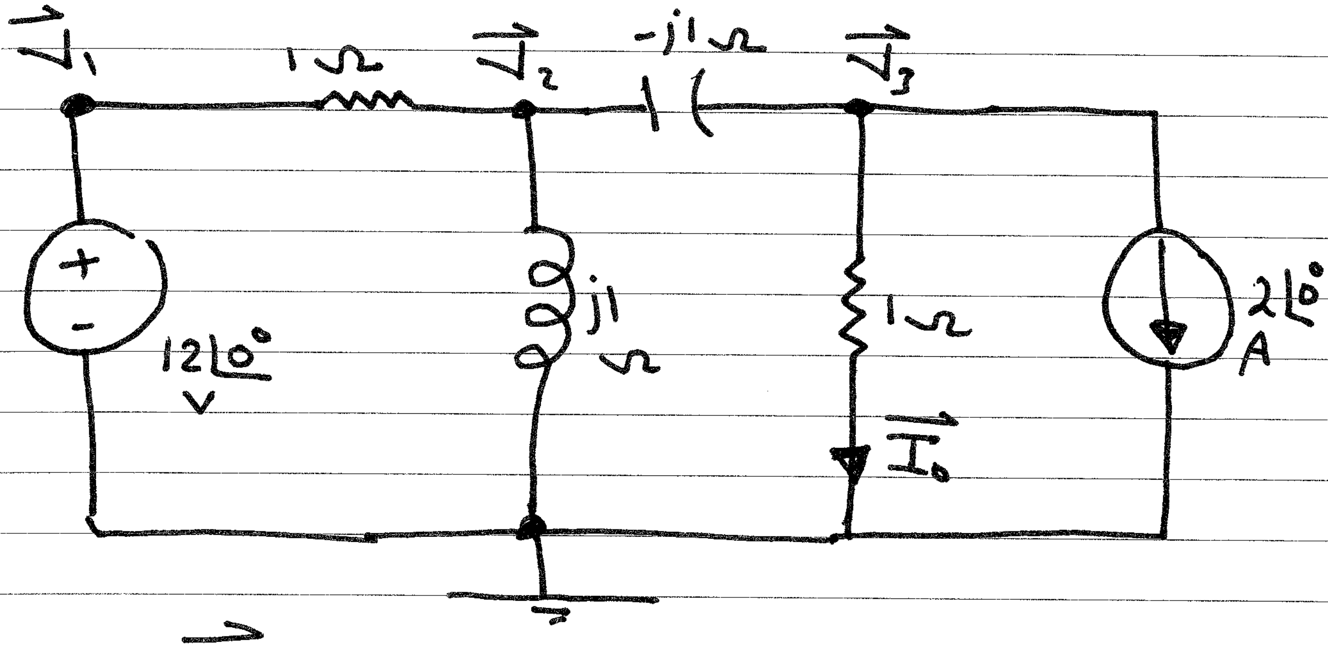
* Transform the Circuit to the phasor or frequency domain.

* Solve the problem using Circuit techniques (nodal analysis, mesh analysis, Superposition, etc .....)

* Transform the resulting phasor to the time domain.



Nodal Analysis



Find $\vec{I}_o$ using Nodal Analysis

$$\vec{I}_o = \frac{\vec{V}_3}{1}$$

$\vec{V}_1 = 12\angle 0^\circ$ Constrain equation

KCL at node 2 :

$$\frac{\vec{V}_2 - \vec{V}_1}{1} + \frac{\vec{V}_2}{j1} + \frac{\vec{V}_2 - \vec{V}_3}{-j1} = 0$$

$$-\vec{V}_1 + \vec{V}_2 - j\vec{V}_3 = 0$$

KCL at node 3 :

$$-2\angle 0^\circ = -\frac{1}{-j1}\vec{V_2} + \left(\frac{1}{-j1} + 1\right)\vec{V_3}$$

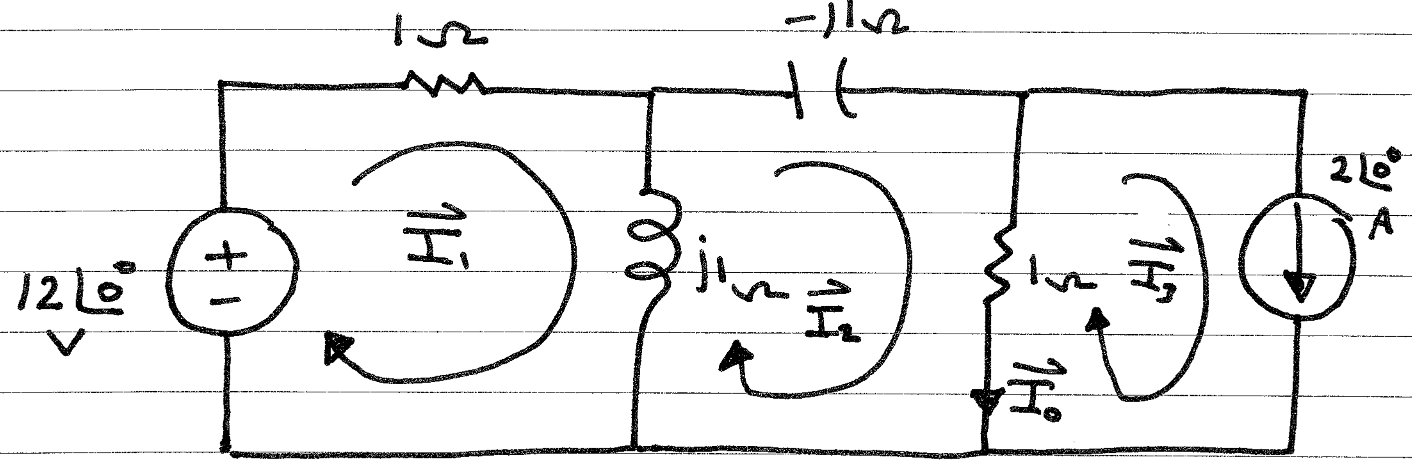
$$-2\angle 0^\circ = -j\vec{V_2} + (1+j)\vec{V_3}$$

Solving for $\vec{V_3}$;

$$\vec{V_3} = \left(\frac{8}{5} + j\frac{26}{5}\right) V$$

$$\therefore \ \vec{I_0} = \frac{\vec{V_3}}{1} = \left(\frac{8}{5} + j\frac{26}{5}\right) A$$

Mesh Analysis



Find $\vec{I_0}$ using Mesh Analysis

$$\vec{I_0} = \vec{I_2} - \vec{I_3}$$

KVL for mesh 1 :

$$12\angle 0^\circ = (1+j1)\vec{I_1} - j1\,\vec{I_2}$$

KVL for mesh 2 :

