Ch7

Solving linear Systems of DE's

- In this chapter, we will learn how to solve 2x2 linear system of ODE's of the form:

$$x_1' = a_{11}x_1 + a_{12}x_2 \quad , \quad x_1(t_0) = x_1^0$$
$$x_2' = a_{21}x_1 + a_{22}x_2 \quad , \quad x_2(t_0) = x_2^0$$

Homo. System with constant coefficients

where $x_1' = \frac{dx_1}{dt}$ and $x_2' = \frac{dx_2}{dt}$

- We can write this linear system using matrix form:

$$x' = Ax \quad , \overset{\circ}{x} = x(0) = \begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix} \quad \text{where} \qquad (1)$$

$$x' = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} \quad , \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad , \quad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

- Note that $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ is vector of unknowns

A is called the coefficient matrix

$x_0 = x(0) = \begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix}$ is vector of initial conditions

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Question How to solve the linear system of two ODE's ?

Answer Assume exponensial solution of the form

$$x(t) = \underset{\sim}{\xi}\, e^{rt}$$

where $r \in \mathbb{R}$ is an eigenvalue and

$\underset{\sim}{\xi} \in \mathbb{R}^2$ is the corresponding nonzero eigenvector given by

$$\underset{\sim}{\xi} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad y_1, y_2 \in \mathbb{R}$$

- Hence, $\frac{dx}{dt} = x' = r \underset{\sim}{\xi}\, e^{rt}$
- Now substitute x and x' in (1) $\Rightarrow$

$$x' = Ax \Rightarrow r \underset{\sim}{\xi}\, e^{rt} = A \underset{\sim}{\xi}\, e^{rt}$$

$$A \underset{\sim}{\xi}\, e^{rt} - r \underset{\sim}{\xi}\, e^{rt} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$A \underset{\sim}{\xi} - r \underset{\sim}{\xi} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\boxed{(A - rI)\underset{\sim}{\xi} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}} - *^2 \quad \text{where } I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- since $\underset{\sim}{\xi}$ is nonzero vector $\Rightarrow$ we must have

$$\boxed{|A - rI| = 0} - *^1 \quad \text{where } |\;| \text{ means determinante}$$

- This means the square matrix $A - rI$ is singular

• To solve the linear system of two ODE's (1):

→ First solve $*^1$ for the eigenvalues $r_1 \neq r_2$

→ Second solve $*^2$ for the corresponding eigenvectors ξ_1 and ξ_2

→ 1^{st} solution $X_1(t) = \xi_1 e^{r_1 t}$

→ 2^{nd} solution $X_2(t) = \xi_2 e^{r_2 t}$

→ General solution $X(t) = c_1 X_1(t) + c_2 X_2(t)$

$$= c_1 \xi_1 e^{r_1 t} + c_2 \xi_2 e^{r_2 t}$$

→ To find c_1 and c_2 we use the initial vector x^0

Remark There are three possible cases for the values of the eigenvalues r_1 and r_2:

1. If $r_1 \neq r_2 \in \mathbb{R}$, then we will study the solution in section 7.5

2. If $r_{1,2} = \lambda \pm \mu i$, then we will study the solution in section 7.6

3. If $r_1 = r_2 = r \in \mathbb{R}$, then we will study the solution in section 7.8

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4. $x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is trivial solution for (1) or Eq. solution.

But we look for nontrivial solution for (1).

7.5 Real Different Eigenvalues

$r_1 \neq r_2 \in \mathbb{R}$

Exp Find the general solution for the linear system:

$$x_1' = x_1 + x_2 \quad , \quad x_1(0) = 3$$
$$x_2' = 4x_1 + x_2 \quad , \quad x_2(0) = -2$$

- Note that $x' = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\overset{\circ}{x} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$, $A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$

- First we solve $*^1$ for r_1 and r_2 $\Rightarrow$

$$|A - rI| = 0 \quad \Rightarrow \left| \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} - r \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} 1-r & 1 \\ 4 & 1-r \end{vmatrix} = 0$$

$$(1-r)^2 - 4 = 0 \quad \Rightarrow (1-r)^2 = 4 \quad \Rightarrow |1-r| = 2$$

either $1-r = 2 \quad \Rightarrow r_1 = -1$
or $\quad 1-r = -2 \quad \Rightarrow r_2 = 3$

