Chapter 3 : Fourier Series

3.1 : Trigonometric representation of Periodic Signal

$x(t)$ is periodic with fundamental period T_0

$$f_0 = \frac{1}{T_0} \quad , \quad w_0 = 2\pi f_0 = \frac{2\pi}{T_0}$$

Then $x(t)$ can be resolved into infinite sum of sine and cosine terms

$$x(t) = a_0 + a_1 \cos(w_0 t) + a_2 \cos(2w_0 t) + \cdots + b_1 \sin(w_0 t) + b_2 \sin(2w_0 t) + \cdots$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos(n w_0 t) + \sum_{b=1}^{\infty} b_n \sin(n w_0 t)$$

a_0 : is the average of dc value of the signal

a_n and b_n : are called Trigonometric fourier series coefficients

$$a_0 = \frac{1}{T_0} \int_{T_0} x(t)\, dt$$

$$a_n = \frac{2}{T_0} \int_{T_0} x(t) \cos(n w_0 t)\, dt$$

$$b_n = \frac{2}{T_0} \int_{T_0} x(t) \sin(n w_0 t)\, dt$$

* usefull integrations

① $\int_T \sin(m\omega_0 t)\, dt = 0$ for any integer m

$= \frac{-1}{m\omega_0}\left[\cos(m\omega_0 t)\right]_0^{T_0} = \frac{-1}{m\omega_0}\left[\cos(m\omega_0 t - \cos(0)\right]$

$= \frac{-1}{m\omega_0}\left[\cos(2\pi m) - \cos(0)\right] = \frac{-1}{m\omega_0}[1-1] = 0$

② $\int_T \cos(n\omega_0 t) = 0$ for non-zero integer n

③ $\int_T \sin(n\omega_0 t)\cos(m\omega_0 t)\, dt = 0$ for any integer m, n

$\int_0^{T_0} \sin(n\omega_0 t)\cos(m\omega_0 t)\, dt = \frac{1}{2}\int_0^{T_0}\left[\sin(n+m)\omega_0 t + \sin(m-n)\omega_0 t\right] dt$

$= \frac{1}{2}\int_0^{T_0} \sin(n+m)\omega_0 t\, dt + \frac{1}{2}\int_0^{T_0} \sin(n-m)\omega_0 t\, dt = 0$

④ $\int_T \sin(n\omega_0 t)\sin(m\omega_0 t) = \begin{cases} 0, & m \neq |n| \\ \frac{T_0}{2}, & m = n \neq 0 \end{cases}$

$\int_0^{T_0} \sin(n\omega_0 t)\sin(m\omega_0 t)\, dt = \frac{1}{2}\int_0^{T_0}\left[\cos(n-m)\omega_0 t - \cos(n+m)\omega_0 t\right] dt$

$= \frac{1}{2}\int_0^{T_0}\cos(n-m)\omega_0 t\, dt - \frac{1}{2}\int_0^{T_0}\cos(n+m)\omega_0 t\, dt$

If $m \neq |n| \Rightarrow$

$= 0 - 0 = 0$

If $m = n$

$$= \frac{1}{2}\int_0^{T_0} \cos(0) - \frac{1}{2}\int_0^{T_0} \cancel{\cos(2m\omega_0 t)}^{\,0}$$

$$= \frac{T_0}{2}$$

(5) $$\int_T \cos(n\omega_0 t)\cos(m\omega_0 t)\,dt = \begin{cases} 0, & m \neq |n| \\ \frac{T_0}{2}, & m = \pm n \neq 0 \end{cases}$$

$$\int_0^{T_0} \cos(n\omega_0 t)\cos(m\omega_0 t)\,dt = \frac{1}{2}\int_0^{T_0} \left[\cos(n+m)\omega_0 t + \cos(n-m)\omega_0 t\right] dt$$

$$= \frac{1}{2}\int_0^{T_0} \cos(n+m)\omega_0 t\,dt + \frac{1}{2}\int_0^{T_0} \cos(n-m)\omega_0 t\,dt$$

If $m \neq |n|$

$$= \frac{1}{2}\int_0^{T} \cancel{\cos(n+m)\omega_0 t\,dt}^{\,0} + \frac{1}{2}\int_0^{T_0} \cancel{\cos(n-m)\omega_0 t\,dt}^{\,0}$$

If $m = \pm n$

$$= \frac{1}{2}\int_0^{T_0} \cos(0) + \frac{1}{2}\int_0^{T_0} \cancel{\cos(n+m)\omega_0 t\,dt}^{\,0}$$

$$= \frac{T_0}{2}$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)$$

① To evaluate a_0

If we take integral for both sides on full period

$$\int_{T_0} x(t)\,dt = \int_{T_0} a_0\,dt + \cancelto{0}{\int_{T_0} \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t)\,dt} + \cancelto{0}{\int_{T_0} \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)\,dt}$$

$$\int_{T_0} x(t)\,dt = \int_{T_0} a_0\,dt = a_0 \int_{T_0} 1\,dt = a_0 T_0$$

$$\therefore\ a_0 = \frac{1}{T_0} \int_{T_0} x(t)\,dt$$

② To evaluate a_n

$$\int_{T_0} x(t)\cos(m\omega_0 t)\,dt = \cancelto{0}{\int_{T_0} a_0 \cos(m\omega_0 t)\,dt} + \cancelto{\frac{T_0}{2}}{\int_{T_0} \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t)\cos(m\omega_0 t)\,dt}$$

$$+ \cancelto{0}{\int_{T_0} \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)\cos(m\omega_0 t)\,dt}$$

$$\int_{T_0} x(t)\cos(m\omega_0 t)\,dt = a_n \frac{T_0}{2}$$

$$\therefore\ a_n = \frac{2}{T_0} \int_{T_0} x(t)\cos(m\omega_0 t)\,dt \qquad n = m$$