$$0 = -j1\,\vec{I_1} + (1+j1-j1)\vec{I_2} - \vec{I_3}$$

$$0 = -j1\,\vec{I_1} + \vec{I_2} - \vec{I_3}$$

$$\vec{I_3} = 2\angle 0^\circ \text{ A}$$ Constrain equation

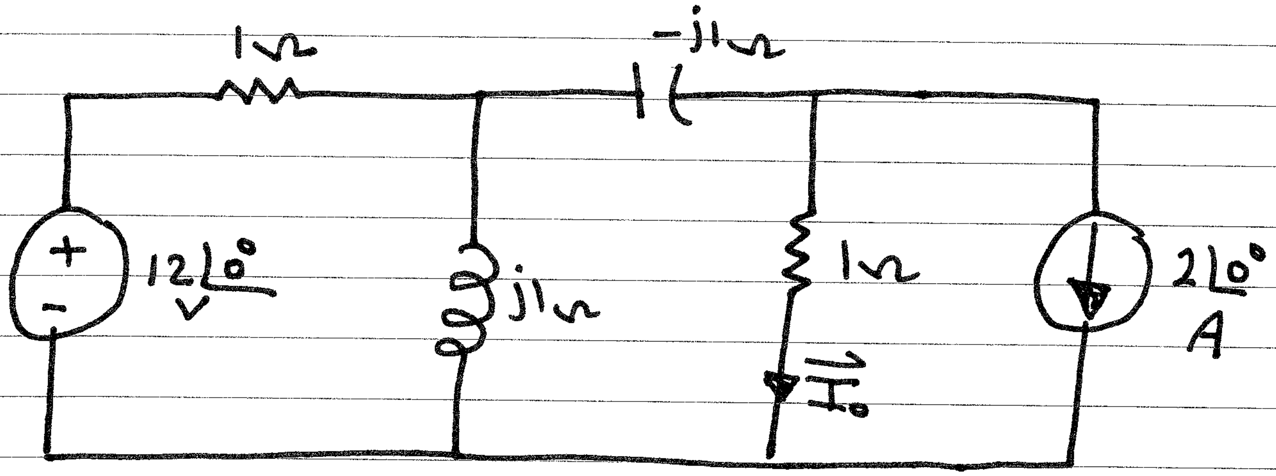
Solving for $\vec{I_2}$ and $\vec{I_3}$

$$\vec{I_2} = \left(\frac{18}{5} + j\,\frac{26}{5}\right) \text{A}$$

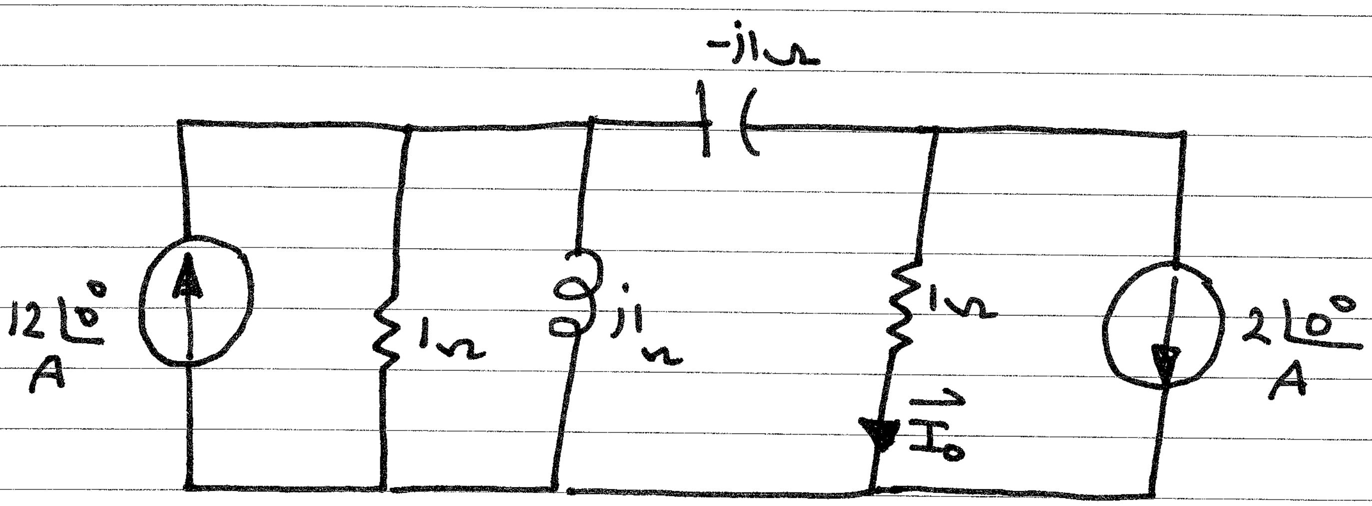
$$\vec{I}_3 = 2\angle 0^\circ \ A$$

$$\therefore \ \vec{I}_0 = \left(\frac{8}{5} + j\,\frac{26}{5}\right) A$$

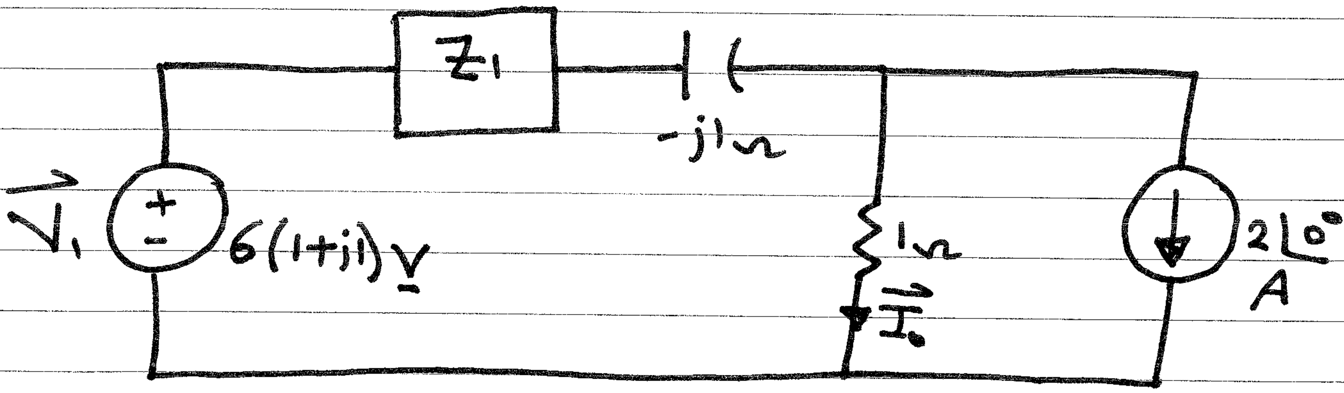
Source Transformation



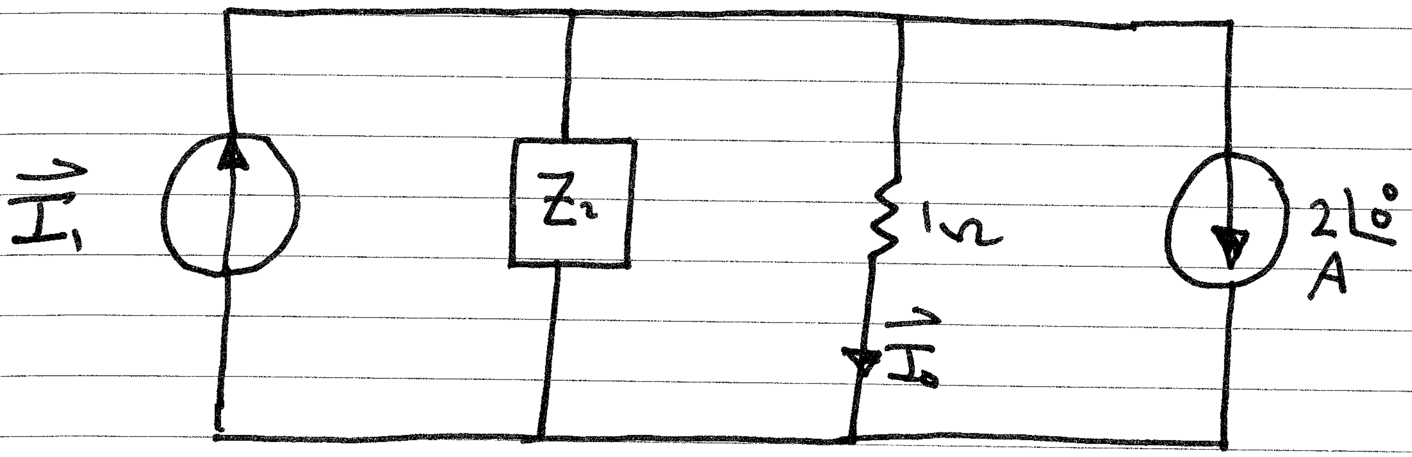
Find $\vec{I_o}$ using Source Transformation



