16.3 Path Independence, Conservative Fields, and Potential Functions

* Path Independence:

- Recall that if A and B are two points in an open region D in space where a vector field $\vec{F}$ is defined, then the line integral of $\vec{F}$ along curve C from A to B depends on the path C (see Remark[2] page 137).
- The question now is: For which kind of vector fields $\vec{F}$ makes the line integral the same for all paths C from A to B?

Def. Let $\vec{F}$ be a vector field defined on an open region D in space.

- Suppose that for any two points A and B in D the line integral $\int_C \vec{F}\cdot d\vec{r}$ along a path C from A to B in D is the same for all paths from A to B.
- Then, the line integral $\int_C \vec{F}\cdot d\vec{r}$ is path independent in D and the vector field $\vec{F}$ is conservative on D.

Note that
- the word conservative comes from physics and refers to fields in which the principle of conservation of energy holds.
- when a line integral is path independent over

all paths C from A to B, we write the integral $\int_C \vec{F}\cdot d\vec{r} = \int_A^B \vec{F}\cdot d\vec{r}$ to remember the path-independence property.

Def If $\vec{F}$ is a vector field defined on D and $\vec{F} = \nabla f$ for some scalar function f on D, then f is called a potential function for $\vec{F}$.

Exp Find a potential function f for the vector field
$\vec{F} = 2x\,\vec{i} + 3y\,\vec{j} + 4z\,\vec{k}$

$$\vec{F} = \nabla f$$

$$2x\,\vec{i} + 3y\,\vec{j} + 4z\,\vec{k} = f_x\,\vec{i} + f_y\,\vec{j} + f_z\,\vec{k}$$

$f_x = \frac{\partial f}{\partial x} = 2x \Rightarrow f(x) = x^2 + g(y,z)$

$f_y = \frac{\partial f}{\partial y} = 3y = g_y \Leftrightarrow g(y) = \frac{3}{2}y^2 + c$

$f_z = \frac{\partial f}{\partial z} = 4z = g_z \Leftrightarrow g(z) = 2z^2 + c$

$f(x,y,z) = x^2 + \frac{3}{2}y^2 + 2z^2 + C$
where C is a constant.

Note that • once we find a potential function f for a field $\vec{F}$, we can evaluate all the line integrals in the domain of $\vec{F}$ over any path between A and B by

$$\int_A^B \vec{F}\cdot d\vec{r} = \int_A^B \nabla f \cdot d\vec{r} = f(B) - f(A)$$

as a result of the Fundemental Theorem of Calculus:

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

- Under some conditions (listed below):
  $\vec{F}$ is conservative iff $\vec{F} = \nabla f$ where f is a potential function for $\vec{F}$.

- $\vec{F}$ is conservative on D is equivalent to say the line integral of $\vec{F}$ over every closed path in D is zero.

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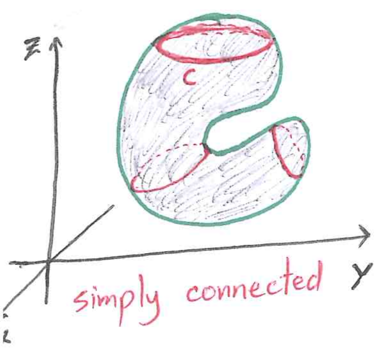
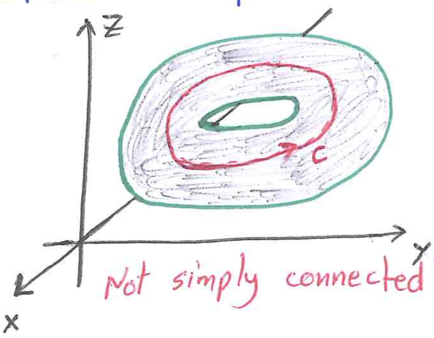
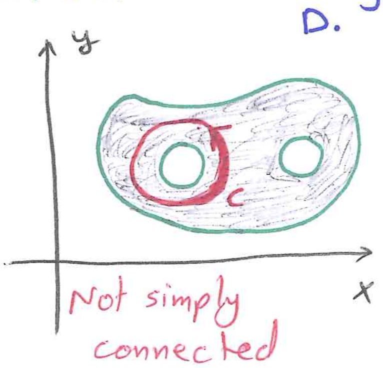
* Assumptions on Curves, Vector Fields and Domains:

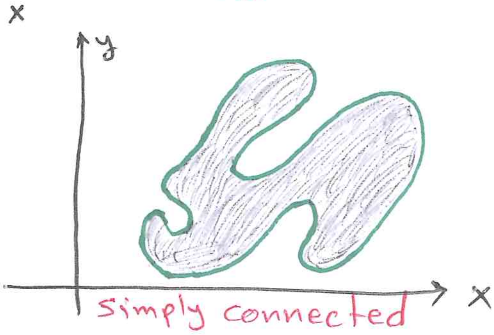
The results in this section hold under the following conditions:

- The curve C is piecewise smooth.
- The vector field $\vec{F}$ has components M, N, P that have continuous first partial derivatives.
- The domain D is an open region in space: every point in D is a center of an open ball lies entirely in D.
- The domain D is connected: any two points in D can be joined by a smooth curve lies in D.
- The domain D is simply connected: every loop in D can be contracted to a point in D without ever leaving D.









* Line Integrals in Conservative Fields

The following result is analogous to the Fundemental Theorem of Calculus which gives a way to evaluate the line integrals of gradient fields.

Th (Fundemental Theorem of Line Integrals)

- Let C be a smooth curve joining the point A to the point B in the plane (or space).
- Assume C parametrized by $\vec{r}(t)$.
- Let f be a diff function with a continuous gradient vector $\vec{F} = \nabla f$ on a domain D containing C. Then

$$\int_C \vec{F} \cdot d\vec{r} = f(B) - f(A)$$

Proof • Let $C: \ \vec{r}(t) = g(t)\vec{i} + h(t)\vec{j} + k(t)\vec{k}$, $a \le t \le b$ be a smooth curve joining A to B in region D.

- Since f is diff along the curve C $\Rightarrow$

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \quad \cdots\cdots \ *$$

- Noting that $x = g(t)$, $y = h(t)$ and $z = k(t)$ $\Rightarrow$

$$\frac{d\vec{r}}{dt} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}$$

- Hence, $*$ becomes $\dfrac{df}{dt} = \nabla f \cdot \dfrac{d\vec{r}}{dt}$ but $\vec{F} = \nabla f$ $\Rightarrow$

$$= \vec{F} \cdot \frac{d\vec{r}}{dt}$$

$$\boxed{\begin{aligned}\vec{r}(a) &= A \\ \vec{r}(b) &= B\end{aligned}}$$

- Therefore,

$$\int_C \vec{F} \cdot d\vec{r} = \int_{t=a}^{t=b} \vec{F} \cdot \frac{d\vec{r}}{dt}\,dt = \int_a^b \frac{df}{dt}\,dt = f(x,y,z)\Big|_a^b = f(g(b),h(b),k(b)) - f(g(a),h(a),k(a))$$

$$= f(B) - f(A)$$

Exp Suppose the force field $\vec{F} = \nabla f$ is the gradient of the function $f(x,y,z) = \dfrac{-1}{x^2+y^2+z^2}$. Find the work done by $\vec{F}$ in moving an object along a smooth curve C joining (1, 0, 0) to (0,0,2) that does not pass through the origin.

$$\text{work} = \int_C \vec{F}\cdot d\vec{r} = f(0,0,2) - f(1,0,0) = \frac{-1}{4} - (-1) = \frac{3}{4}$$

Th²(Conservative Fields are Gradient Fields)

Let $\vec{F} = M\vec{i} + N\vec{j} + P\vec{k}$ be a vector field whose components M, N, P are continuous on an open connected region D in space.
Then $\vec{F}$ is conservative iff $\vec{F} = \nabla f$ for a diff function f.

That is $\vec{F} = \nabla f$ iff for any two points A and B in D, the line integral $\int_C \vec{F}\cdot d\vec{r}$ is independent of the path C joining A to B in D.