If $n \neq m$ a_n not defined

③ To evaluate bn

$$\int_{T_0} x(t)\sin(m\omega_0 t)\,dt = \cancel{\int_{T_0} a_0 \sin(m\omega_0 t)\,dt}^{\,0} + \cancel{\int_{T_0}\sum_{n=1}^{\infty} a_n \cos(n\omega_0 t)\sin(m\omega_0 t)\,dt}^{\,0}$$

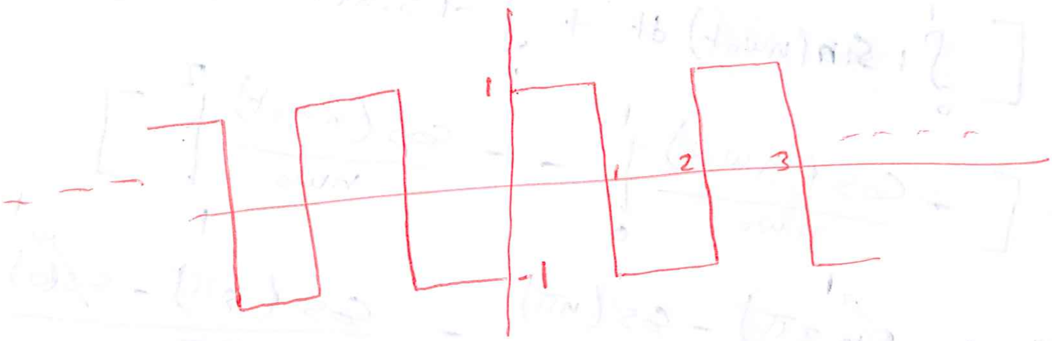
$$+ \int_{T_0}\sum_{n=1}^{\infty} \cancel{b_n \sin(n\omega_0 t)\sin(m\omega_0 t)\,dt}^{\,T_0/2}$$

$$\int_{T_0} x(t)\sin(m\omega_0 t)\,dt = b_n \frac{T_0}{2}$$

$$\therefore\ b_n = \frac{2}{T_0}\int_{T_0} x(t)\sin(n\omega_0 t)\,dt \qquad n = m$$

If $n \neq m$, bn is not defined

Examples- Find the trigonometric Fourier series Coefficients



$$T_0 = 2 \qquad \omega_0 = \frac{2\pi}{2} = \pi$$

$$a_0 = \frac{1}{T_0}\int_{T_0} x(t)\,dt$$

$$= \frac{1}{2}\left[\int_0^1 1\,dt + \int_1^2 -1\,dt\right] = \frac{1}{2}\left[t\Big|_0^1 + -t\Big|_1^2\right]$$

$$= \frac{1}{2}\left[1+(-2+1)\right] = \frac{1}{2}(0) = 0$$

$$a_n = \frac{2}{T_0}\int_T x(t)\cos(n\omega_0 t)\,dt$$

$$= \frac{2}{T_0}\left[\int_0^1 1\cos(n\omega_0 t)\,dt + \int_1^2 -1\cos(n\omega_0 t)\,dt\right]$$

$$= \frac{2}{T_0}\left[\frac{\sin(n\omega_0 t)}{(n\omega_0)}\Big|_0^1 - \frac{\sin(n\omega_0 t)}{(n\omega_0)}\Big|_1^2\right]$$

$$= \frac{2}{2}\left[\frac{\sin(n\pi)-\sin(0)}{(n\pi)} - \frac{\sin(n\pi 2)-\sin(n\pi)}{(n\pi)}\right]$$

$= 0$ n even or odd

$$b_n = \frac{2}{T_0}\int_{T_0} x(t)\sin(n\omega_0 t)\,dt$$

$$= \frac{2}{T_0}\left[\int_0^1 1\sin(n\omega_0 t)\,dt + \int_1^2 -1\sin(n\omega_0 t)\,dt\right]$$

$$= \frac{2}{2}\left[-\frac{\cos(n\omega_0 t)}{n\omega_0}\Big|_0^1 - -\frac{\cos(n\omega_0 t)}{n\omega_0}\Big|_1^2\right]$$