$Z_1 = 1\Omega \| j1\Omega = \left(\frac{1}{2}+j\frac{1}{2}\right)\Omega$

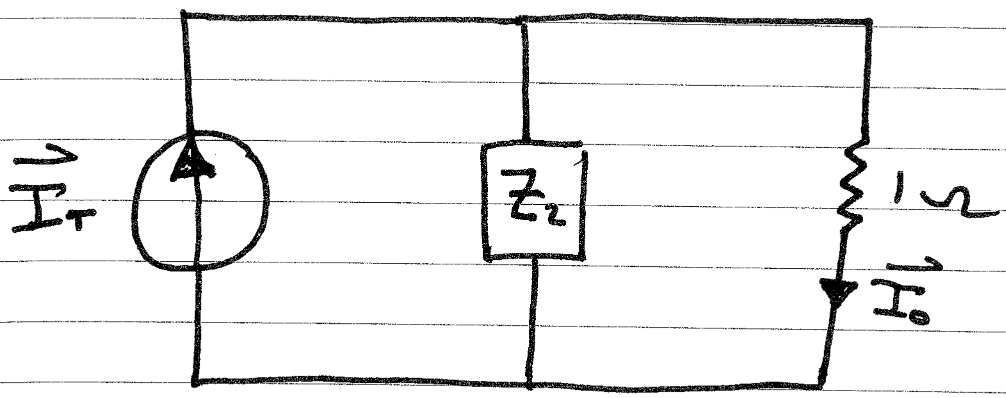


$\vec{V_1} = 12\angle 0^\circ \cdot Z_1 = 6(1+j1)$ V



$$\vec{I}_1 = \frac{\vec{V}_1}{Z_2} = \frac{12(1+j1)}{1-j1}$$

$$\dot{Z}_2 = -j1 + Z_1 = \left(\frac{1}{2} - j\frac{1}{2}\right)\Omega$$

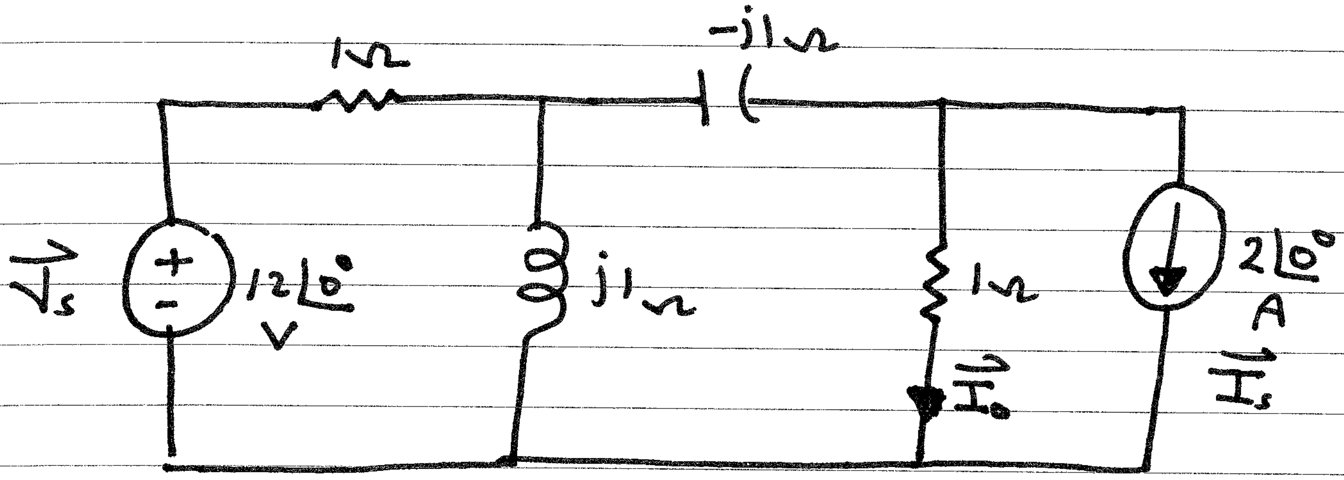


$$\vec{I}_T = \vec{I}_1 - 2\angle 0^\circ$$

$$\vec{I}_T = \left(\frac{10+j14}{1-j1}\right) A$$

$$\vec{I}_o = \frac{Z_2}{Z_2+1}\,\vec{I}_T = \left(\frac{8}{5} + j\frac{26}{5}\right) A$$

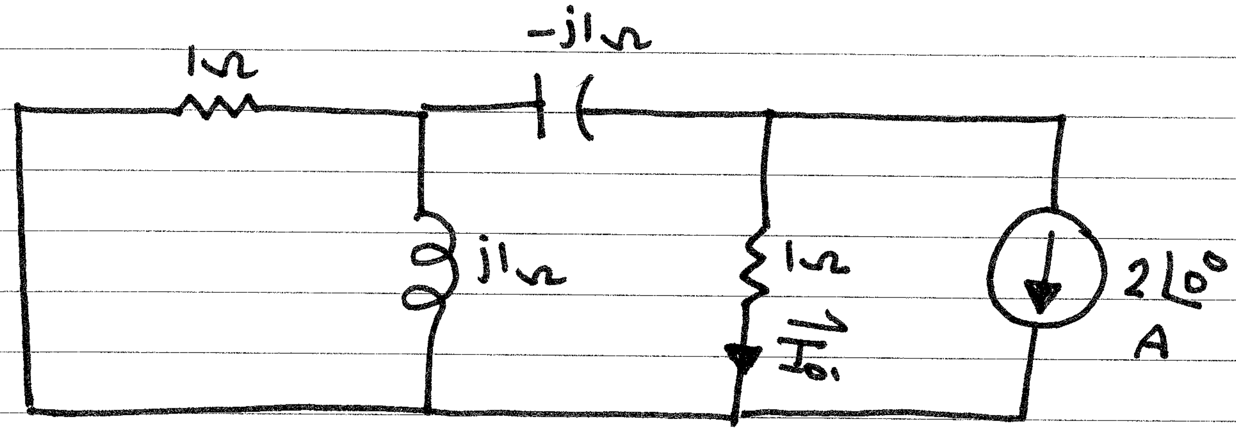
Superposition



Find $\vec{I}_o$ using Superposition

$$\vec{I}_o = \vec{I}_{o1} + \vec{I}_{o2}$$

1) let $\vec{V}_s$ off, and $\vec{I}_s$ on

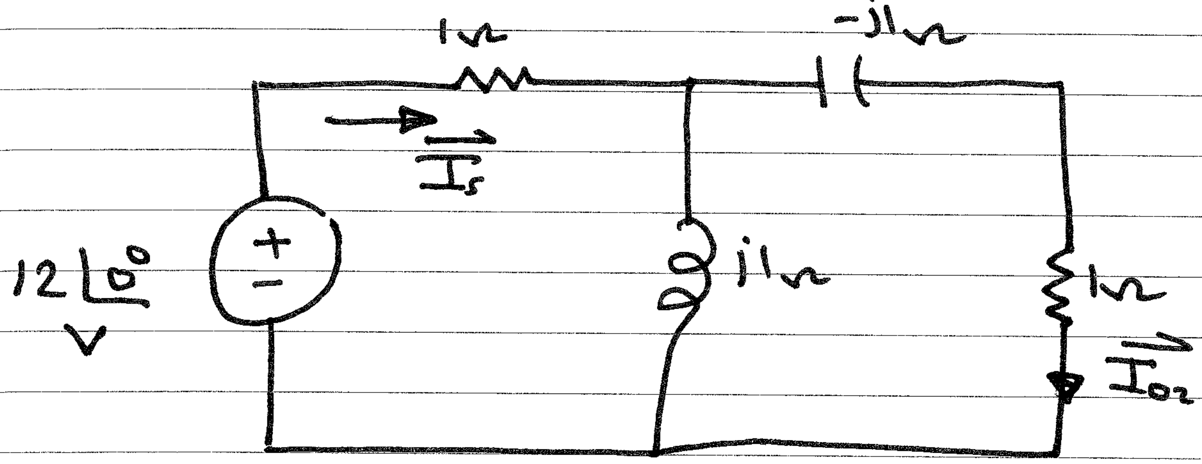


$$\vec{I}_{o1} = -2\angle 0^\circ \; \frac{Z_1}{Z_1 + 1}$$

$$Z_1 = -j1 + 1 \parallel j1 = -j1 + \frac{j1}{1+j1}$$

$$\vec{I}_{o1} = \frac{-2}{2+j1} \text{ A}$$

2) let $\vec{I_s}$ off, and $\vec{V_s}$ on



$$\vec{I_s} = \frac{12\angle 0^\circ}{Z_{eq}}$$

$$Z_{eq} = 1 + j1 \parallel (1 - j1) = (2 + j1)\ \Omega$$

$$\therefore \vec{I_s} = \frac{12\angle 0^\circ}{2 + j1}\ A$$

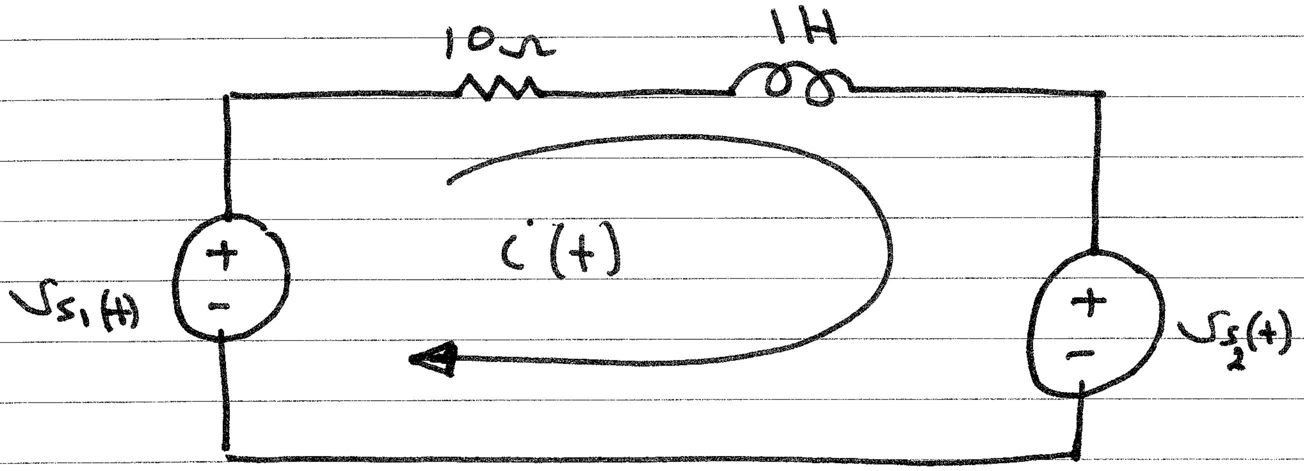
$$\vec{I_{o2}} = \vec{I_s}\ \frac{j1}{j1 + 1 - j1}$$

