# Ch.2 Determinate of Matrix

By : Rawan Alfares

# Determinate Of Matrix

$A_{n\times n}$ : Is $A$ nonSingular?

$A_{n\times n}$ nonSingular $\Leftrightarrow A \cong I$ $\quad [A \to \to \to \cdots \to I]$ reduction

$\boxed{1}$ $A_{1\times1} = a_{11}$ , non Singular $\to a_{11} \neq 0$.

let determinate of $A$ = $\det(A) = a_{11}$

So $A_{1\times1}$ is nonSingular $\Longleftrightarrow \det(A) \neq 0$.

Ex. $A = [5]$ .

$\det(A) = 5$ , So $A$ is nonSingular.

$A^{-1} = [\frac{1}{5}]$

EX. $A[0]$ , is Singular , $\det(0) = 0$

$\boxed{2}$ $A_{2\times2} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ , $A \cong I \Longleftrightarrow a_{11}a_{22} - a_{21}a_{12} \neq 0$

$A$ is nonSingular iff $a_{11}a_{22} - a_{21}a_{12} \neq 0$

Def : $\det(\underset{2\times2}{A}) = a_{11}a_{22} - a_{12}a_{21}$ .

$$\underset{2\times2}{A^{-1}} = \frac{1}{\det(A)} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

EX. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ , $\det(A) = 4 - 6 = -2$ , $A$ is nonsingular.

$$A^{-1} = \frac{-1}{2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

EX. $\begin{bmatrix} 6 & 3 \\ 4 & 2 \end{bmatrix}$ , $\det(A) = 12 - 12 = 0$

so, $A$ is singular. (No inverse).

$\boxed{3}$ $A_{3\times3}$ , $A \cong I \Longleftrightarrow$

راح يطلع كتير طويل

وطبعًا $A_{4\times4}$ راح يطلع أطول

فبدنا نلاقي طريقة أسهل

لحساب ال det ...

**Def 8-** let $A = (a_{ij})_{n\times n}$, we define the Cofactor $A_{ij}$ of $a_{ij}$ as $A_{ij} = (-1)^{i+j} \det(M_{ij})$ where $M_{ij}$ is the $(n-1)\times(n-1)$ matrix obtained by deleting the $i$th row and $j$th column { $M_{ij}$ called minor $a_{ij}$ }.

* for each element at the matrix has an own Cofactor.

EX. $A_{3\times3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$A_{11} = (-1)^{1+1} \det(M_{11}) = (-1)^2 \det\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} = a_{22}a_{33} - a_{23}a_{32}$

delete the 1st row and 1st column.

$A_{12} = (-1)^3 \det\begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} = -a_{21}a_{33} + a_{23}a_{31}$

$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

∴ $A_{3\times3}$ : $\det(A) = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{1}$

→ every element in the first row multiply by it's Cofactor give you the determinate for $A_{3\times3}$

EX. $\begin{bmatrix} 1 & 2 & 3 \\ -2 & 0 & 1 \\ 1 & -1 & 4 \end{bmatrix}$, $\det(A) = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$

$= 1\times(-1)^2 \begin{vmatrix} 0 & 1 \\ -1 & 4 \end{vmatrix} + 2\times(-1)^3 \begin{vmatrix} -2 & 1 \\ 1 & 4 \end{vmatrix} + 3\times(-1)^4 \begin{vmatrix} -2 & 0 \\ 1 & -1 \end{vmatrix}$

$= 1 + -2(-8-1) + 3(2)$

$= 25 \neq 0$, so $A$ is nonsingular

find $A^{-1}$?

→ no answer.

→ det منحلل بس بنقدر أحدد إذا matrix is singular or not.

* وهون أخذنا الـ first row ولو أخذنا 2nd row أو 3rd row it gonna be the same.

* $a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23}$

* $a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33}$

* وبرضو لو أخذنا (3rd/2nd/1st) Column gonna be the same.

$a_{11}A_{11} + a_{21}A_{21} + a_{32}A_{32} \ldots$ and so on

Ingeneral :- $A_{n\times n} = (a_{ij})_{n\times n}$, We define

$\det(A) = a_{11}A_{11} + a_{12}A_{12} + \dots + a_{1n}A_{1n}$ (1st row).

$\det(A) = a_{i1}A_{i1} + a_{i2}A_{i2} + \dots + a_{in}A_{in}$ (ith row).

→ Cofactor expantion of $\det(A)$ in terms of ith row.