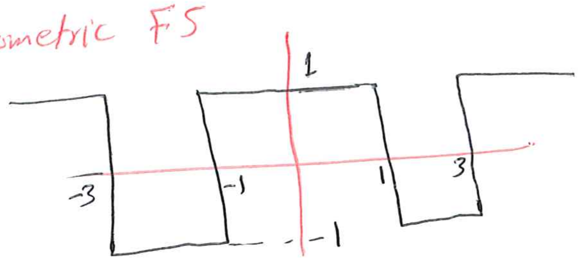
$$= \left[\frac{\overset{1}{\cos(n2\pi)}-\cos(n\pi)}{n\pi} - \frac{\cos(n\pi)-\overset{1}{\cos(0)}}{n\pi}\right]$$

$$= \frac{1}{n\pi}\left[1-\cos(n\pi) - \cos(n\pi)+1\right]$$

$$= \frac{2}{n\pi}\left(1-\cos(n\pi)\right) \qquad \cos(n\pi) = \begin{cases} 1 & n \text{ is even} \\ -1 & n \text{ is odd}\end{cases}$$

$$= \frac{4}{n\pi} \quad , n \text{ odd}$$

Example :- Find the trigonometric FS representation of x(t)



$T_0 = 4 \qquad w_0 = \frac{2\pi}{4} = \frac{\pi}{2}$

x(t) is symmetrical around the x axis so $a_0 = 0$

x(t) is even so $b_n = 0$

$$\therefore\ x(t) = \sum_{n=1}^{\infty} a_n \cos n w_0 t$$

$$a_n = \frac{2}{T_0} \int_{T_0} x(t) \cos n w_0 t \, dt$$

$$= \frac{2}{4} \left[\int_{-1}^{1} (1) \cos n \frac{\pi}{2} t \, dt + \int_{1}^{3} (-1) \cos n \frac{\pi}{2} t \, dt \right]$$

$$= \frac{1}{2} \left[\frac{2}{n\pi} \sin \frac{n\pi}{2} t \Big|_{-1}^{1} - \frac{2}{n\pi} \sin \frac{n\pi}{2} t \Big|_{1}^{3} \right]$$

$$= \frac{1}{n\pi} \left[\sin \frac{n\pi}{2} + \sin \frac{n\pi}{2} - \sin \frac{3n\pi}{2} + \sin \frac{n\pi}{2} \right]$$

$$= \frac{1}{n\pi} \left[3 \sin \frac{n\pi}{2} - \sin \frac{3n\pi}{2} \right] \qquad \sin \frac{3n\pi}{2} = -\sin \frac{n\pi}{2}$$

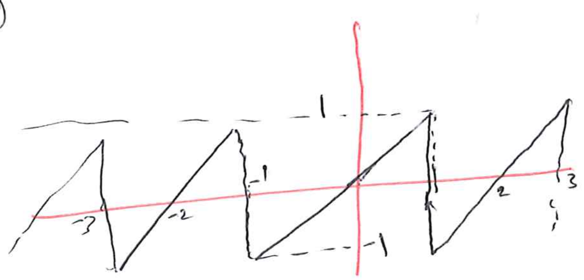
$$= \frac{4}{n\pi} \sin \frac{n\pi}{2} \qquad a_n = \begin{cases} 0, & \text{for } n \text{ even} \\ 1, & \text{for } n = 1, 5, 9, 13 \\ -1, & \text{for } n = 3, 7, 11 \ldots \end{cases} \Bigg\} n \text{, odd}$$

$$x(t) = \sum_{n=1}^{\infty} \frac{4}{n\pi} \sin \frac{n\pi}{2} \cos \frac{n\pi}{2} t \quad , \ \underline{n \text{ odd}}$$

for example $\quad a_5 = 1 \qquad a_{11} = -1$

Suggested problems :-

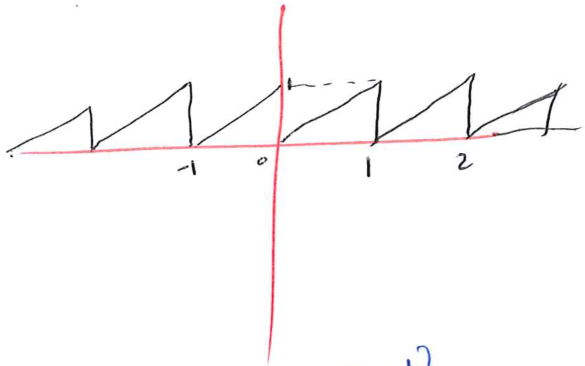
(1)



$T_0 = 2 \qquad w_0 = \pi \qquad a_0 = 0 \qquad a_n = 0$

$b_n = \frac{-2}{n\pi} \cos n\pi$

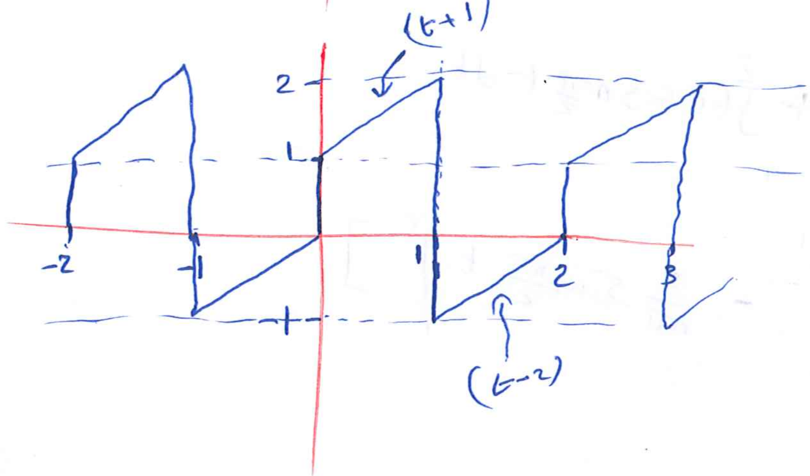
(2)