Exp Find the work done by the conservative field $\vec{F} = yz\,\vec{i} + xz\,\vec{j} + xy\,\vec{k} = \nabla f$ where $f(x,y,z) = xyz$ along any smooth curve C joining the points $A(-1,3,9)$ to $B(1,6,-4)$.

$$\text{work} = \int_C \vec{F}\cdot d\vec{r} = \int_A^B \nabla f\cdot d\vec{r} = f(B) - f(A)$$

$$= f(1,6,-4) - f(-1,3,9)$$

$$= -24 - -27$$

$$= 3$$

Th[3] (Loop Property of Conservative Fields)

$\oint_C \vec{F}\cdot d\vec{r} = 0$ around every loop (closed curve C) in D iff the field $\vec{F}$ is conservative on D.

* How do we know whether a given vector field $\vec{F}$ is conservative?

Th*' (Component Test for Conservative Fields)

- Let $\vec{F} = M(x,y,z)\vec{i} + N(x,y,z)\vec{j} + P(x,y,z)\vec{k}$ be a vector field defined on a connected and simply connected domain.
- Assume the components M, N, P have continuous first partial derivatives.
- Then, $\vec{F}$ is conservative iff $P_y = N_z$, $M_z = P_x$ and $N_x = M_y$.

Exp show that $\vec{F} = y\vec{i} + (x+z)\vec{j} - y\vec{k}$ is not conservative.

$$P_y = -1 \neq 1 = N_z$$

Exp show that $\vec{F} = (e^x \cos y + yz)\vec{i} + (xz - e^x \sin y)\vec{j} + (xy+z)\vec{k}$ is conservative and find a potential function for $\vec{F}$.

- $M = e^x \cos y + yz$, $N = xz - e^x \sin y$, $P = xy + z$

$P_y = x = N_z$, $M_z = y = P_x$, $N_x = z - e^x \sin y = M_y$

Note that the domain of $\vec{F}$ is all of space which is connected and simply connected. Furthermore, the partial derivatives are continuous, so $\vec{F}$ is conservative. Hence, $\exists$ a function f s.t $\nabla f = \vec{F}$. To find the potential function $f \Longrightarrow$

by Th2

$f_x = M = e^x \cos y + yz \Rightarrow f(x,y,z) = e^x \cos y + yzx + g(y,z)$

$f_y = N = xz - e^x \sin y$

$f_z = P = xy + z$

- To find $g(y,z) \Rightarrow f_y = N$

$-e^x \sin y + xz + g_y = xz - e^x \sin y$

$g_y = 0 \Rightarrow g(y,z) = h(z) \Rightarrow$

$f(x,y,z) = e^x \cos y + yzx + h(z)$

- To find $h(z) \Rightarrow$

$f_z = P$

$xy + h' = xy + z \Rightarrow h'(z) = z \Rightarrow h(z) = \frac{z^2}{2} + c$

Hence, $f(x,y,z) = e^x \cos y + yzx + \frac{z^2}{2} + c$

Exp Consider the vector field $\vec{F} = \frac{-y}{x^2+y^2}\vec{i} + \frac{x}{x^2+y^2}\vec{j}$

1. Show that $\vec{F}$ satisfies the equations in the Component Test.

$M = \frac{-y}{x^2+y^2}$, $N = \frac{x}{x^2+y^2}$, $P = 0$

$P_y = 0 = N_z$, $P_x = 0 = M_z$, $M_y = \frac{y^2 - x^2}{(x^2+y^2)^2} = N_x$

2. Show that $\vec{F}$ is not conservative on its natural domain.

- $\vec{F}$ is not simply connected on its natural domain (the complement of the z-axis)
- since $x^2 + y^2$ never zero, the natural domain contains loops that can not be contracted to a point.
- One such loop is the unit circle C in the xy-plane parametrized by:

$\vec{r}(t) = (\cos t)\vec{i} + (\sin t)\vec{j}$, $0 \le t \le 2\pi$

- To show that $\vec{F}$ is not conservative, we now apply Th³ and compute the line integral $\oint_C \vec{F}\cdot d\vec{r}$ around the loop C :