$\det(A) = a_{1j}A_{1j} + a_{2j}A_{2j} + \dots + a_{nj}A_{nj}$ (j-th Column).

→ cofactor expantion of $\det(A)$ in terms of jth column.

EX. $A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ -1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ -1 & 1 & 2 & 4 \end{bmatrix}$, find $\det(A)$

* نختار 3rd Column
عشان أغلبه أصفار
فأسهل علي أحسبه
بس لو اخترت اشي ثاني
راح يطلع نفس الجواب

$$\det(A) = \cancel{a_{13}A_{13}} + \cancel{a_{23}A_{23}} + \cancel{a_{33}A_{33}} + a_{43}A_{43}$$

$$= 2 \times (-1)^7 \begin{vmatrix} 1 & 2 & -1 \\ -1 & 1 & 2 \\ 0 & 1 & 3 \end{vmatrix}_{3\times 3}$$

$$= -2 \times \left[ 1 \times \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix} + 0 \right]$$

$$= -2 - 14 = -16 \neq 0$$, so Non Singular matrix.

## properties for determinate

$\boxed{1}$ if $A$ is $n\times n$ - matrix, then $\det(A) = \det(A^T)$

$\boxed{2}$ if $A$ is $n\times n$ triangular matrix, then $\det(A) = a_{11}a_{22}a_{33}\dots a_{nn}$
[the product of diagonal elements of $A$)

EX. $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & -5 \end{bmatrix}$, $\det(A) = 1\begin{vmatrix} 3 & 1 \\ 0 & -5 \end{vmatrix} = -15 \neq 0$

$= \underbrace{(1)(3)(-5)}_{\text{Product of diagonals.}} \neq 0$

 

Ex. $A = \begin{bmatrix} 2 & 1 & 3 & -5 \\ 0 & 2 & 4 & 2 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & 6 \end{bmatrix}$, $\det(A) = (2)(2)(-3)(6)$.

this property for both upper and lower triangular... "any triangular matrix".

$\boxed{3}$ if $A_{n\times n}$ has a row or column of zeros, then $\det(A) = 0$ → singular.

Ex. $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & 3 \\ 1 & 0 & -5 \end{bmatrix}$, $\det(A) = 0$.

$\boxed{4}$ if $A$ has two identical [rows or columns], then $\det(A) = 0$ → singular

Ex. $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -5 & 6 \\ 1 & 2 & 3 \end{bmatrix}$, $\det(A) = 0$

$A = \begin{bmatrix} -1 & 1 & -1 \\ 2 & 5 & 2 \\ 3 & -1 & 3 \end{bmatrix}$, $\det(A) = 0$

$\det(EA) = \det(E)\det(A)$

$\det(E) = \begin{cases} -1 , & E \text{ Type I} \\ \alpha , & E \text{ Type II} \\ 1 , & E \text{ Type III} \end{cases}$

$\boxed{1}$ Row operation I. ($R_i \leftrightarrow R_j$)

$A \xrightarrow{ERO's} EA$, E is from type I

∻ $\det(EA) = -\det(A)$

∻ $\det(E) = -1$.

∻ $\det(EA) = \det(E)\det(A) = -\det(A)$.

$\boxed{2}$ Row operation II ($\alpha R_i$ ).

$A \xrightarrow{ERO's} E_2 A$, E is from type 2.

∻ $\det(EA) = \det(E)\det(A) = \alpha \det(A)$

## 3) Row operation III :-

$\Rightarrow \det(EA) = \det(E)\det(A) = \det(A)$.

EX for type I : $A = \begin{bmatrix} 2 & 4 \\ 6 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 6 & 5 \\ 2 & 4 \end{bmatrix}$

find $|A| = 10 - 24 = -14$

$|B| = 24 - 10 = 14$.

EX. for type II.

$A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 4 \\ 4 & 5 \end{bmatrix}$

$|A| = -3$.

$|B| = -6 = 2|A|$

$= 2(-3)$.

EX. for type III.