$$\vec{I_{o2}} = \vec{I_s} \cdot j1 = \frac{12}{1 - j2}\ A$$

$$\therefore \vec{I_o} = \vec{I_{o1}} + \vec{I_{o2}}$$

$$\vec{I_o} = \left(\frac{8}{5} + j\,\frac{26}{5}\right) A$$

The Power of Superposition



$$v_{s1}(t) = 100 \cos 10t \ \underline{v}$$

$$v_{s2}(t) = 50 \cos (20t - 10^\circ) \ \underline{v}$$

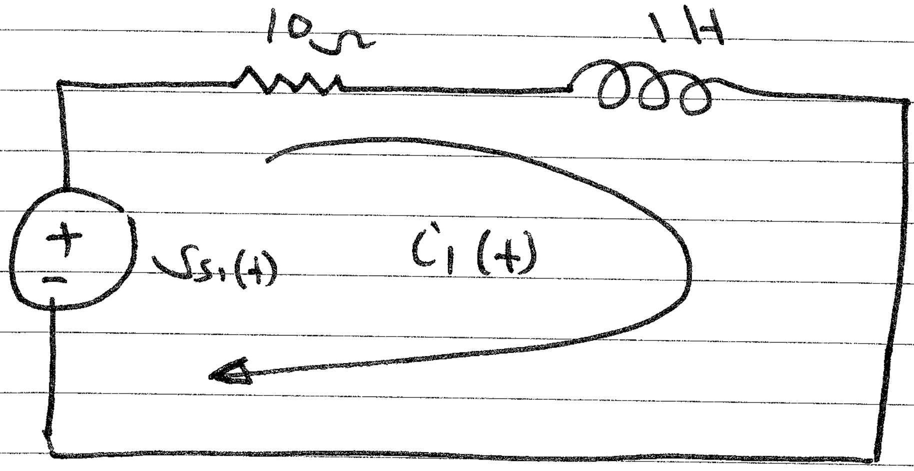
note that $\omega_1 = 10$ r/s and

$\omega_2 = 20$ r/s

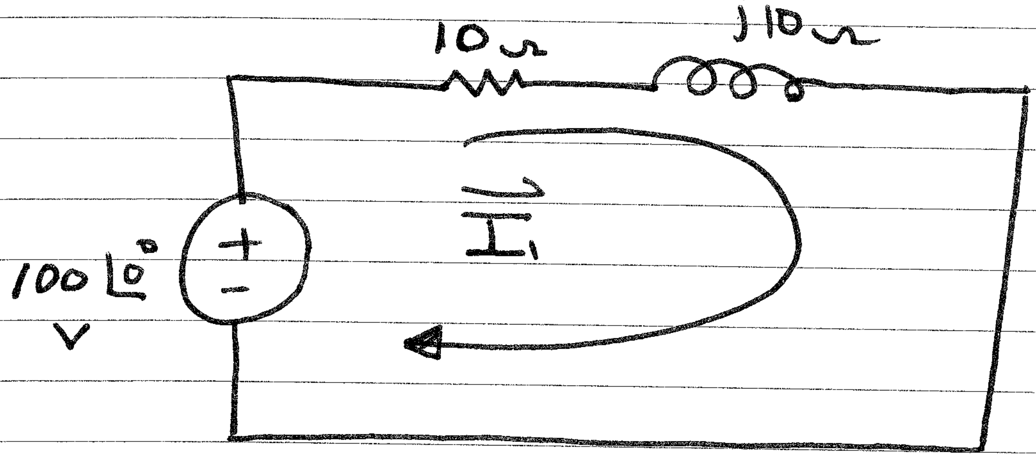
$\therefore$ Superposition is the Only method of analysis.

$$i(t) = i_1(t) + i_2(t)$$

1) Let $V_{s2}(t)$ OFF, and $V_{s1}(t)$ on



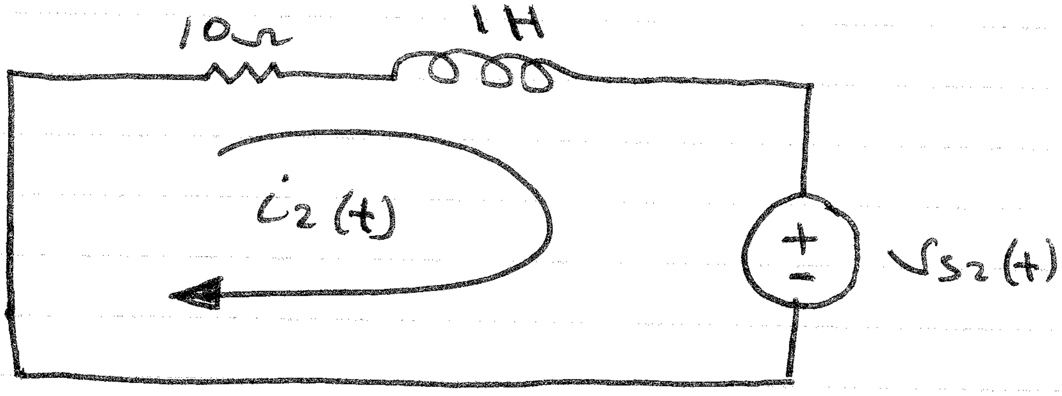
$$V_{s1}(t) = 100 \cos 10t \ \underline{V}$$



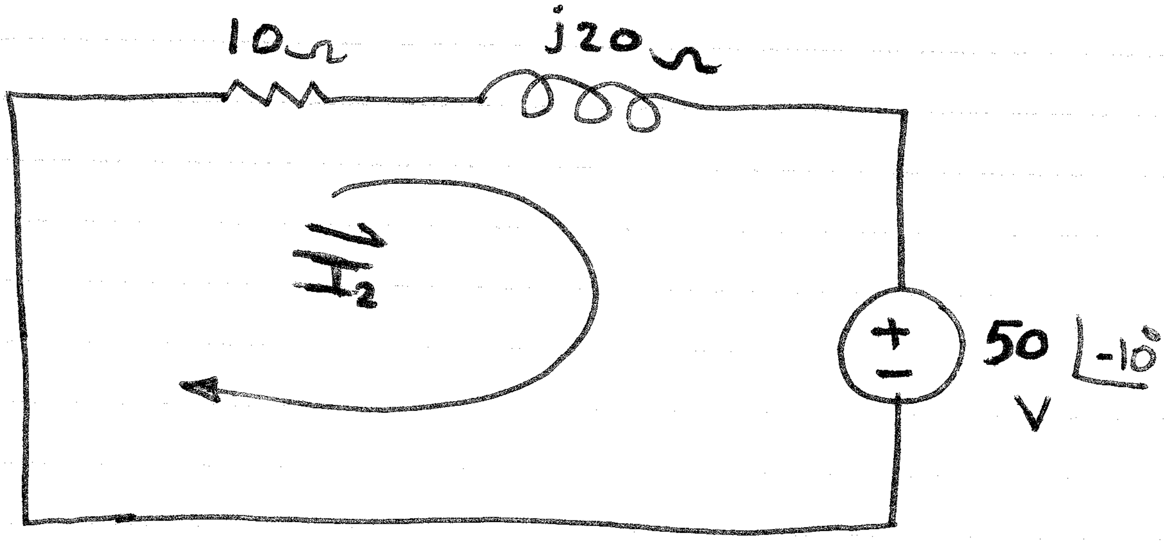
$$\vec{I}_1 = \frac{100\angle 0^\circ}{10 + j10} = 7.07\angle -45^\circ \ A$$

$$\therefore \ i_1(t) = 7.07 \cos\left(10t - 45^\circ\right) A$$

2) let $v_{s1}(t)$ OFF, and $v_{s2}(t)$ On



$$v_{s2}(t) = 50 \cos(20t - 10^\circ)\,V$$



$$\vec{I_2} = \frac{-50\angle -10^\circ}{10 + j20} = \frac{50\angle 170^\circ}{10 + j20}$$