$\Big\}$ eigenvalues are real different

- To find the eigenvector $\xi_1 = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ corresponding to the eigenvalue $r_1 = -1$ we solve $*^2$:

$$(A - r_1 I)\xi_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \begin{pmatrix} 1-r_1 & 1 \\ 4 & 1-r_1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1--1 & 1 \\ 4 & 1--1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- It's enough to take $2y_1 + y_2 = 0 \quad \Rightarrow y_2 = -2y_1$

Take $y_1 = 1 \Rightarrow y_2 = -2 \Rightarrow \xi_1 = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

Hence, the 1^{st} solution is $x_1(t) = \xi_1 e^{r_1 t}$

$$= \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}$$

. To find the eigenvector $\xi_2 = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ corresponding to the eigenvalue $r_2 = 3$ we solve $*^2$:

$$(A - r_2 I)\xi_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1-r_2 & 1 \\ 4 & 1-r_2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1-3 & 1 \\ 4 & 1-3 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

. It's enough to set $-2z_1 + z_2 = 0 \Rightarrow z_2 = 2z_1$

Take $z_1 = 1 \Rightarrow z_2 = 2 \Rightarrow \xi_2 = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Hence, the 2^{nd} solution is $x_2(t) = \xi_2 e^{r_2 t}$

$$= \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t}$$

. Hence, the gen. sol. is :

$$x(t) = c_1 x_1(t) + c_2 x_2(t)$$

$$= c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t}$$

. To find the constants c_1 and c_2 we use the $\underline{\underline{IC}}$:

$$x(0) = c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 3 \\ -2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\left.\begin{aligned} 3 &= c_1 + c_2 \\ -2 &= -2c_1 + 2c_2 \end{aligned}\right\} \Rightarrow \begin{aligned} c_1 &= 2 \\ c_2 &= 1 \end{aligned}$$

So the gen. sol. becomes:

$$x(t) = 2\begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t} + \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{3t}$$

$$= \begin{pmatrix} 2 \\ -4 \end{pmatrix} e^{-t} + \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{3t}$$

Remark 1 $x_1(t) = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}$ and $x_2(t) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t}$

are two independent solution since

$$W(x_1(t), x_2(t))(t) = \begin{vmatrix} e^{-t} & e^{3t} \\ -2e^{-t} & 2e^{3t} \end{vmatrix} = 2e^{2t} - -2e^{2t} = 4e^{2t} \neq 0$$

Hence, $\{x_1(t), x_2(t)\}$ form fundemental set of solution

2 • If r_1 and r_2 are negative, then origin is asymptotically stable Eq. point

• If r_1 and r_2 are positive, then origin is unstable Eq. point

• If $r_1 r_2 < 0$, then origin is saddle point which is unstable Eq. point

 or 

Exp Find two independent solutions for the system

$$x' = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} x$$

- First solve $*^1$ for r_1 and r_2 $\Rightarrow |A - rI| = 0$

$$\begin{vmatrix} 1-r & 2 \\ 2 & 4-r \end{vmatrix} = 0 \Rightarrow (1-r)(4-r) - 4 = 0$$

$$4 - r - 4r + r^2 - 4 = 0$$

$$r^2 - 5r = 0$$

$$r(r-5) = 0$$

$$r_1 = 0 \ , \ r_2 = 5$$ eigenvalues are real different

- To find the eigenvector $\xi_1 = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ corresponding to the eigenvalue $r_1 = 0$ we solve $*^2$:

$$(A - r_1 I)\xi_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1-r_1 & 2 \\ 2 & 4-r_1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$ $\Rightarrow$ Enough to set $y_1 + 2y_2 = 0$

$$y_1 = -2y_2$$

Take $y_2 = 1 \Rightarrow y_1 = -2 \Rightarrow \xi_1 = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