$T_0 = 1 \qquad a_0 = \frac{1}{2} \qquad a_n = 0$ ← integration by parts

$b_n = \frac{-1}{n\pi}$ ← integration by parts

(3)



$T_0 = 2 \qquad a_0 = \frac{1}{T_0} = \frac{1}{2}$

$a_n = 0 \qquad b_n = \begin{cases} \frac{4}{n\pi}, & n \text{ odd} \\ \frac{-2}{n\pi}, & n \text{ even} \end{cases}$

* The Complex Exponential Fourier Series

The trigonometric fourier series representation can be modified using Euler identity

$$\sin(nw_ot) = \frac{e^{jnw_ot} - e^{-jnw_ot}}{j2}$$

$$\cos(nw_ot) = \frac{e^{jnw_ot} + e^{-jnw_ot}}{2}$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nw_ot) + \sum_{n=1}^{\infty} b_n \sin(nw_ot)$$

$$= a_0 + \sum_{n=1}^{\infty} \frac{a_n}{2}\left[e^{jnw_ot} + e^{-jnw_ot}\right] + \sum_{n=1}^{\infty} \frac{b_n}{j2}\left[e^{jnw_ot} - e^{-jnw_ot}\right]$$

$$= a_0 + \sum_{n=1}^{\infty}\left[\frac{a_n - jb_n}{2}\right] e^{jnw_ot} + \underbrace{\sum_{n=1}^{\infty}\left[\frac{a_n + jb_n}{2}\right] e^{-jnw_ot}}_{\text{replacing } n \text{ by } -n}$$

$$= a_0 + \sum_{n=1}^{\infty}\left[\frac{a_n - jb_n}{2}\right] e^{jnw_ot} + \sum_{n=-\infty}^{-1}\left[\frac{a_{-n} + jb_{-n}}{2}\right] e^{jnw_ot}$$

$$= X_0 + \sum_{n=1}^{\infty} X_n e^{jnw_ot} + \sum_{n=-\infty}^{-1} X_{-n} e^{jnw_ot}$$

$$= \sum_{n=-\infty}^{\infty} X_n e^{jnw_ot}$$

where

$$X_n = \begin{cases} \dfrac{a_n - jb_n}{2}, & n > 0 \\ \dfrac{a_{-n} + jb_{-n}}{2}, & n < 0 \end{cases}$$

$$a_n = 2\,\mathrm{Re}\{X_n\}$$

$$b_n = -2\,\mathrm{Im}\{X_n\}$$

$X_n = X_{-n}^{*}$ when $|X_n| = |X_{-n}|$ and $\theta_n = -\theta_{-n}$

$$X_n = |X_n|\,e^{j\theta n}$$

$$X_n = \frac{1}{2}(a_n - jb_n), \quad n \geq 0$$

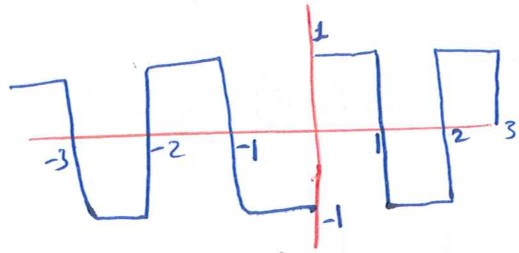
$$= \frac{1}{2}\left[\frac{2}{T_0}\int_{T_0} x(t)\cos(n w_0 t)\,dt - j\,\frac{2}{T_0}\int_{T_0} x(t)\sin n w_0 t\,dt\right]$$

$$= \frac{1}{T_0}\left[\int_{T_0} x(t)\left[\cos(n w_0 t) - j\sin(n w_0 t)\right]dt\right]$$

$$= \frac{1}{T_0}\int_{T_0} x(t)\,e^{-jnw_0t}\,dt \qquad n = -\infty, \dots, \infty$$

Example:-

Find the Complex Fourier Series coefficients



$T_0 = 2 \quad \omega_0 = \pi \quad a_0 = 0 \quad a_n = 0$

$b_n = \frac{4}{n\pi}$, n odd

$$X_n = \frac{a_n - j b_n}{2}$$

$$= \frac{0 - j\frac{4}{n\pi}}{2} = \frac{-j2}{n\pi}$$

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}$$

$$X_n = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jn\omega_0 t}\, dt$$

$$= \frac{1}{2}\left[\int_{-1}^{0} (-1)\, e^{-jn\omega_0 t}\, dt + \int_{0}^{1} 1\, e^{-jn\omega_0 t}\, dt\right]$$

$$= \frac{1}{2}\,\frac{1}{-jn\pi}(-1)\, e^{-jn\pi t}\Big|_{-1}^{0} + \frac{1}{2}\,\frac{1}{-jn\pi}\left[e^{-jn\pi t}\right]\Big|_{0}^{1}$$

$$= \frac{1}{2jn\pi}\left[1 - e^{jn\pi}\right] - \frac{1}{2jn\pi}\left[e^{-jn\pi} - 1\right]$$