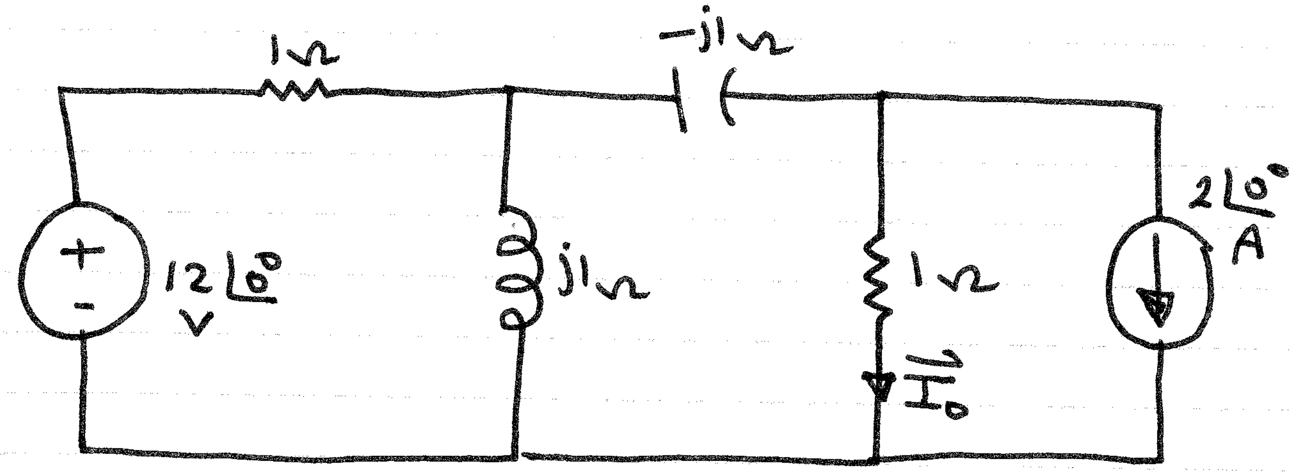
$$\vec{I_2} = 2.24\angle 106.57^\circ \; A$$

$$\therefore \; i_2(t) = 2.24 \cos(20t + 106.57^\circ)\,A$$

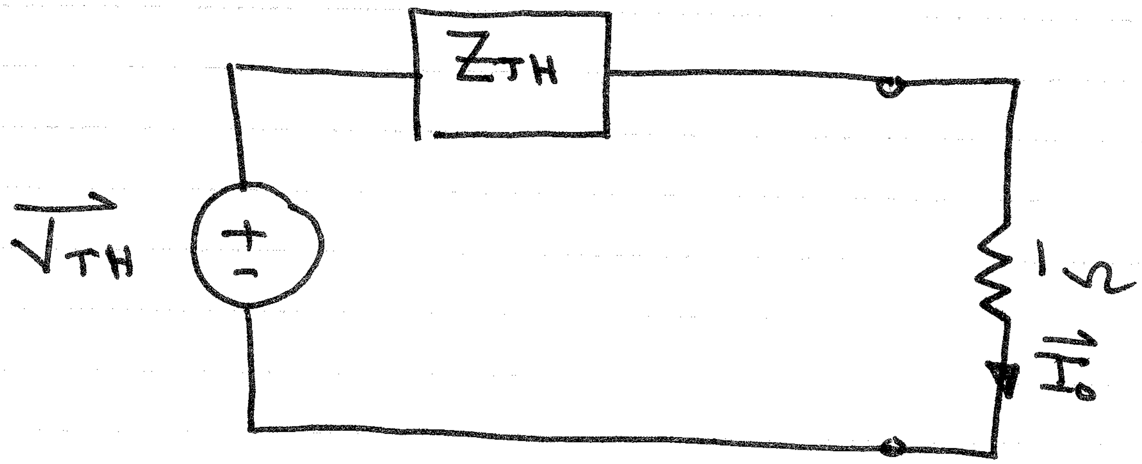
$$\therefore\ i(t) = i_1(t) + i_2(t)$$

$$i(t) = 7.07 \cos(10t - 45^\circ)\ A$$
$$+ 2.24 \cos(20t + 106.57^\circ)\ A$$

Thevenin's and Norton's Theorems

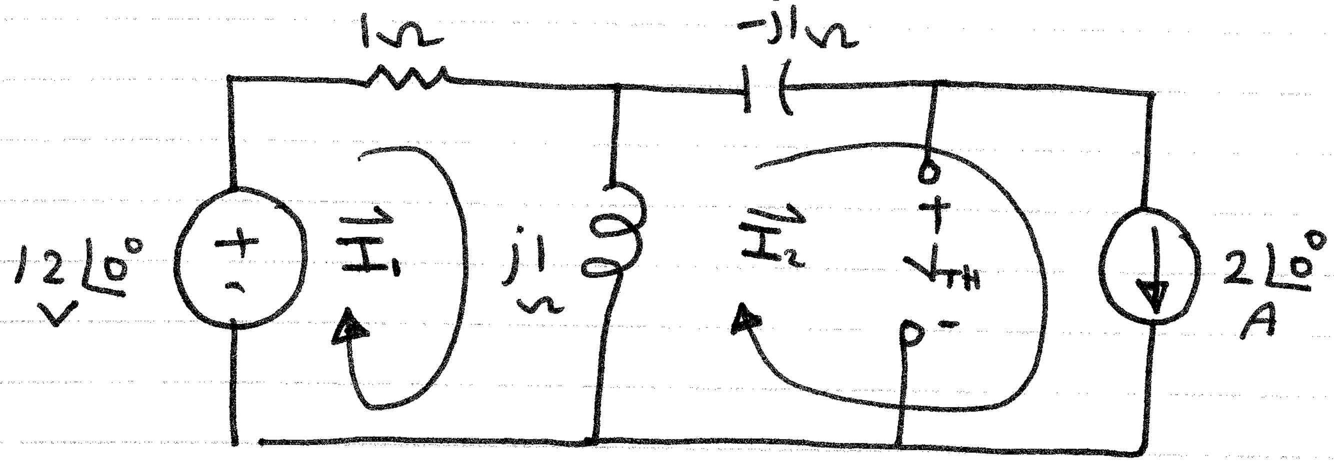


Find $\vec{I_o}$ using Thevenin's theorem



$$\vec{I_o} = \frac{\vec{V_{TH}}}{Z_{TH} + 1\Omega}$$

1) To find $\vec{V}_{TH}$



$$\vec{V}_{TH} = -(-j1\Omega)\vec{I}_2 + j1\Omega(\vec{I}_1 - \vec{I}_2)$$

$\vec{I}_2 = 2\angle 0°$ constrain equation

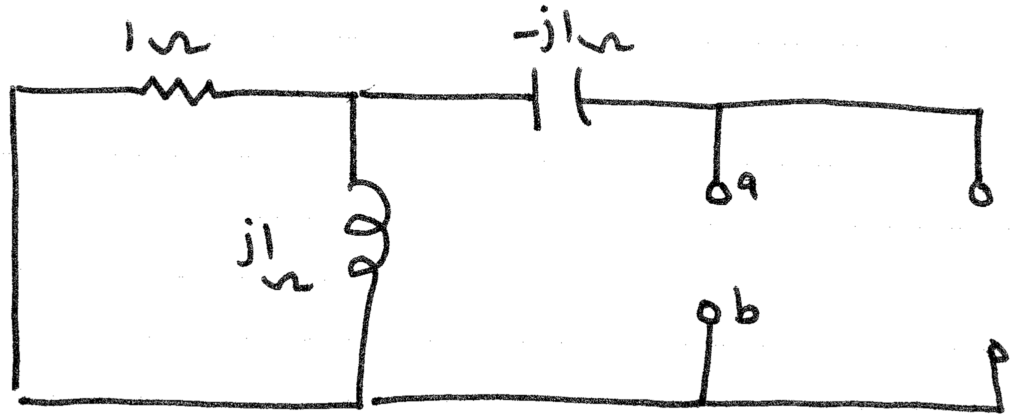
KVL for mesh 1 :