Hence, the 1^{st} solution is $x_1(t) = \xi_1 e^{r_1 t} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{0t}$

$$= \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

- To find the eigenvector $\xi_2 = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ corresponding to the eigenvalue $r_2 = 5$ we solve $*^2$:

$$(A - r_2 I)\xi_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1-r_2 & 2 \\ 2 & 4-r_2 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \text{Enough to set } 2z_1 - z_2 = 0$$
$$\Rightarrow z_2 = 2z_1$$

Take $z_1 = 1 \Rightarrow z_2 = 2 \Rightarrow \xi_2 = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Hence, the 2^{nd} solution is $x_2(t) = \xi_2 e^{r_2 t} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{5t}$

- To show $x_1(t)$ and $x_2(t)$ are independent $\Rightarrow$

$$W(x_1(t), x_2(t))(t) = \begin{vmatrix} -2 & e^{5t} \\ 1 & 2e^{5t} \end{vmatrix} = -4e^{5t} - e^{5t} = -5e^{5t} \neq 0$$

Hence, $x_1(t)$ and $x_2(t)$ are independent solutions and so $\{x_1(t), x_2(t)\}$ forms fundemental set of solutions

- The gen. sol. is $x(t) = c_1 x_1(t) + c_2 x_2(t)$

$$= c_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{5t}$$

Exp Find the eigenvalues of the system:

$$x' = \begin{pmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{pmatrix} x$$

Solve $x' \Rightarrow |A - rI| = 0 \Rightarrow \begin{vmatrix} -3-r & \sqrt{2} \\ \sqrt{2} & -2-r \end{vmatrix} = 0$

$(-3-r)(-2-r) - 2 = 0$

$6 + 3r + 2r + r^2 - 2 = 0$

$r^2 + 5r + 4 = 0$

$(r+1)(r+4) = 0$

$r_1 = -1, \quad r_2 = -4$

eigenvalues

One can find

$\xi_1 = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} \Rightarrow x_1(t) = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} e^{-t}$

$\xi_2 = \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} \Rightarrow x_2(t) = \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} e^{-4t}$

7.6 Complex Eigenvalues

$$r_{1,2} = \lambda \pm \mu i$$

<u>Exp</u> solve this system of DE's

$$x' = \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix} x$$

. First solve $*^1$ for the eigenvalues r_1 and r_2:

$$|A - rI| = 0 \Rightarrow \begin{vmatrix} 1-r & -1 \\ 5 & -3-r \end{vmatrix} = 0$$

$$(1-r)(-3-r) - -5 = 0$$

$$-3 - r + 3r + r^2 + 5 = 0 \quad \Rightarrow r^2 + 2r + 2 = 0$$

$$r_{1,2} = \frac{-2 \pm \sqrt{4-8}}{2} = \frac{-2 \pm 2i}{2}$$

$$= -1 \pm i \quad \text{complex eigenvalues}$$

. To find the eigenvector $\xi_1 = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ corresponding to the eigenvalue $r_1 = -1-i$ we solve $*^2$:

$$(A - r_1 I)\xi_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1-r_1 & -1 \\ 5 & -3-r_1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1-(-1-i) & -1 \\ 5 & -3-(-1-i) \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 2+i & -1 \\ 5 & -2+i \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$(2+i)y_1 - y_2 = 0 \Rightarrow y_2 = (2+i)y_1$$

Take $y_1 = 1 \Rightarrow y_2 = 2+i \Rightarrow \xi_1 = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2+i \end{pmatrix}$

. It can be shown that the 2nd eigenvector $\xi_2 = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2-i \end{pmatrix}$ which is conjugate of ξ_1. To see that we solve $*^2 \Rightarrow$

$$(A - r_2 I)\xi_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1-r_2 & -1 \\ 5 & -3-r_2 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Note $r_2 = -1+i$

$$\begin{pmatrix} 2-i & -1 \\ 5 & -2-i \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$(2-i)z_1 - z_2 = 0 \Rightarrow z_2 = (2-i)z_1$$