$\rightarrow$ $\vec{r}(t) = (\cos t)\,\vec{i} + (\sin t)\,\vec{j}$ with $0 \le t \le 2\pi$

$x = \cos t$

$y = \sin t$

$$\rightarrow \quad \vec{F} = \frac{-y}{x^2+y^2}\,\vec{i} + \frac{x}{x^2+y^2}\,\vec{j}$$

$$= \frac{-\sin t}{\sin^2 t + \cos^2 t}\,\vec{i} + \frac{\cos t}{\sin^2 t + \cos^2 t}\,\vec{j}$$

$$= (-\sin t)\,\vec{i} + (\cos t)\,\vec{j}$$

$$= \frac{d\vec{r}}{dt}$$

$$\rightarrow \oint_C \vec{F}\cdot d\vec{r} = \oint_C \vec{F}\cdot\frac{d\vec{r}}{dt}\,dt = \int_0^{2\pi}\left(\sin^2 t + \cos^2 t\right)dt = 2\pi$$

$\rightarrow$ since the line integral is not zero $\Rightarrow$ $\vec{F}$ is not conservative by Th³.

Exact Differential Form :

- Recall that the work and circulation integrals $\int_C \vec{F}\cdot d\vec{r}$
- If $\vec{F}$ is conservative, then $\int_C \vec{F}\cdot d\vec{r} = \int_A^B \nabla f\cdot d\vec{r} = f(B) - f(A) = \int_A^B df$
- So we can write $\int_C \vec{F}\cdot d\vec{r} = \int_A^B \nabla f\cdot d\vec{r}$

$$= \int_C \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz$$

$$= \int_C M\,dx + N\,dy + P\,dz$$

Remember :

$\nabla f = f_x\,\vec{i} + f_y\,\vec{j} + f_z\,\vec{K}$

$\vec{r}(t) = x\,\vec{i} + y\,\vec{j} + z\,\vec{K}$

$d\vec{r} = dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{K}$

Def • The differential form is any expression of the form:

$$M(x,y,z)\,dx + N(x,y,z)\,dy + P(x,y,z)\,dz$$

• The differential form is exact on a domain D in space if:

$$M\,dx + N\,dy + P\,dz = f_x\,dx + f_y\,dy + f_z\,dz = df$$

for some scalar function f on D.

Th*² (Component Test for Exactness of $M\,dx + N\,dy + P\,dz$)

• The differential form $M\,dx + N\,dy + P\,dz$ is exact on a connected and simply connected domain iff

$$P_y = N_z \;,\quad M_z = P_x \quad \text{and} \quad N_x = M_y$$

• This is equivalent to say $\vec{F} = M\vec{i} + N\vec{j} + P\vec{k}$ is conservative.

Exp show that the differential form in the integral

$$\int_{(1,1,2)}^{(3,5,0)} yz\,dx + xz\,dy + xy\,dz$$

is exact. Find the integral.

• $\vec{F} = M\vec{i} + N\vec{j} + P\vec{k}$

$= yz\,\vec{i} + xz\,\vec{j} + xy\,\vec{k}$

with $P_y = x = N_z$, $M_z = y = P_x$

and $N_x = z = M_y$. Hence,

$M\,dx + N\,dy + P\,dz$ is exact.

• $f_x = M = yz \qquad f_y = N = xz$

$\qquad f_z = P = xy$

• $f = \int f_x\,dx = yzx + g(y,z)$

$f_y = \cancel{xz} + g_y = \cancel{xz} \Leftrightarrow g(y,z) = h(z)$

• $f(x,y,z) = xyz + h(z)$

$f_z = xy + h'(z) = xy \Leftrightarrow h = c$

• $f(x,y,z) = xyz + c$

So $\displaystyle\int_{(1,1,2)}^{(3,5,0)} yz\,dx + xz\,dy + xy\,dz = f(3,5,0) - f(1,1,2) = 0 - 2 = -2$