$$12\angle 0° = (1+j1)\vec{I}_1 - j1\vec{I}_2$$

$$\therefore \vec{I}_1 = \left(\frac{12+j2}{1+j1}\right) A$$

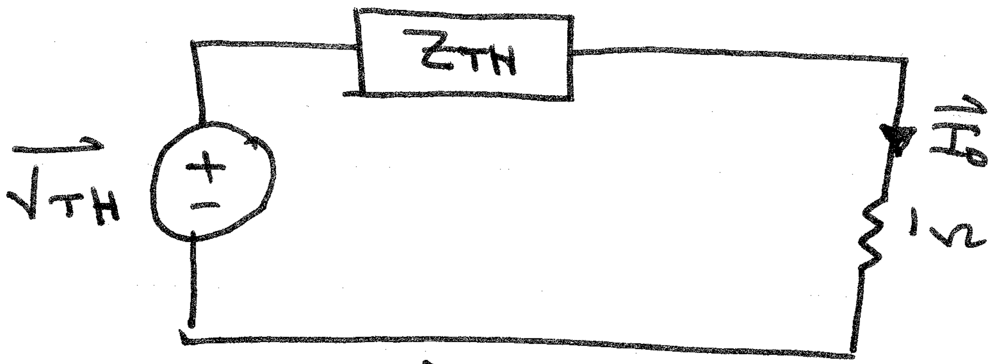
$$\therefore \vec{V}_{TH} = \left(\frac{-2+j12}{1+j1}\right) V$$

2) To find Z_{TH}, set all the independent sources to zero



$$Z_{TH} = -j1 + (1 \parallel j1)$$

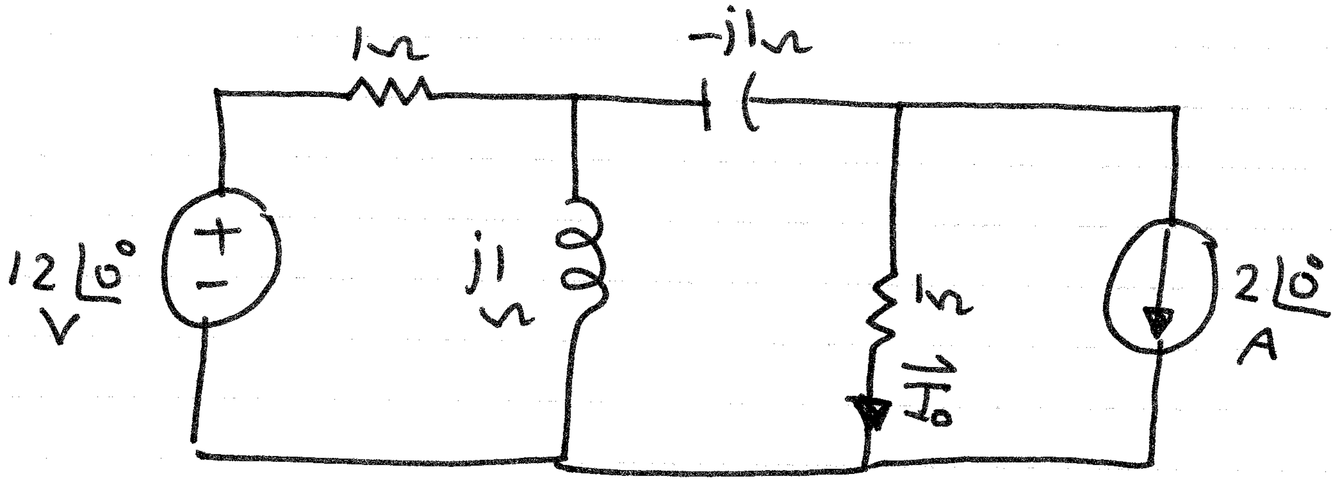
$$Z_{TH} = \left(\frac{1}{2} - j\frac{1}{2}\right)\Omega$$



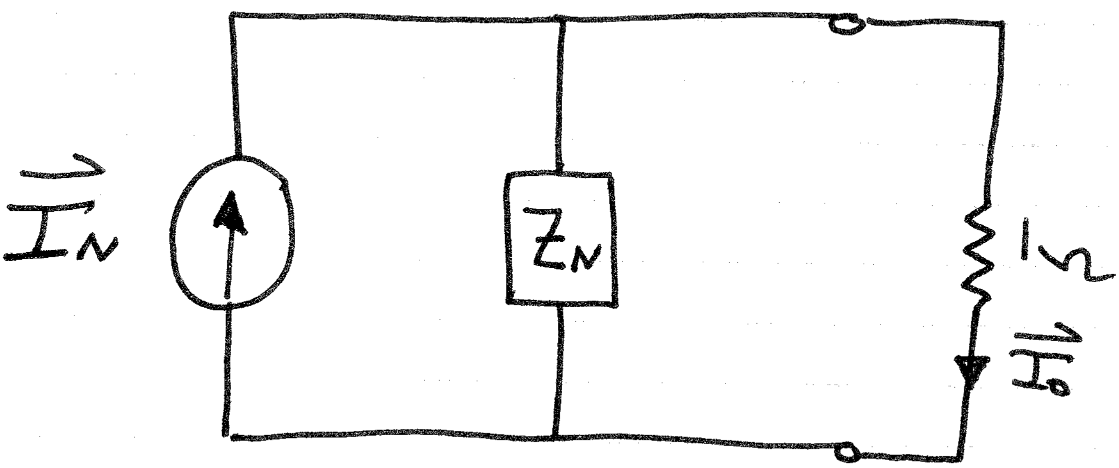
$$\vec{I}_o = \frac{\vec{V}_{TH}}{Z_{TH} + 1\Omega}$$

$$I_o = \left(\frac{8}{5} + j\frac{26}{5}\right) A$$

Norton's Theorem

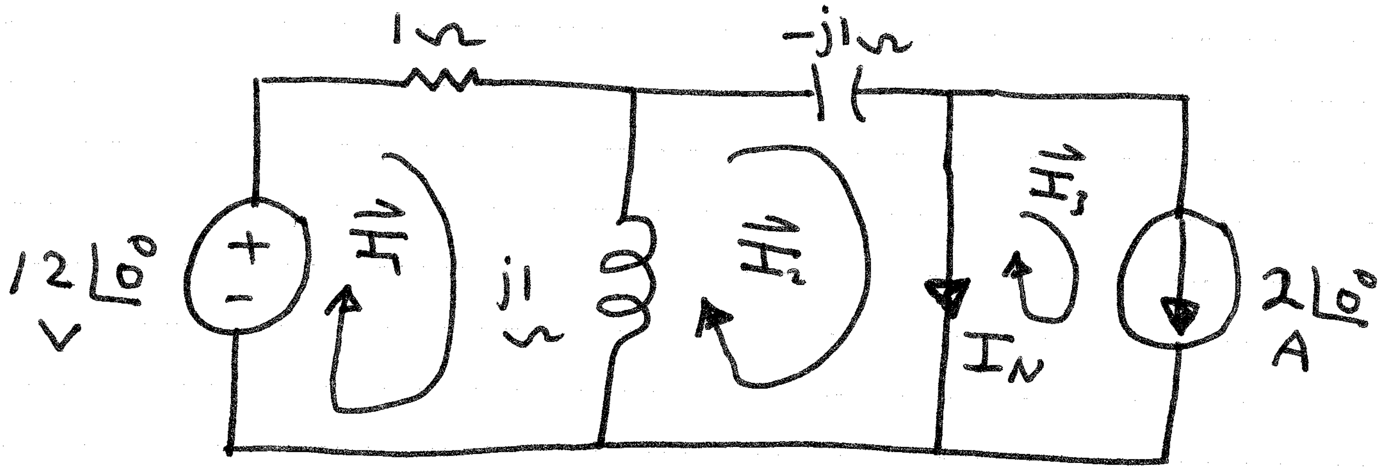


Find $\vec{I_o}$ using Norton's theorem



$$\vec{I_o} = \vec{I_N} \frac{Z_N}{Z_N + 1\Omega}$$

1) To find I_N



$$\vec{I_N} = \vec{I_2} - \vec{I_3}$$

$$\vec{I_3} = 2\angle 0^\circ \text{ A}$$ Constrain equation

KVL for mesh 1 :

