5.4 Euler Equations

Cauchy-Euler Equation A linear differential equation of the form

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 x \frac{dy}{dx} + a_0 y = g(x),$$

where the coefficients $a_n, a_{n-1}, \ldots, a_0$ are constants, is known as a **Cauchy-Euler equation.** The differential equation is named in honor of two of the most prolifi mathematicians of all time. **Augustin-Louis Cauchy** (French, 1789–1857) and **Leonhard Euler** (Swiss, 1707–1783). The observable characteristic of this type of equation is that the degree $k = n, n-1, \ldots, 1, 0$ of the monomial coefficients x^k matches the order k of differentiation $d^k y/dx^k$:

same same

$$a_n \underset{\uparrow}{x^n} \frac{d^{\overset{\downarrow}{n}} y}{dx^n} + a_{n-1} \underset{\uparrow}{x^{n-1}} \frac{d^{\overset{\downarrow}{n-1}} y}{dx^{n-1}} + \cdots .$$

Method of Solution We try a solution of the form $y = x^m$, where m is to be determined. Analogous to what happened when we substituted e^{mx} into a linear equation with constant coefficients, when we substitute x^m, each term of a Cauchy-Euler equation becomes a polynomial in m times x^m, since

For example, when we substitute $y = x^m$, the second-order equation becomes

$$ax^2 \frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = am(m-1)x^m + bmx^m + cx^m = (am(m-1) + bm + c)x^m.$$

Thus $y = x^m$ is a solution of the differential equation whenever m is a solution of the **auxiliary equation**

$$am(m-1) + bm + c = 0 \qquad \text{or} \qquad am^2 + (b-a)m + c = 0. \tag{2}$$

There are three different cases to be considered, depending on whether the roots of this quadratic equation are real and distinct, real and equal, or complex. In the last case the roots appear as a conjugate pair.

Case I: Distinct Real Roots Let m_1 and m_2 denote the real roots of (1) such that $m_1 \neq m_2$. Then $y_1 = x^{m_1}$ and $y_2 = x^{m_2}$ form a fundamental set of solutions. Hence the general solution is

$$y = c_1 x^{m_1} + c_2 x^{m_2}. \tag{3}$$

EXAMPLE 1 Distinct Roots

Solve $x^2 \dfrac{d^2y}{dx^2} - 2x \dfrac{dy}{dx} - 4y = 0.$

SOLUTION Rather than just memorizing equation (2), it is preferable to assume $y = x^m$ as the solution a few times to understand the origin and the difference between this new form of the auxiliary equation and that obtained in Section 4.3. Differentiate twice,

$$\frac{dy}{dx} = mx^{m-1}, \qquad \frac{d^2y}{dx^2} = m(m-1)x^{m-2},$$

and substitute back into the differential equation:

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = x^2 \cdot m(m-1)x^{m-2} - 2x \cdot mx^{m-1} - 4x^m$$

$$= x^m(m(m-1) - 2m - 4) = x^m(m^2 - 3m - 4) = 0$$

if $m^2 - 3m - 4 = 0$. Now $(m+1)(m-4) = 0$ implies $m_1 = -1$, $m_2 = 4$, so $y = c_1x^{-1} + c_2x^4$.

Case II: Repeated Real Roots

If the roots of (2) are repeated (that is, $m_1 = m_2$), then we obtain only one solution—namely, $y = x^{m_1}$.

write the Cauchy-Euler equation in the standard form

$$\frac{d^2y}{dx^2} + \frac{b}{ax}\frac{dy}{dx} + \frac{c}{ax^2}y = 0$$

and make the identifications $P(x) = b/ax$ and $\int (b/ax)\,dx = (b/a)\ln x$. Thus

$$\begin{aligned} y_2 &= x^{m_1}\int \frac{e^{-(b/a)\ln x}}{x^{2m_1}}\,dx \\ &= x^{m_1}\int x^{-b/a}\cdot x^{-2m_1}\,dx && \leftarrow e^{-(b/a)\ln x} = e^{\ln x^{-b/a}} = x^{-b/a} \\ &= x^{m_1}\int x^{-b/a}\cdot x^{(b-a)/a}\,dx && \leftarrow -2m_1 = (b-a)/a \\ &= x^{m_1}\int \frac{dx}{x} = x^{m_1}\ln x. \end{aligned}$$

The general solution is then

$$y = c_1x^{m_1} + c_2x^{m_1}\ln x. \tag{4}$$

EXAMPLE 2 Repeated Roots

Solve $4x^2 \dfrac{d^2y}{dx^2} + 8x\dfrac{dy}{dx} + y = 0$.

SOLUTION The substitution $y = x^m$ yields

$$4x^2\frac{d^2y}{dx^2} + 8x\frac{dy}{dx} + y = x^m(4m(m-1) + 8m + 1) = x^m(4m^2 + 4m + 1) = 0$$

when $4m^2 + 4m + 1 = 0$ or $(2m + 1)^2 = 0$. Since $m_1 = -\frac{1}{2}$, it follows from (4) that the general solution is $y = c_1x^{-1/2} + c_2x^{-1/2}\ln x$.

Case III: Conjugate Complex Roots If the roots of (2) are the conjugate pair $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$, where α and $\beta > 0$ are real, then a solution is

$$y = C_1 x^{\alpha+i\beta} + C_2 x^{\alpha-i\beta}.$$

But when the roots of the auxiliary equation are complex, as in the case of equations with constant coefficients, we wish to write the solution in terms of real functions only. We note the identity

$$x^{i\beta} = (e^{\ln x})^{i\beta} = e^{i\beta \ln x},$$

which, by Euler's formula, is the same as

$$x^{i\beta} = \cos(\beta \ln x) + i \sin(\beta \ln x).$$

Similarly,
$$x^{-i\beta} = \cos(\beta \ln x) - i \sin(\beta \ln x).$$

we conclude that

$$y_1 = x^{\alpha} \cos(\beta \ln x) \quad \text{and} \quad y_2 = x^{\alpha} \sin(\beta \ln x)$$

constitute a fundamental set of real solutions of the differential equation. Hence the general solution is

$$y = x^{\alpha}[c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x)]. \tag{5}$$

EXAMPLE 3 An Initial-Value Problem

Solve $4x^2y'' + 17y = 0$, $y(1) = -1$, $y'(1) = -\frac{1}{2}$.

SOLUTION The y' term is missing in the given Cauchy-Euler equation; nevertheless, the substitution $y = x^m$ yields

$$4x^2y'' + 17y = x^m(4m(m-1) + 17) = x^m(4m^2 - 4m + 17) = 0$$

when $4m^2 - 4m + 17 = 0$. From the quadratic formula we find that the roots are $m_1 = \frac{1}{2} + 2i$ and $m_2 = \frac{1}{2} - 2i$. With the identifications $\alpha = \frac{1}{2}$ and $\beta = 2$ we see from (5) that the general solution of the differential equation is

$$y = x^{1/2}[c_1 \cos(2 \ln x) + c_2 \sin(2 \ln x)].$$

By applying the initial conditions $y(1) = -1, y'(1) = -\frac{1}{2}$ to the foregoing solution and using $\ln 1 = 0$, we then find, in turn, that $c_1 = -1$ and $c_2 = 0$. Hence the solution of the initial-value problem is $y = -x^{1/2} \cos(2 \ln x)$.