$A = \begin{bmatrix} 1 & 4 \\ 5 & -5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 4 \\ 0 & -25 \end{bmatrix}$

$|A| = -25$

$|B| = -25 = |A|$.

ex. Find $\begin{vmatrix} 1 & 3 & 4 \\ 2 & -1 & 1 \\ 3 & -2 & 5 \end{vmatrix} \underset{-2R_1+R_2}{=} \begin{vmatrix} 1 & 3 & 4 \\ 0 & -7 & -7 \\ 3 & -2 & 5 \end{vmatrix} \underset{-3R_1+R_3}{=} \begin{vmatrix} 1 & 3 & 4 \\ 0 & -7 & -7 \\ 0 & -11 & -7 \end{vmatrix} = (1)(-1)^2 \begin{vmatrix} -7 & -7 \\ -11 & -7 \end{vmatrix}$

$= 49 - 77 = -28$

**Example 8-**

$$\begin{vmatrix} 1 & 2 & -1 & 1 \\ 2 & -1 & 1 & 0 \\ 1 & -1 & 2 & 1 \\ -1 & 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & -1 & 1 \\ 0 & -5 & 3 & -2 \\ 0 & -3 & 3 & 0 \\ 0 & 3 & -1 & 2 \end{vmatrix}$$

$$= - \begin{vmatrix} 1 & 2 & -1 & 1 \\ 0 & -3 & 3 & 0 \\ 0 & -5 & 3 & -2 \\ 0 & 3 & -1 & 2 \end{vmatrix} = - \begin{vmatrix} 1 & 2 & -1 & 1 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 2 & 2 \end{vmatrix}$$

$$\det(A) = (-)(1)(-3) \begin{vmatrix} -2 & -2 \\ 2 & 2 \end{vmatrix}$$
$$= 0$$

$$= - \begin{vmatrix} 1 & 2 & -1 & 1 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & 0 \end{vmatrix}$$

$$= (-)(1)(-3)(-2)(0)$$
$$= 0$$

or we have row of zeros.

**Example 8-**

$$\begin{vmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 0 & 5 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & -1 \\ 0 & 1 & 1/3 \\ 0 & 5 & 2 \end{vmatrix},$$

$$\frac{1}{3} \begin{vmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 0 & 5 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & -1 \\ 0 & 1 & 1/3 \\ 0 & 5 & 2 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 0 & 5 & 2 \end{vmatrix} = 3 \begin{vmatrix} 1 & 2 & -1 \\ 0 & 1 & 1/3 \\ 0 & 5 & 2 \end{vmatrix}$$

**NOTE 8-**

$\det(EA) = \alpha \det(A)$

after doing row operation = قبل عمل Row operation

**EX 8** $A_{4\times4}$, $\det(A) = -3$

find $\det(2A)$

$\det(2A) = 2^4 \det(A)$.

* if $A = (a_{ij})_{n\times n}$, then $\det(kA) = k^n \det(A)$.

---

**Remainder 8-**

$\det(EA) = \det(E).\det(A)$, $E$ is elementary

Q. what's the $\det(AB)$ ?

Theorem :- let $A, B$ be $n\times n$ - matricies, then $\det(AB) = \det(A)\det(B)$

Proof :- let $A, B$ be $n\times n$ - matricies.

Case 1 :- if $B$ is singular $\Rightarrow \det(B)=0$.

, Q.18 | sec 1.5 $\rightarrow$ if $A$ $n\times n$ matrix, $B_{n\times n}$ is Singular, then $C=AB$ is singular.

$\det(AB)=0$. , Now, $\det(A)\det(B)$
$= \det(A) . 0$
$= 0$

So, $\det(AB) = \det(A)\det(B) = 0$.

Case 2 :- if $B$ is nonSingular, then $B \cong I$

$\Rightarrow$ means that, there are elementary matricies $E_1, E_2 \dots E_k$ (product of elementry matricies).

$B = E_k \dots E_2 E_1 I = E_k \dots E_2 E_1$

substitute

$$
\begin{aligned}
\det(AB) &= \det(A E_k \dots E_2 E_1). \\
&= \det((A E_k \dots E_3 E_2)(E_1)). \\
&= \det((A E_k \dots E_3) \det(E_2) \det(E_1). \\
&\ \vdots \\
&= \det(A)\det(E_k) \dots \det(E_2)\det(E_1). \\
&= \det(A)\det(E_k \dots E_2 E_1). \\
\det(AB) &= \det(A)\det(B)
\end{aligned}
$$

Ex :- 6 / 2.2 let $A$ be non singular matrix, show that $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

$A A^{-1} = I$

$\det(A . A^{-1}) = \det(I)$

$\det(A) . \det(A^{-1}) = \det(I)$

$\det(A) . \det(A^{-1}) = 1$

$\det(A^{-1}) = \dfrac{1}{\det(A)}$ , $\det(A) \neq 0$, because its nonSingular.