$$= \frac{1}{2jn\pi}\left[1 - e^{jn\pi} - e^{-jn\pi} + 1\right]$$

$$e^{jn\pi} = \cos(n\pi) + j\sin(n\pi)$$

$$e^{-jn\pi} = \cos(n\pi) - j\sin(n\pi)$$

$$\therefore X_n = \frac{1}{2jn\pi}\left[2 - 2\cos n\pi\right]$$

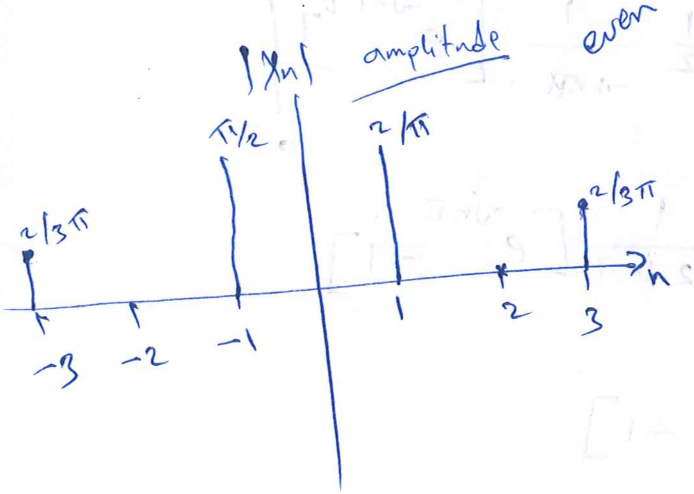
$$= \frac{2}{jn\pi} \Big|_{n \text{ odd}}$$

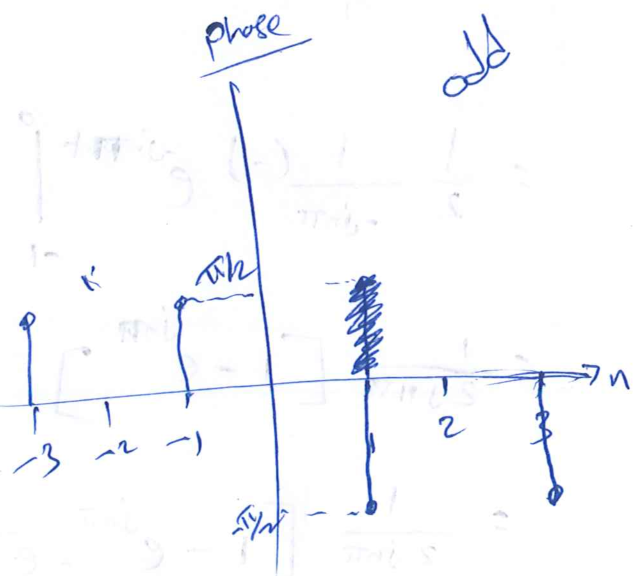
$$x(t) = \sum_{n=-\infty}^{\infty} \frac{2}{jn\pi} e^{jn\pi t}$$

$$X_n = \frac{2}{jn\pi} = \frac{-2j}{n\pi}$$

$$|X_n| = \frac{2}{|n|\pi}$$

$$\Theta_n = \begin{cases} -90 & , n > 0 \\ 90 & , n < 0 \end{cases}$$





If $X_n = \frac{2}{jn\pi}$

$\therefore a_n = 2 \, Re\{X_n\}$

$= 2 * (0) = 0$

$a_0 = 0$

$b_n = 2 \, Im\{X_n\}$

$= 2\left(\frac{2}{jn\pi}\right) = \frac{4}{jn\pi}$

* Compact Fourier series

To write the expression in terms of cosine only (single sided spectrum)

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}$$

$$= X_0 + \underbrace{\sum_{n=-\infty}^{-1} X_n e^{jn\omega_0 t}}_{\text{replace } n \text{ by } -n} + \sum_{n=1}^{\infty} X_n e^{jn\omega_0 t}$$

$$= X_0 + \sum_{n=1}^{\infty} X_{-n} e^{-jn\omega_0 t} + \sum_{n=1}^{\infty} X_n e^{jn\omega_0 t}$$

$$x(t) = X_0 + \sum_{n=1}^{\infty} (X_{-n} e^{-jnw_0t} + X_n e^{jnw_0t})$$

$$X_n = |X_n| e^{j\theta_n}$$

$$\therefore X_{-n} e^{-jnw_0t} + X_n e^{jnw_0t} = |X_n| e^{-j\theta_n} e^{-jnw_0t} + |X_n| e^{j\theta_n} e^{jnw_0t}$$