Take $z_1 = 1 \Rightarrow z_2 = (2-i) \Rightarrow \xi_2 = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2-i \end{pmatrix}$

- Hence, 1^{st} complex solution is $x_1(t) = \xi_1 e^{r_1 t} = \begin{pmatrix} 1 \\ 2+i \end{pmatrix} e^{(-1-i)t}$

  2^{nd} complex solution is $x_2(t) = \xi_2 e^{r_2 t} = \begin{pmatrix} 1 \\ 2-i \end{pmatrix} e^{(-1+i)t}$

- But we need to find real valued solutions:
  So we use Euler Formula to find the
  first real solution ($u(t)$ – real part) and the
  second real solution ($v(t)$ – imaginary part)

- we apply Euler Formula either on $x_1(t)$ or $x_2(t)$:

$$x_1(t) = \begin{pmatrix} 1 \\ 2+i \end{pmatrix} e^{(-1-i)t} = \begin{pmatrix} 1 \\ 2+i \end{pmatrix} e^{-t} e^{-it}$$

Euler Formular
$e^{i\theta} = \cos\theta + i\sin\theta$

$$= \begin{pmatrix} 1 \\ 2+i \end{pmatrix} e^{-t}\left(\cos(-t) + i\sin(-t)\right)$$

$$= e^{-t}\begin{pmatrix} 1 \\ 2+i \end{pmatrix}(\cos t - i\sin t)$$

$$= e^{-t}\begin{pmatrix} \cos t - i\sin t \\ 2\cos t + \sin t + i\cos t - 2i\sin t \end{pmatrix}$$

$$x_1(t) = e^{-t}\begin{pmatrix} \cos t \\ 2\cos t + \sin t \end{pmatrix} + i\, e^{-t}\begin{pmatrix} -\sin t \\ \cos t - 2\sin t \end{pmatrix}$$

$u(t)$ - real part $\qquad$ $v(t)$ - imaginary part

Hence, the gen. sol. is

$$x(t) = c_1\, u(t) + c_2\, v(t)$$

$$= c_1 e^{-t}\begin{pmatrix} \cos t \\ 2\cos t + \sin t \end{pmatrix} + c_2\, e^{-t}\begin{pmatrix} -\sin t \\ \cos t - 2\sin t \end{pmatrix}$$

Remark If we take $x_2(t)$ we get the same $u(t)$ and $v(t) \Rightarrow$

$$x_2(t) = \begin{pmatrix} 1 \\ 2-i \end{pmatrix} e^{(-1+i)t} = e^{-t}\begin{pmatrix} 1 \\ 2-i \end{pmatrix} e^{it} = e^{-t}\begin{pmatrix} 1 \\ 2-i \end{pmatrix}(\cos t + i\sin t)$$

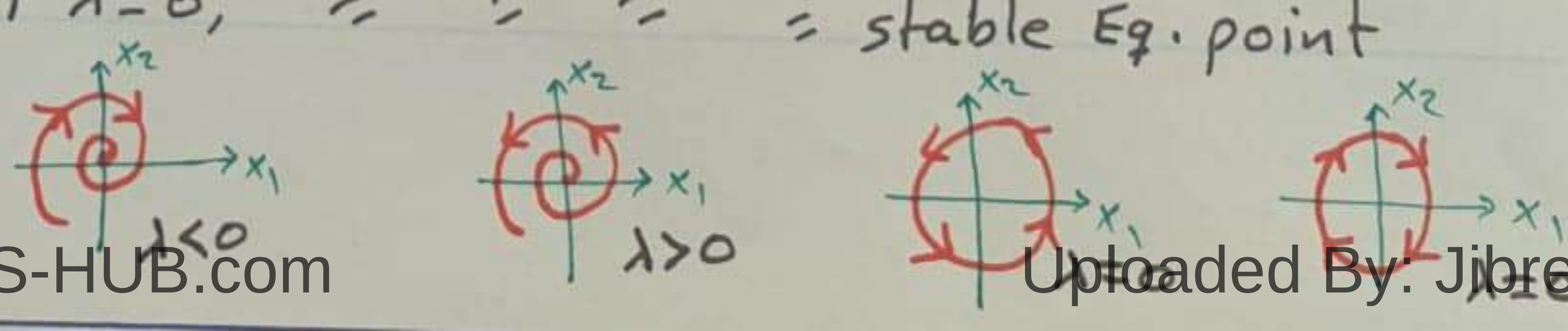
$$= e^{-t}\begin{pmatrix} \cos t + i\sin t \\ 2\cos t + \sin t - i\cos t + 2i\sin t \end{pmatrix}$$