$$12\angle 0^\circ = (1+j1)\vec{I_1} - j1\,\vec{I_2}$$

KVL for mesh 2 :

$$0 = -j1\,\vec{I_1} + (j1 - j1)\,\vec{I_2}$$

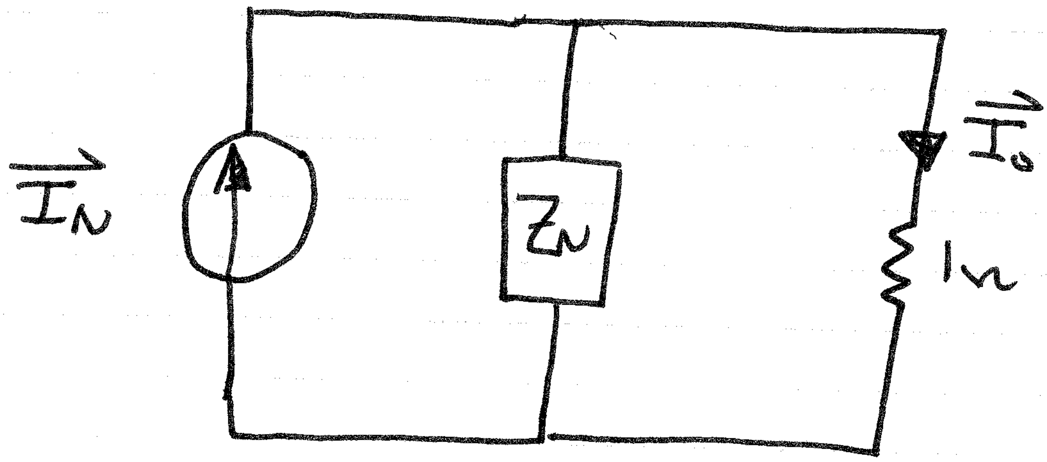
$$0 = -j1\,\vec{I_1}$$

$$\therefore \vec{I_1} = 0$$

$$\therefore \vec{I_2} = 12\angle 90^\circ \text{ A}$$

$$\therefore \vec{I_N} = \vec{I_2} - \vec{I_3} = -2 + j12$$

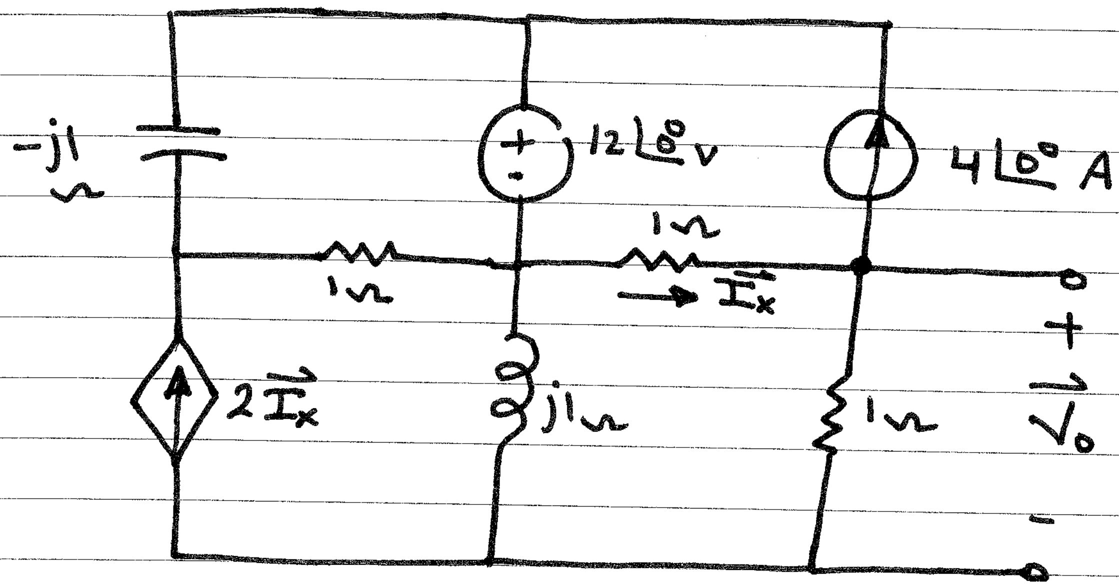
$$Z_N = Z_{TH} = \left(\frac{1}{2} - j\frac{1}{2}\right)\Omega$$



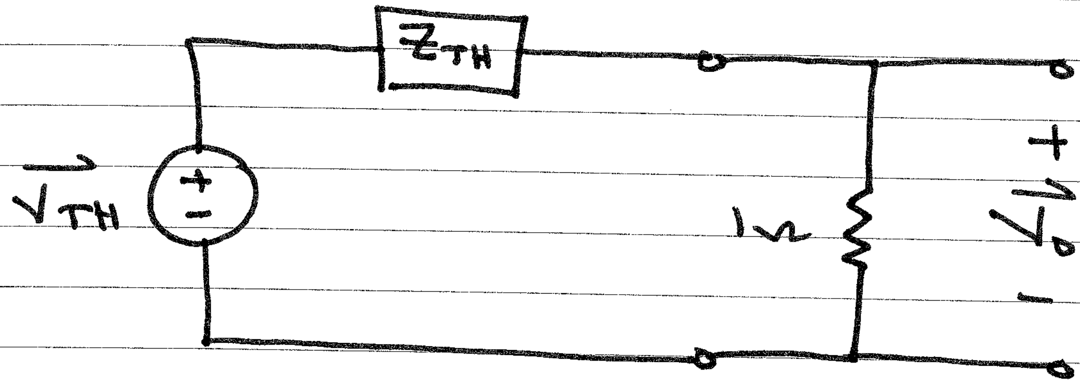
$$\vec{I}_o = \vec{I}_N \frac{Z_N}{Z_N + 1\Omega}$$