Q. 7 | 2.2 , $A, B$ $3\times3$ - matrix, $\det(A) = 4$, $\det(B) = 5$.

1) $\det(AB) = \det(A).\det(B) = 4 \times 5 = 20$.

2) $\det(3A) = 3^3 \times \det(A) = 27 \times 4 = 108$

3) $\det(A^{-1}B) = \det(A^{-1}).\det(B) = \frac{1}{4} . 5 = \frac{5}{4}$

4) $\det(3A^T . B^{-1} . A^2) = 3^3 \times \det(A).\det(B^{-1}).\det(A).\det(A)$
$= 27 \times 64 \times \frac{1}{5} = \boxed{\phantom{x}}$

 

$\text{IV}$ if $A$ and $B$ have different size, then it's not defined.

4 | 2.2 Find all values of $\underline{c}$ that makes $A$ singular.

$\Downarrow$

$\det(A) = 0$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 9 & c \\ 1 & c & 3 \end{bmatrix}$$

$$\det(A) = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 9 & c \\ 1 & c & 3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 8 & c-1 \\ 0 & c-1 & 2 \end{vmatrix}$$

$$= 1 \begin{vmatrix} 8 & (c-1) \\ (c-1) & 2 \end{vmatrix}$$

$$= 16 - (c-1)^2$$

$$= 16 - c^2 + 2c - 1$$

$$0 = -c^2 + 2c - 15$$

$$(c+3)(c-5) = 0$$

$$c = -3, c = 5$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 9 & c \\ 1 & c & 3 \end{bmatrix}$$, is singular when $c = 5$ or $c = -3$

Ex. if $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 5$. Find, $\begin{vmatrix} 2a & 2b & 2c \\ d & e & f \\ g+a & h+b & i+c \end{vmatrix}$

المحددة الصف ∝

II

$$= (2) \begin{vmatrix} a & b & c \\ d & e & f \\ g+a & h+b & i+c \end{vmatrix} \underset{-R_1+R_3}{=} (2) \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = (2)(5) = 10.$$

III ← المحددة

الصف (1)

## Exampels :-

1. $|A+B| = |A| + |B|$ . F

2. $|A^n| = |A|^n$ T

$$\det(A^n) = \det(A.A.A \dots A)$$
$$= \det(A.A\dots).\det(A)$$
$$= \det(A.A)\dots \det(A)$$
$$\vdots$$
$$= \det(A).\det(A).\det(A)\dots\det(A)$$
$$= \det(A)^n = |A|^n$$

#

3. $|KA| = K^n |A|$ T

4. if $A$ is nonsingular, then $\det(A^{-1}) = \dfrac{1}{\det(A)}$ T

5. if $A^2 = A$, then $|A| = 0$ or $|A| = 1$ T

6. if $A^TA = I$ , then $|A| = \mp 1$ T

7. if $A_{n\times n}$ is skew symmetric and $n$ is odd, then $A$ must be singular. T

$A^T = -A$ singular $\det(A) = 0$

$\det(A^T) = \det(-A)$

$\det(A^T) = (-1)^n \det(A)$ , $n$ odd

$\det(A) + \det(A) = 0$

$2\det(A) = 0 \rightarrow \det(A) = 0$

$\rightarrow$ if $n$ is even

$\det(A) = \det(A)$

$0 = 0 \rightarrow$ undifined.

مهم الإثبات

8. if $A_{n\times n}$ is skew-symmetric and $n$ is even, then $A$ must be nonsingular F

9. let $A_{n\times n}$, $B_{n\times n}$, then $AB$ is nonsingular, iff $A$ and $B$ are both nonsing. T

10. if $A, B$ and $C$ are $3\times3$ matricies, $|A| = 9$, $|B| = 2$, $|C| = 3$, then $|4C^TBA^{-1}| = \dfrac{128}{3}$

$$4^3.\det(C).\det(B).\frac{1}{\det(A)} = \frac{64\times\cancel{3}\times2}{\cancel{9}\,3} = \frac{128}{3}$$

## 2.3 Adjoint

Def :- let $A = (a_{ij})_{n\times n}$ any matrix, we defined the adjoint of $A$ as,

$$adj(A) = \begin{bmatrix} A_{11} & A_{12} \cdots & A_{1n} \\ A_{21} & A_{22} \cdots & A_{2n} \\ \vdots & & \\ A_{n1} & \cdots & A_{nn} \end{bmatrix}^T \Longrightarrow$$