$$= |X_n| (e^{j(nw_0t + \theta_n)} + e^{-j(nw_0t + \theta_n)}$$

$$= 2 |X_n| \cos(nw_0t + \theta_n)$$

$$= D_n \cos(nw_0t + \theta_n)$$

where $D_n = 2|X_n|$, called compact Trigonometric Fourier series coefficient

$$x(t) = X_0 + \sum_{n=1}^{\infty} D_n \cos(nw_0t + \theta)$$

$$D_n = 2|X_n| \quad n = 1, 2, \dots \infty$$

$$X_n = \frac{1}{T_0} \int_{T_0} x(t) e^{-jnw_0t} dt$$

$$X_0 = \frac{1}{T_0} \int_{T_0} x(t) dt$$

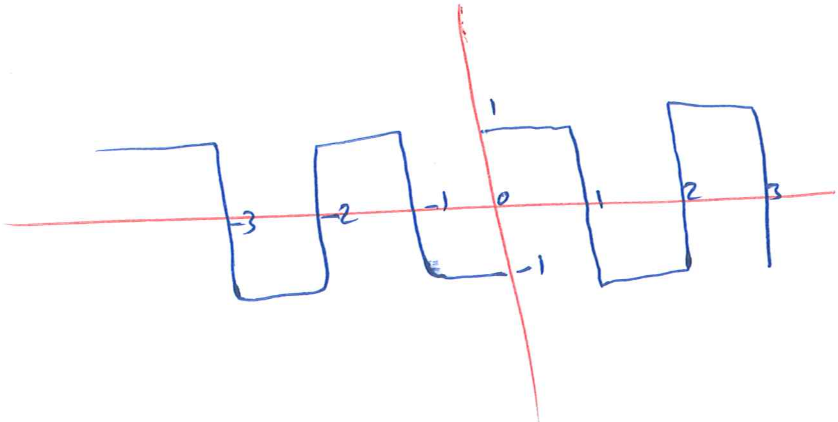
also $X_n = \dfrac{a_n - j b_n}{2} = |X_n| e^{j\theta_n}$

$$X_n = \frac{1}{2} \sqrt{a_n^2 + b_n^2}$$

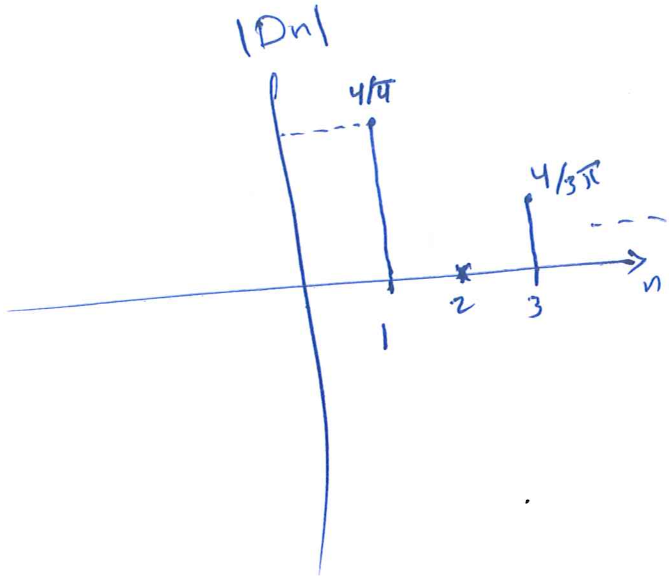
$$\theta_n = \tan^{-1}\left(\frac{-b_n}{a_n}\right)$$

Example

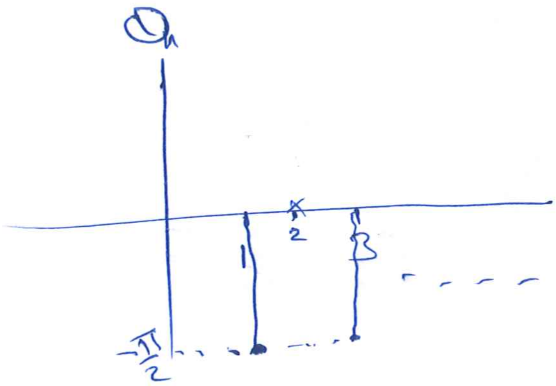
$$X_n = \frac{2}{jn\pi}$$



$$|X_n| = \frac{2}{|n|\pi} \Big|_{n \text{ odd}}$$



$$D_n = 2\,|X_n| = \frac{4}{|n|\pi}$$



* Parseval's Theorem

$$P_{ave} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt \quad \text{in general}$$

$$P_{ave} = \frac{1}{T_0} \int_{T_0} |x(t)|^2 \, dt \quad \text{for a periodic signal}$$

now we can exeress P_{ave} in terms of Fourier coeffecients of $x(t)$:-

$$P_{ave} = \frac{1}{T_0} \int_{T_0} x(t) \, x^*(t) \, dt$$

$$x(t) = \sum_{-\infty}^{\infty} X_n e^{jn\omega_0 t} \Rightarrow x^*(t) = \sum_{-\infty}^{\infty} X_n^* e^{-jn\omega_0 t}$$

$$P_{ave} = \frac{1}{T_0} \int_{T_0} x(t) \sum_{-\infty}^{\infty} X_n^* e^{-jn\omega_0 t} \, dt$$

$$P_{ave} = \sum_{-\infty}^{\infty} X_n^* \left(\frac{1}{T_0} \int_{T_0} x(t) e^{-jn\omega_0 t} \, dt \right)$$