$$\vec{I}_o = \left(\frac{8}{5} + j\,\frac{26}{5}\right) A$$

Thevenin's Theorem

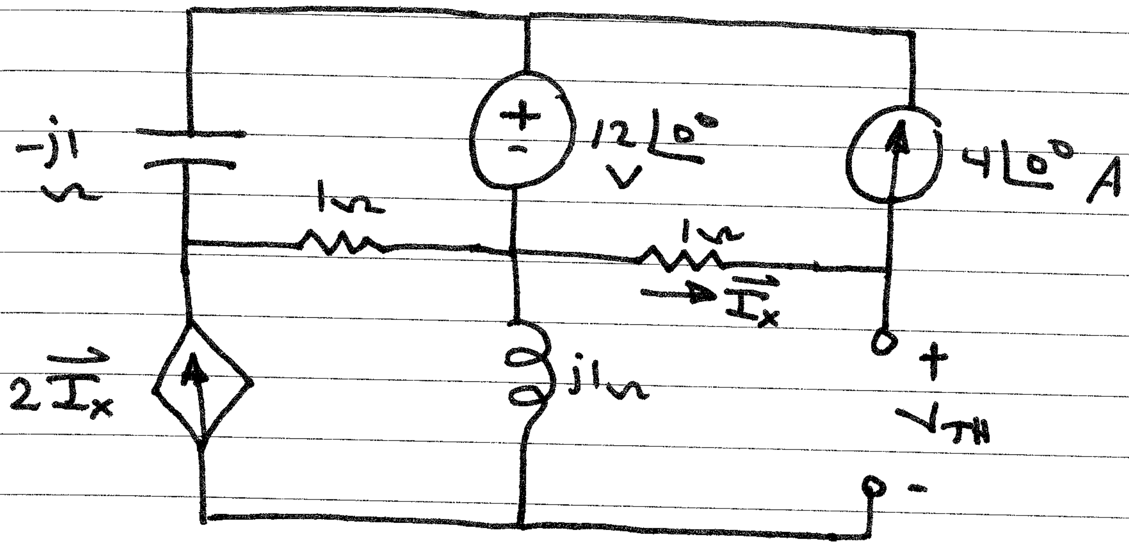


Find $\vec{V}_o$ using Thevenin's theorem



$$\vec{V}_o = \frac{1\Omega}{1\Omega + Z_{TH}} \vec{V}_{TH}$$

1) To find $\vec{V}_{TH}$



$$V_{TH} = -1\Omega\, \vec{I}_x + j1\Omega\,(2\vec{I}_x)$$

$$\vec{I}_x = 4\angle 0^\circ \text{ A}$$

$$\therefore\ \vec{V}_{TH} = (-4 + j8)\ \text{V}$$

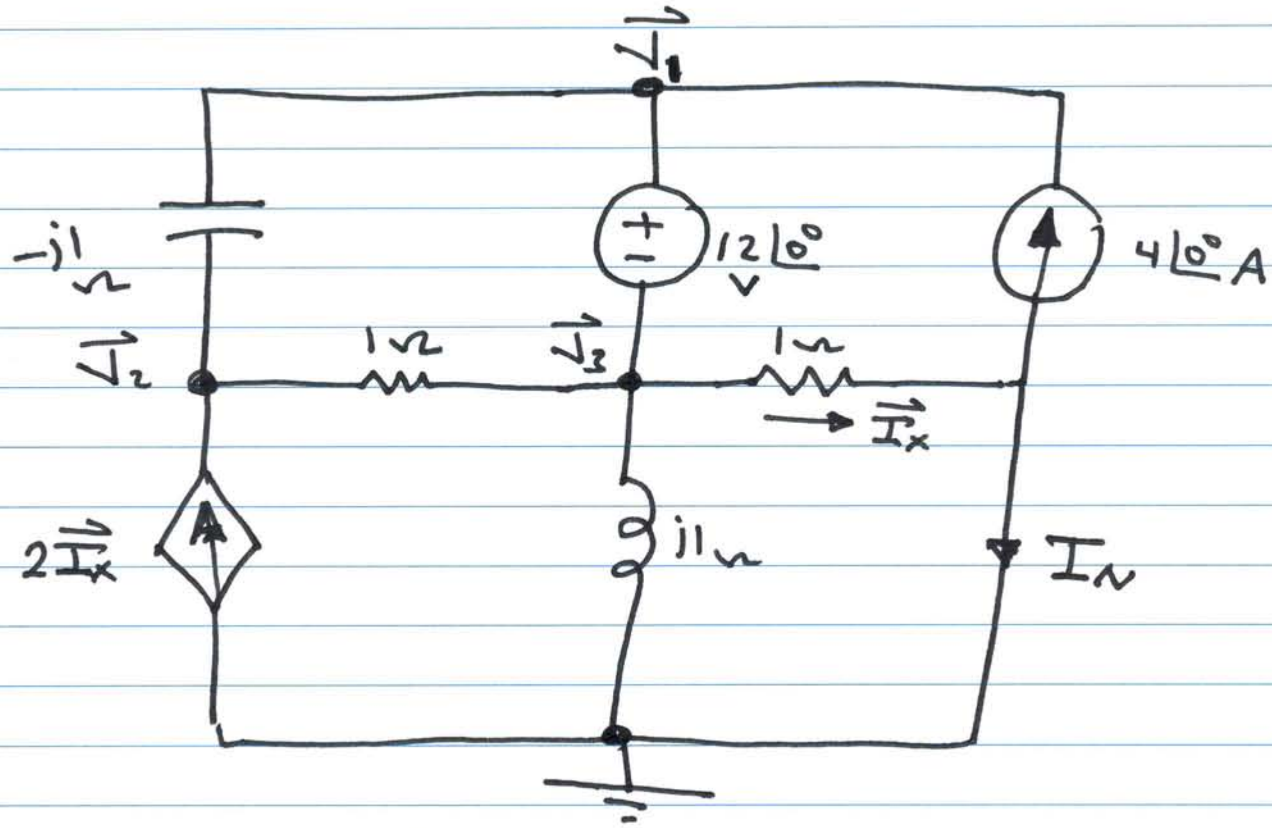
2) To find Z_{TH}

a) $Z_{TH} = \dfrac{\vec{V}_{TH}}{\vec{I}_N}$

b) $Z_{TH} = \dfrac{\vec{V}_T}{\vec{I}_T}$ | all independent sources are set to zero

a) $Z_{TH} = \dfrac{\vec{V}_{TH}}{\vec{I}_N}$

To find $\vec{I}_N$



$$\vec{I}_N = \vec{I}_x - 4\angle 0^\circ$$

$$\vec{I}_x = \frac{\vec{V}_3}{1\Omega} = \vec{V}_3$$

Nodal Analysis

$\vec{V}_1 - \vec{V}_3 = 12\angle 0^\circ$ Constrain equation

KCL at node 2 :

$$2\vec{I}_x = \left(1 + \frac{1}{-j1}\right)\vec{V}_2 + jV_1 - 1V_3$$

KCL for the Supernode (1,3)

$$4\angle 0^\circ = \left(\frac{1}{-j1}\right)V_1 + \left(1+1+\frac{1}{j1}\right)V_3 - \left(1+\frac{1}{-j1}\right)V_2$$

Solving for $\vec{V}_3$

$$\vec{V}_3 = \frac{4j}{1-j1}$$

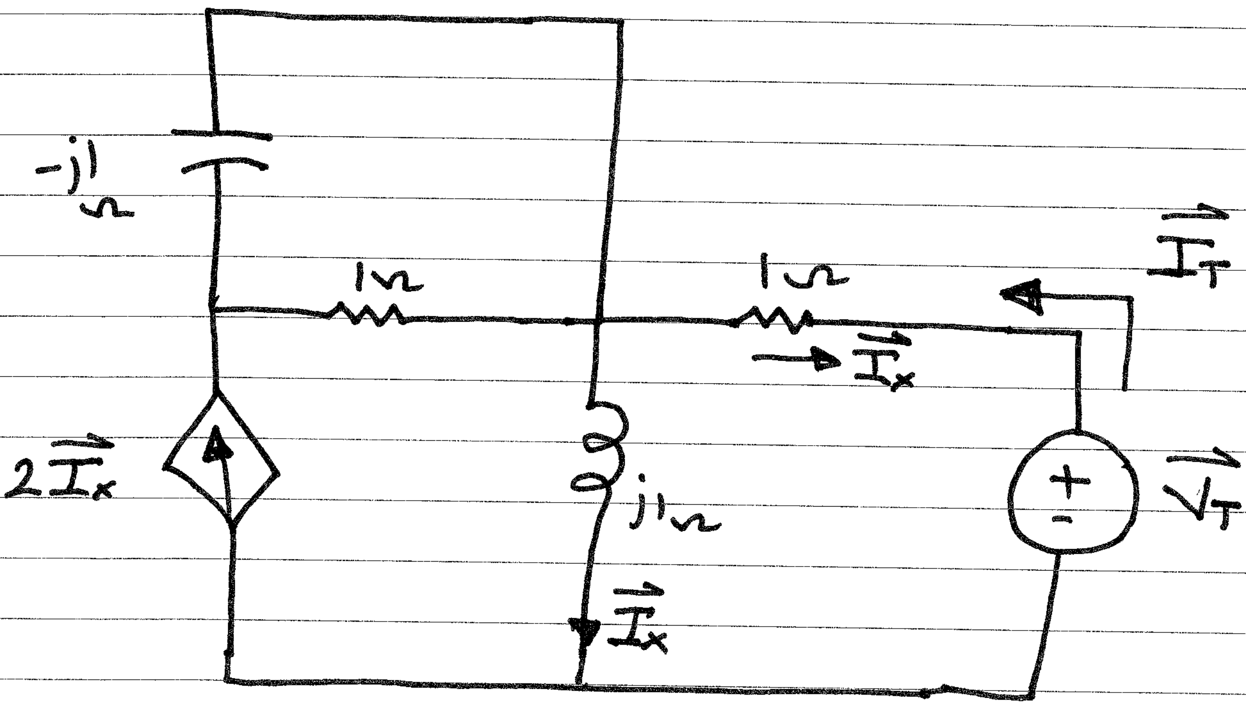
$$\therefore I_N = -\left(\frac{8+j4}{1+j1}\right)$$

$$\therefore Z_{TH} = \frac{\vec{V}_{TH}}{\vec{I}_N}$$

$$Z_{TH} = (1-j1)\ \Omega$$

$$\therefore \vec{V}_o = \frac{-4+j8}{1+1-j} = 4\angle 143.13^\circ\ V$$

b) $Z_{TH} = \dfrac{\vec{V}_T}{\vec{I}_T} \Big|$ independent source are zero



$$\vec{V}_T = -1(\vec{I}_x) + j1(\vec{I}_x)$$

$$\vec{V}_T = (-1+j1)\vec{I}_x$$

$$\vec{I}_x = -\vec{I}_T$$

$$\vec{V}_T = (-1+j1)\vec{I}_T$$

$$\therefore Z_{TH} = \frac{\vec{V}_T}{\vec{I}_T} = (1-j1)\ \Omega$$