بنعوض مكان كل element ال cofactor تبعها. وباخذ transpose

* Remainder :-

$$\text{Cofactor } A_{ij} = (-1)^{i+j} |M_{ij}|$$

Ex. $A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 3 \\ -2 & 2 & 5 \end{bmatrix}$, Find $adj(A)$

| | | |
|---|---|---|
| $A_{11} = \begin{vmatrix} 1 & 3 \\ 2 & 5 \end{vmatrix} = -1$ | $A_{21} = -\begin{vmatrix} 0 & 2 \\ 2 & 5 \end{vmatrix} = -4$ | $A_{31} = \begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix} = -2$ |
| $A_{12} = -\begin{vmatrix} -1 & 3 \\ -2 & 5 \end{vmatrix} = -1$ | $A_{22} = +\begin{vmatrix} 1 & 2 \\ -2 & 5 \end{vmatrix} = 9$ | $A_{32} = -\begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix} = -5$ |
| $A_{13} = \begin{vmatrix} -1 & 1 \\ -2 & 2 \end{vmatrix} = 0$ | $A_{23} = -\begin{vmatrix} 1 & 0 \\ -2 & 2 \end{vmatrix} = -2$ | $A_{33} = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1$ |

$$Adj(A) = \begin{bmatrix} -1 & -1 & 0 \\ -4 & 9 & -2 \\ -2 & -5 & 1 \end{bmatrix}^T = \begin{bmatrix} -1 & -4 & -2 \\ -1 & 9 & -5 \\ 0 & -2 & 1 \end{bmatrix}$$

$$A.\,adj(A) = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 3 \\ -2 & 2 & 5 \end{bmatrix} \cdot \begin{bmatrix} -1 & 4 & -2 \\ -1 & 9 & -5 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$= det(A) = -1$

**Theorem :** if $A = (a_{ij})_{n\times n}$, then $A.\,adj(A) = det(A)\, I = adj(A).\,A$

* Case 1 :- if $A$ is nonsingular, ($A^{-1}$ exist, $det(A) \neq 0$)

$$\frac{1}{det(A)} A\,adj(A) = \frac{det(A)}{det(A)} = I.$$

$$\underbrace{A\left[\frac{1}{det(A)} adj(A)\right]}_{\text{inverse of } A} = I$$

$$A^{-1} = \frac{1}{det(A)} adj(A)$$

**Ex.** $A = \begin{bmatrix} 2 & 1 & 2 \\ 3 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

(1) Find adj(A).
(2) Find det(A).
(3) Find $A^{-1}$ if exist

(1) adj(A) $\Rightarrow A_{11} = \begin{vmatrix} 2 & 2 \\ 2 & 3 \end{vmatrix} = 2$, $A_{12} = -\begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix} = -7$, $A_{13} = +\begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix} = 4$

$A_{21} = \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = +1$, $A_{22} = \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} = 4$, $A_{23} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = -3$

$A_{31} = \begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} = -2$, $A_{32} = \begin{vmatrix} 2 & 2 \\ 3 & 2 \end{vmatrix} = +2$, $A_{33} = \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = 1$

$$\text{Adj}(A) = \begin{bmatrix} 2 & -7 & 4 \\ 1 & 4 & -3 \\ -2 & +2 & 1 \end{bmatrix}$$

(2) $\det(A) = \begin{vmatrix} \overset{+}{2} & \overset{-}{1} & \overset{+}{2} \\ 3 & 2 & 2 \\ 1 & 2 & 3 \end{vmatrix} = 2(2) - 1(7) + 2(4)$
$= 5$

(3) $A^{-1} = \dfrac{\text{adj}(A)}{\det(A)} = \dfrac{1}{5} \begin{bmatrix} 2 & -7 & 4 \\ 1 & 4 & -3 \\ -2 & +2 & 1 \end{bmatrix}$

* if $A_{n\times n}x = b$ is $n\times n$ system, $A$ is nonsingular $\Rightarrow Ax = b$ has a unique solution and solution $x = A^{-1}b$ or use GEM

## creamer's methods

↳ another method to find Solution for x.

* لكن الطريقة هاي ماتنربط بكل المصفوفات، خاصة إذا كانت Singular
لإنك تتقسم على المحددة و Singular فيها المحددة = صفر.