$$= \sum_{n=-\infty}^{\infty} X_n^* X_n$$

$$= \sum_{n=-\infty}^{\infty} |X_n|^2$$

$$= X_0^2 + |X_1|^2 + |X_2|^2 + \cdots + |X_{-1}|^2 + |X_{-2}|^2 + \cdots$$

$$= X_0^2 + 2\sum_{n=1}^{\infty} |X_n|^2$$

$\therefore$ P_{ave} for a periodic signal $x(t)$:-

① in terms of time domain

$$P_{ave} = \frac{1}{T_0} \int_{T_0} |x(t)|^2 \, dt$$

② in frequency domain

$$P_{ave} = \sum_{n=\infty}^{\infty} |X_n|^2$$

$$= |X_0|^2 + 2\sum_{n=1}^{\infty} |X_n|^2 \quad \text{for real signal}$$

Example :- Find P_{ave} for $x(t) = \underset{=}{\overset{A}{4}} \sin(50\pi t)$

$\omega_0 = 50\pi \quad f_0 = 25 \quad T_0 = \frac{1}{25}$

Solution:- in time domain

$$P_{ave} = \frac{1}{T_0} \int_{T_0} |x(t)|^2 \, dt$$

$$= 25 \int_0^{T_0} \left(4 \sin(50\pi t)\right)^2 dt$$

$$= 25 \int_0^{T_0} 16 \sin^2(50\pi t) \, dt$$

$$= 25 * 16 \int_0^{T_0} \left[\frac{1}{2} - \frac{1}{2}\cos(100\pi t)\right] dt$$

$$= 25 * 8 \left[t - \frac{1}{100\pi}\sin(100\pi t)\right]_0^{T_0}$$

$$= 25 * 8 \left[T_0 - \frac{1}{100\pi}\sin(100\pi T_0)\right] =$$

$$= 25 * 8 \left[\frac{1}{25} - \frac{1}{100\pi}\sin(4\pi)\right] \quad 85 \quad = 8 - 0 = 8\,W \quad = \frac{A^2}{2} = \frac{16}{2}$$

To find P_{ave} using Parseval's theorem

$$P_{ave} = |X_0|^2 + 2\sum_{n=1}^{\infty} |X_n|^2$$

$$X_n = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jnw_0t}\, dt$$

$$x(t) = 4 \sin 50\pi t$$

$$= 4 \sin wt \qquad w = 50\pi$$

$$= 4\left[\frac{e^{jwt} - e^{-jwt}}{2j}\right]$$

$$= -j2\, e^{jwt} + j2\, e^{-jwt}$$

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jnw_0t}$$

$$= X_0 + X_1 e^{jw_0t} + X_2 e^{j2w_0t} + X_3 e^{j3w_0t} + \cdots$$

$$+ X_{-1} e^{-jw_0t} + X_{-2} e^{-j2w_0t} + X_{-3} e^{-j3w_0t} + \cdots$$

It is clear that

$X_0 = 0 \qquad X_1 = -j2 \qquad X_2 = 0 \qquad X_3 = 0, \ldots$

$X_{-1} = j2 \qquad X_{-2} = 0 \qquad X_{-3} = 0 \ldots$

$X_1 = -j2 = 2\angle{-90}$

$X_{-1} = j2 = 2\angle{90}$

$\longrightarrow |X_n| = |X^*_{-n}| = 2$

$\theta_{xn} = -\theta^*_{xn}$

$|X_1| = 2$

$P_{ave} = |X_0|^2 + 2\sum_{n=1}^{\infty} |X_n|^2$

$= 0 + 2(2)^2 = 8\ W$

Example:- $X(t) = 10\cos(10\pi t + \pi/7) + 4\sin(30\pi t + \pi/8)$

Find P_{ave} = ??

$= 10\,(e^{j(10\pi t + \pi/7)} - e^{-j10\pi t}$

$X_1 = -j2 = 2\angle{-90}$ $\Longrightarrow$ $|X_n| = |X_{-n}| = 2$

$X_{-1} = j2 = 2\angle{90}$

$\theta_{x_n} = -\theta_{x_{-n}}$

$|X_1| = 2$

$$P_{ave} = |X_0|^2 + 2\sum_{n=1}^{\infty} |X_n|^2$$

$$= 0 + 2(2)^2 = 8W$$

Examples:- $x(t) = 10\cos(10\pi t + \pi/7) + 4\sin(30\pi t + \pi/8)$

find $P_{ave} = ??$

$$x(t) = 10\cos(10\pi t + \pi/7) + 4\sin(30\pi t + \pi/8)$$

$w_0 = 10\pi$

$$x(t) = 10\left(\frac{e^{j(10\pi t + \pi/7)} + e^{-j(10\pi t + \pi/7)}}{2}\right) + 4\left(\frac{e^{j(30\pi t + \pi/8)} - e^{-j(30\pi t + \pi/8)}}{j2}\right)$$

$$= 5e^{j(10\pi t + \pi/7)} + 5e^{-j(10\pi t + \pi/7)} - j2e^{j(30\pi t + \pi/8)} + j2e^{-j(30\pi t + \pi/8)}$$

Using $w_0 = 10\pi$ we can rewrite $x(t)$ as

$$= \underset{\uparrow\,X_1}{5}e^{j(10\pi t + \pi/7)} + \underset{\uparrow\,X_{-1}}{5}e^{-j(10\pi t + \pi/7)} \underset{\uparrow\,X_3}{-j2}e^{j(3(10\pi)t + \pi/8)} + \underset{\uparrow\,X_{-3}}{j2}e^{-j(3(10\pi)t + \pi/8)}$$