$$= e^{-t}\begin{pmatrix} \cos t \\ 2\cos t + \sin t \end{pmatrix} - i\, e^{-t}\begin{pmatrix} -\sin t \\ \cos t - 2\sin t \end{pmatrix}$$

$u(t)$ - real part $\qquad$ $v(t)$ - imaginary part

the negative sign goes in c_2

Remark
- If $\lambda < 0$, then the origin is asymptotically stable spiral Eq. point
- If $\lambda > 0$, " " " " = unstable spiral Eq. point
- If $\lambda = 0$, " " " " = stable Eq. point



$\boxed{7.8}$ Repeated Eigenvalues

$r_1 = r_2 = r$

- If we solve $*^1$ $|A - rI| = 0$ and we get $r_1 = r_2 = r$, then we solve $*^2$ $(A - rI)\xi = \begin{pmatrix}0\\0\end{pmatrix}$ to find $\xi_1 = \begin{pmatrix}y_1\\y_2\end{pmatrix}$
- Now to find the second eigenvector $\xi_2 = \begin{pmatrix}z_1\\z_2\end{pmatrix}$ we solve $*^3$

$$\boxed{(A - rI)\xi_2 = \xi_1} \quad \cdots *^3$$

- Hence, the 1st solution is $x_1(t) = \xi_1 e^{rt}$

  the 2nd solution is $x_2(t) = \xi_2 e^{rt} + \xi_1 t e^{rt}$

- The gen. sol. is $x(t) = c_1 x_1(t) + c_2 x_2(t)$

$$= c_1 \xi_1 e^{rt} + c_2\left(\xi_1 t e^{t} + \xi_2 e^{rt}\right)$$

Exp solve this linear system: $x_1' = x_1 - 4x_2$, $x_1(0) = 3$

$x_2' = 4x_1 - 7x_2$, $x_2(0) = 2$

- First find the eigenvalues by solving $*^1 \Rightarrow |A - rI| = 0$

$$\begin{vmatrix} 1-r & -4 \\ 4 & -7-r \end{vmatrix} = 0 \quad \Rightarrow (1-r)(-7-r) - -16 = 0$$

$$-7 - r + 7r + r^2 + 16 = 0$$

$$r^2 + 6r + 9 = 0$$

$$(r+3)(r+3) = 0$$

$$r_1 = r_2 = r = -3$$ repeated eigenvalues

- Now we find the eigenvector $\xi_1 = \begin{pmatrix}y_1\\y_2\end{pmatrix}$ corresponding to the eigenvalue $r = -3$ by solving $*^2 \Rightarrow$

$$(A - rI)\xi_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1-r & -4 \\ 4 & -7-r \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow 4y_1 - 4y_2 = 0$$
$$\Rightarrow y_1 = y_2$$

Take $y_1 = 1 \Rightarrow y_2 = 1 \Rightarrow \xi_1 = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

- Hence, 1^{st} solution is $x_1(t) = \xi_1 e^{rt} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t}$

- To find the 2^{nd} eigenvecter $\xi_2 = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ we solve $*^3 \Rightarrow$

$$(A - rI)\xi_2 = \xi_1 \Rightarrow \begin{pmatrix} 1-r & -4 \\ 4 & -7-r \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow 4z_1 - 4z_2 = 1$$
$$\Rightarrow z_1 = \frac{1}{4} + z_2$$

Take $z_2 = K \Rightarrow z_1 = \frac{1}{4} + K \Rightarrow \xi_2 = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} + K \\ K \end{pmatrix}$

$$= \begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix} + \underbrace{K\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{}$$