**theorem :-** if $A = (a_{ij})_{n\times n}$ is nonsingular, and $b \in \mathbb{R}^n$, then the unique solution of $Ax = b$ is given by $x_i = \dfrac{\det(A_i)}{\det(A)}$, $i = 1, \dots, n$

where $A_i$ is the matrix obtained from $A$ by replacing the $i^{th}$ column of $A$ by $b$.

**Ex.** Use Cramer's Rule to Solve

$x_1 + 2x_2 + x_3 = 5$
$2x_1 + 2x_2 + x_3 = 6$
$x_1 + 2x_2 + 3x_3 = 9$

$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $b = \begin{bmatrix} 5 \\ 6 \\ 9 \end{bmatrix}$



$\det(A) = 1(0) + -2(-1) + 3(-2) = -4 \neq 0$, So $A$ is nonSingular.

$x_1 = \frac{\det(A_1)}{\det(A)}$ بنعوض b مكان أول كولوم في A

$= \frac{\begin{vmatrix} 5 & 2 & 1 \\ 6 & 2 & 1 \\ 9 & 2 & 3 \end{vmatrix}}{-4} = 1(-6) - (-8) + 3(-2)$

$= \frac{-4}{-4} = 1$

$x_2 = \frac{\begin{vmatrix} 1 & 5 & 1 \\ 2 & 6 & 1 \\ 1 & 9 & 3 \end{vmatrix}}{-4} = \frac{1(12) - (4) + 3(-4)}{-4} = \frac{-4}{-4} = 1$

$x_3 = \frac{\begin{vmatrix} 1 & 2 & 5 \\ 2 & 2 & 6 \\ 1 & 2 & 9 \end{vmatrix}}{-4} = \frac{1(6) - 2(8) + (2)}{-4} = \frac{-8}{-4} = 2$

So Solution $X = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

---

**properties for A and adj(A)** "$A\,adj(A) = \det(A).I$"

① if $A$ is singular ($\det(A) = 0$), then $A.adj(A) = 0_{n\times n}$.

② if $A$ is nonsingular, then $\det(adj(A)) = \det(A)^{n-1}$, $n>1$.

✗ Proof :- $A$ is nonsingular, $\det(A) \neq 0$.

we know $A\,adj(A) = \det(A)\,I$

$\Rightarrow \det(A\,adj(A)) = \det(\det(A)\,I)$. $\det(A) \to k$, Constant.

$= \det(A)\det(adj(A)) = k^n \det(I)$

$\frac{\cancel{\det(A)}\det(adj(A))}{\cancel{\det(A)}} = \frac{\det^n(A)}{\det(A)}$, $\det(A) \neq 0$, $A$ is non singular.

$\det(adj(A)) = \det^{n-1}(A)$

#

③ if $A$ is nonsingular, then $adj(A)$ is nonsingular, $n>1$

✗ Proof :- if $A$ is nonsingular, $\det(A) \neq 0$,

$\det(adj(A)) = \det(A)^{n-1} \neq 0$, since $\det(A) \neq 0$

then $adj(A)$ is nonsingular.

* $adj(A) = |A| . A^{-1}$

* $A\ adj(A) = det(A) . I$

* $det(adj(A)) = det(A)^{n-1}$

* $(adj\ A)^{-1} = adj(A^{-1}) = |A^{-1}| . A$

* تم اشتقاقهم من المعادلة الأساسية

$A . Adj(A) = det(A) . I$.

ex. let $A$ be $n \times n$ matrix (non singular) then $adj(adj(A)) = |A|^{n-2} . A$

# Proof :- $adj\ A = |A| . A^{-1}$

$adj(A) = |A| . A^{-1}$

$$adj(adj\ A) = adj(|A| . A^{-1})$$

$$= \left| |A| . A^{-1} \right| . \left( |A| . A^{-1} \right)^{-1}$$

$$= |A|^n . |A^{-1}| . \frac{1}{|A|} . |A^{-1}|^{-1}$$

$$= |A|^n . |A^{-1}| . \frac{1}{|A|} . A = |A|^n . \frac{1}{|A|} . \frac{1}{|A|} . A = |A|^{n-2} . A$$

Ex. Show that if $|A| = 1$, then $adj(adj(A)) = A$.

$adj(A) = det(A) . A^{-1}$

$adj(adj(A)) = adj(det(A) . A^{-1})$

$$adj(adj(A)) = adj(A^{-1})$$

$$= det(A^{-1}) . A$$

$$= \frac{1}{det(A)} . A = A$$ #

ربنا تقبل منا إنك أنت السميع العليم

روان فارس