$|X_1| = 5$ $\quad$ $|X_3| = 2$

$|X_{-1}| = 5$ $\quad$ $|X_{-3}| = 2$

$P_{ave} = |X_0|^2 + 2 \sum_{n=1}^{\infty} |X_n|^2$

$= 2(5^2 + 2^2) = 2(29) = 58\ W$

Example :- $x(t) = \sum_{n=-\infty}^{\infty} \frac{1}{1+j\pi n} e^{j(3\pi n t/2)}$

a) Determine the numerical Value of T_0

b) What is the average Value of $x(t)$ over the interval $(0, T_0)$

c) Determine the amplitude of the third-harmonic component

d) Determine the phase of the third-harmonic component.

e) Write down an expression for the third harmonic term in the fourier series.

a) $x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}$

$\therefore\ \omega_0 = \frac{3\pi}{2} = \frac{2\pi}{T_0}$

$\Rightarrow T_0 = \frac{4\pi}{3\pi} = \frac{4}{3}$

b) $X_n = \frac{1}{1+j\pi n}$ $\quad$ $X_0 = \frac{1}{1+0} = 1$

c) $X_3 = \frac{1}{1+j\pi(3)}$ $\therefore$ $|X_3| = \frac{1}{\sqrt{(1)^2+(3\pi)^2}}$

d) $\theta_{X_3} = -\tan^{-1}\left(\frac{3\pi}{1}\right) = -\tan^{-1}(3\pi)$

e) in the compact FS

$$x(t) = X_0 + \sum_{n=1}^{\infty} D_n \cos(n\omega_0 t + \theta)$$

$D_n = 2|X_n| = 2|X_3| =$

$$x(t) = \cdots + 2|X_3| \cos(3\omega_0 t + \theta_{X_3}) + \cdots$$

Example 8 - Consider the following Signal

$$x(t) = (2+j2)e^{-j30\pi t} - j3e^{-j20\pi t} + 5 + j3e^{j20\pi t} + (2-j2)e^{j30\pi t}$$

a) what is the average value of the signal

$$x(t) = 2e^{-j30\pi t} + j2e^{-j30\pi t} - j3e^{-j20\pi t} + \underbrace{5}_{X_0} + j3e^{j20\pi t} + 2e^{j20\pi t} - j2e^{j30\pi t}$$

$\omega_1 = 30\pi$
$\omega_2 = 20\pi$ $\Rightarrow$ $\omega_0 = 10\pi$

$$\therefore x(t) = \underbrace{2e^{-j3(10\pi)t} + j2e^{-j3(10\pi)t}}_{X_{-3}} \underbrace{-j3}_{X_{-2}} e^{-j2(10\pi)t} + \underbrace{5}_{X_0} + \underbrace{j3}_{X_2} e^{j2(10\pi)t} + \underbrace{2e^{j3(10\pi)t} - j2e^{j3(10\pi)t}}_{X_3}$$

$X_0 = 5$

2- Determine the expression of complex coefficient FS

$x(t) = \cancel{8/7+j3}$

$$\underbrace{(2+j2)}_{X_{-3}} e^{-j3(10\pi)t} - \underbrace{j3}_{X_{-2}} e^{-j2(10\pi)t} + \underbrace{5}_{X_0} + \underbrace{j3}_{X_2} e^{j2(10\pi)t} + \underbrace{(2-j2)}_{X_3} e^{j3(10\pi)t}$$

3- Justify that x(t) is a real signal

$$x(t) = 5 + 2e^{j3(10\pi t)} + 2e^{-j3(10\pi t)} + j2e^{-j3(10\pi t)} - j2e^{j3(10\pi t)} - j3e^{-j2(10\pi t)} + j3e^{j2(10\pi t)}$$

2- Determine the expression of complex coefficient FS

$$x(t) = \underbrace{(2+j2)}_{X_{-3}} e^{-j3(10\pi t)} \underbrace{-j3}_{X_{-2}} e^{-j2(10\pi t)} + \underbrace{5}_{X_0} + \underbrace{j3}_{X_2} e^{j2(10\pi t)} + \underbrace{(2-j2)}_{X_3} e^{j3(10\pi t)}$$

3- Justify that $x(t)$ is a real signal and write the correspondi[ng] compact FS representation

$$D_n = 2|X_n| =$$

$$D_2 = 2|X_2| = 2(3) = 6 \qquad \theta_2 = 90^\circ = \pi/2$$

$$D_3 = 2|X_3| = 2\left(\sqrt{2^2+2^2}\right) = 2\sqrt{8} \qquad \theta_3 = \tan^{-1} -1 = -45 = -\pi/4$$

$$x(t) = X_0 + \sum_{n=1}^{\infty} D_n \cos(n\omega_0 t + \theta_n)$$

$$= 5 + 6\cos\left(2(10\pi)t + \pi/2\right) + 2\sqrt{8}\cos\left(3(10\pi t) - \pi/4\right)$$

4) plot the two-sided amplitude and phase [s]pectra of the signal