To be cancelled since it is multiple of ξ_1

- Hence, 2^{nd} solution is $x_2(t) = \xi_2 e^{rt} + \xi_1 t e^{rt}$

$$x_2(t) = \begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix} e^{-3t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^{-3t}$$

- Thus, the gen. sol. becomes:

$$x(t) = c_1 x_1(t) + c_2 x_2(t)$$
$$= c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t} + c_2 \left[\begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix} e^{-3t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^{-3t} \right]$$

To find c_1 and c_2 $\Rightarrow$ we use IC's:

$$X(0) = c_1\begin{pmatrix}1\\1\end{pmatrix} + c_2\begin{pmatrix}\frac{1}{4}\\0\end{pmatrix} = \begin{pmatrix}3\\2\end{pmatrix}$$

- origin is called improper node in repeated roots $r_1 = r_2 = r$
- If $r<0$, then origin is Asy. stable Eq. point.
- If $r>0$, then origin is unstable Eq. point.

$$C_1 + \frac{1}{4}C_2 = 3$$
$$C_1 + 0\,C_2 = 2 \Rightarrow \boxed{C_1 = 2} \Rightarrow \boxed{C_2 = 4}$$

Hence, $X(t) = \begin{pmatrix}2\\2\end{pmatrix}e^{-3t} + \begin{pmatrix}1\\0\end{pmatrix}e^{-3t} + \begin{pmatrix}4\\4\end{pmatrix}t\,e^{-3t}$

Exp Solve $f'(t) = f(t) + g(t)$, $f(0) = 1$
$g'(t) = 2f(t) - g(t)$, $g(0) = -1$

$x_1(t) = f(t)$
$x_2(t) = g(t)$

S1 Find eigenvalues using $*^1 \Rightarrow |A - rI| = 0$

$$\begin{vmatrix}1-r & 1\\ 2 & -1-r\end{vmatrix} = 0 \Rightarrow (1-r)(-1-r) - 2 = 0$$
$$-1 - r + r + r^2 - 2 = 0$$
$$r^2 - 3 = 0$$
$$(r - \sqrt{3})(r + \sqrt{3}) = 0$$
$$r_1 = \sqrt{3}\ ,\ r_2 = -\sqrt{3}$$ real different eigenvalues

- To find $\xi_1 = \begin{pmatrix}y_1\\y_2\end{pmatrix}$ for the corresponding $r_1 = \sqrt{3}$, we solve $*^2$:

$$(A - r_1 I)\xi_1 = \begin{pmatrix}0\\0\end{pmatrix} \Rightarrow \begin{pmatrix}1-\sqrt{3} & 1\\ 2 & -1-\sqrt{3}\end{pmatrix}\begin{pmatrix}y_1\\y_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$(1-\sqrt{3})y_1 + y_2 = 0 \Rightarrow y_2 = (\sqrt{3}-1)\,y_1$$

Take $y_1 = 1 \Rightarrow y_2 = \sqrt{3} - 1 \Rightarrow \xi_1 = \begin{pmatrix}y_1\\y_2\end{pmatrix} = \begin{pmatrix}1\\\sqrt{3}-1\end{pmatrix}$

Hence, 1^{st} solution is $X_1(t) = \xi_1 e^{r_1 t} = \begin{pmatrix}1\\\sqrt{3}-1\end{pmatrix}e^{\sqrt{3}t}$

. To find $\xi_2 = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ for the corresponding $r_2 = -\sqrt{3}$, we solve $*^2$:

$$(A - r_2 I)\xi_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1+\sqrt{3} & 1 \\ 2 & -1+\sqrt{3} \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow (1+\sqrt{3})z_1 + z_2 = 0$$

$$z_2 = -(1+\sqrt{3})z_1$$

Take $z_1 = 1 \Rightarrow z_2 = -(1+\sqrt{3}) \Rightarrow \xi_2 = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1-\sqrt{3} \end{pmatrix}$

Hence, 2^{nd} solution is $x_2(t) = \xi_2 e^{r_2 t} = \begin{pmatrix} 1 \\ -1-\sqrt{3} \end{pmatrix} e^{-\sqrt{3}t}$

. Thus, gen. sol. is $x(t) = c_1 x_1(t) + c_2 x_2(t)$

$$= c_1 \begin{pmatrix} 1 \\ \sqrt{3}-1 \end{pmatrix} e^{\sqrt{3}t} + c_2 \begin{pmatrix} 1 \\ -1-\sqrt{3} \end{pmatrix} e^{-\sqrt{3}t}$$

. To find c_1 and c_2 $\Rightarrow$ we use IC's :

$$x(0) = c_1 \begin{pmatrix} 1 \\ \sqrt{3}-1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1-\sqrt{3} \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$\boxed{c_1 + c_2 = 1}$ — ①

$(\sqrt{3}-1)c_1 - (1+\sqrt{3})c_2 = -1 \Rightarrow \sqrt{3}c_1 - \sqrt{3}c_2 - (c_1 + c_2) = -1$

$$\sqrt{3}c_1 - \sqrt{3}c_2 - 1 = -1$$

$$\sqrt{3}c_1 - \sqrt{3}c_2 = 0$$

$$\boxed{c_1 = c_2} \text{ — ②}$$

$$c_1 = c_2 = \frac{1}{2}$$

Hence, the gen. sol. becomes:

$$x(t) = \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{3}-1 \end{pmatrix} e^{\sqrt{3}t} + \frac{1}{2}\begin{pmatrix} 1 \\ -1-\sqrt{3} \end{pmatrix} e^{-\sqrt{3}t}$$

$$= \begin{pmatrix} \dfrac{e^{\sqrt{3}t} + e^{-\sqrt{3}t}}{2} \\ \left(\dfrac{e^{\sqrt{3}t} - e^{-\sqrt{3}t}}{2}\right)\sqrt{3} - \left(\dfrac{e^{\sqrt{3}t} + e^{-\sqrt{3}t}}{2}\right) \end{pmatrix} = \begin{pmatrix} \cosh\sqrt{3}t \\ \sqrt{3}\sinh\sqrt{3}t - \cosh\sqrt{3}t \end{pmatrix}$$

52 $f'(t) = f(t) + g(t)$, $f(0) = 1$
$g'(t) = 2f(t) - g(t)$, $g(0) = -1$

$L\{f'(t)\} = L\{f(t)\} + L\{g(t)\}$
$L\{g'(t)\} = 2L\{f(t)\} - L\{g(t)\}$

$sF(s) - f(0) = F(s) + G(s)$
$sG(s) - g(0) = 2F(s) - G(s)$

$F(s)(s-1) = 1 + G(s) \Rightarrow \boxed{G(s) = (s-1)F(s) - 1}$ — 1
$G(s)(s+1) = -1 + 2F(s)$ ... 2

substitute 1 in 2 $\Rightarrow \big((s-1)F(s) - 1\big)(s+1) = -1 + 2F(s)$

$(s-1)(s+1)F(s) - (s+1) + 1 - 2F(s) = 0$
$(s^2-1)F(s) - s \cancel{-1} \cancel{+1} - 2F(s) = 0$

$(s^2-3)F(s) = s \Rightarrow F(s) = \dfrac{s}{s^2-3}$

$f(t) = L^{-1}\left(\dfrac{s}{s^2-3}\right) = \cosh\sqrt{3}\,t$

$f'(t) = \sqrt{3}\sinh\sqrt{3}\,t \Rightarrow$ But $f'(t) = f(t) + g(t)$
$\sqrt{3}\sinh\sqrt{3}\,t = \cosh\sqrt{3}\,t + g(t)$

Hence, $g(t) = \sqrt{3}\sinh\sqrt{3}\,t - \cosh\sqrt{3}\,t$

Note that from 51 we have the gen. sol. is

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} f(t) \\ g(t) \end{pmatrix} = \begin{pmatrix} \cosh\sqrt{3}\,t \\ \sqrt{3}\sinh\sqrt{3}\,t - \cosh\sqrt{3}\,t \end{pmatrix}$$