Ch. 3 Differentiation

Q1. Find the derivative $\frac{dy}{dx}$??

ⓐ $f(x) = \dfrac{\sqrt{x} - 1}{\sqrt{x} + 1}$

$$f'(x) = \frac{(\sqrt{x}+1)\frac{1}{2\sqrt{x}} - (\sqrt{x}-1)\cdot\frac{1}{2\sqrt{x}}}{(\sqrt{x}+1)^2}$$

$$= \frac{\frac{\sqrt{x}+1-\sqrt{x}+1}{2\sqrt{x}}}{(\sqrt{x}+1)^2} = \frac{2}{2\sqrt{x}(\sqrt{x}+1)^2} = \frac{1}{\sqrt{x}(\sqrt{x}+1)^2}$$

ⓑ $f(x) = \left(\frac{1}{x} - x\right)(x^2+1)$

$$f'(x) = \left(\frac{1}{x} - x\right)2x + (x^2+1)\cdot\left(\frac{-1}{x^2} - 1\right)$$

$$= 2 - 2x^2 + -1 - x^2 - \frac{1}{x^2} - 1$$

$$= \boxed{-3x^2 - \frac{1}{x^2}}$$

ⓒ $g(x) = \sec(2x+1)\cot(x^2)$

$$g'(x) = \sec(2x+1) \cdot -\csc^2(x^2) * 2x$$

$$+ \cot(x^2) \cdot 2\sec(2x+1)\tan(2x+1).$$

$$g'(x) = 2\sec(2x+1)\left[\cot(x^2)\tan(2x+1) - x\csc^2(x^2)\right]$$

(d) $f(t) = \dfrac{1 + \csc(t)}{1 - \csc(t)}$

$$= \frac{1 + 1/\sin(t)}{1 - 1/\sin(t)} = \frac{\frac{\sin t + 1}{\sin t}}{\frac{\sin t - 1}{\sin t}}$$

$$f(t) = \frac{\sin(t) + 1}{\sin(t) - 1}$$

$$f'(t) = \frac{(\sin(t) - 1)(\cos(t)) - (\sin(t) + 1) \cdot (\cos(t))}{(\sin(t) - 1)^2}$$

$$= \frac{-2\cos(t)}{(\sin t - 1)^2}$$

(e) $f(x) = x^3 \sin x \cos x$

$$f(x) = x^3 \frac{\sin(2x)}{2}$$

$$f'(x) = x^2 \cdot \cancel{\frac{2}{2}} \cos(2x) + \frac{\sin(2x)}{2} \cdot 3x^2$$

$$= x^2 \cos(2x) + \frac{3x^2}{2} \sin(2x)$$

(f) $x^{1/2} + y^{1/2} = 1$

$\sqrt{x} + \sqrt{y} = 1$

$\sqrt{y} = 1 - \sqrt{x}$

$\dfrac{y'}{2\sqrt{y}} = \dfrac{-1}{2\sqrt{x}}$

$y' = \dfrac{-2\sqrt{y}}{2\sqrt{x}} = \dfrac{-\sqrt{y}}{\sqrt{x}} = \dfrac{\sqrt{x} - 1}{\sqrt{x}}$

Q2 (i) $y = \cot^2 x$

$\dfrac{dy}{dx} = 2\cot(x) \cdot -\csc^2(x) = -2\cot x \csc^2 x$

(ii) $x^2 + y^2 = x$ Note $y = \sqrt{x - x^2}$

$2x + 2y\dfrac{dy}{dx} = 1$

$2y\dfrac{dy}{dx} = 1 - 2x$

$\dfrac{dy}{dx} = \dfrac{1 - 2x}{2\sqrt{x - x^2}}$

(iii) $y = \dfrac{\sin x}{1 - \cos x}$

$$\frac{dy}{dx} = \frac{(1-\cos x)\cdot\cos x - \sin^2 x}{(1-\cos x)^2}$$

$$\frac{\cos x - \cos^2 x - \sin^2 x}{(1-\cos x)^2} = \frac{\cos x - 1}{(1-\cos x)^2}$$

$$= \frac{1}{\cos x - 1}$$

$\boxed{Q_3}$ points = ??

curve $y = 2x^3 - 3x^2 - 12x + 20$

where the tangent is parallel to x_axis

Solution

$y' =$ slope of tangent = slope of x_axis $= 0$

$y' = 6x^2 - 6x - 12 = 0$

$x^2 - x - 2 = 0$

$(x+1)(x-2) = 0$

$\boxed{x = -1}$ $\boxed{x = 2}$

$\downarrow$ $\downarrow$

Points :- $(-1, 27)$, $(2, 0)$

Q4

$$f(x) = \begin{cases} \sin(2x) & x \leq 0 \\ ax & x > 0 \end{cases}$$

(i) $f(x)$ is continuous at $x=0$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x)$$

$$\lim_{x \to 0^+} ax = \lim_{x \to 0^-} \sin(2x)$$

$$0 = 0$$ ---> a is any real number

(ii) $f(x)$ is diff. at $x=0$

$$f'(x) = \begin{cases} 2\cos(2x) & x < 0 \\ a & x > 0 \end{cases}$$

$$f'(0)^+ = f'(0)^-$$

$$a = 2\cos(0)$$

$$\boxed{a = 2}$$

Extra question: Determine the equation of the normal line to the curve $4y^2 = x^4$ at the point $(2,-2)$

$$4y^2 = x^4$$

$$8yy' = 4x^3$$

$$y'\Big|_{(2,-2)} = \frac{x^3}{2y}\Big|_{(2,-2)} = \frac{8}{2 \cdot -2} = -2$$

Eq of the Normal Line → Slope $\boxed{\frac{1}{2}}$; point $\boxed{(2,-2)}$

The equation of Normal line $y+2 = \frac{1}{2}(x-2)$

$$\boxed{y = \frac{1}{2}x - 3}$$

$\boxed{Q_5}$ Find the normals to the curve $xy + 2x - y = 0$ that are parallel to the line $2x + y = 0$

The equation of Normal → Slope -2
The equation of Normal → point (x_1, y_1), (x_2, y_2)

- The normal line // line : $y = -2x$

Slope of normal line $= -2$

Slope of the curve $= \frac{1}{2} = y'$

$$\boxed{xy + 2x - y = 0} \text{ - - - - } (1)$$

$$xy' + y + 2 - y' = 0$$

$$y' = \frac{-2 - y}{x - 1} = \frac{1}{2}$$

$$x - 1 = -4 - 2y$$

$$\boxed{x = -3 - 2y} \text{ - - - - } (2)$$

To Find the points (x_1, y_1), (x_2, y_2) of intersection the line with curve :

Substitue equation (2) in equation (1)

$$xy + 2x - y = 0$$

$$(-3 - 2y)y + 2(-3 - 2y) - y = 0$$

$$-3y - 2y^2 - 6 - 4y - y = 0$$

$$-2y^2 - 8y - 6 = 0$$

$$y^2 + 4y + 3 = 0$$

$$(y+1)(y+3) = 0$$

$$\boxed{y = -1} \text{ OR } \boxed{y = -3}$$

$$\downarrow \qquad \downarrow$$

$$\boxed{x = -1} \qquad \boxed{x = 3}$$

So the Intersection points between the normal line and the curve ~~curve~~ are : $(-1, 1)$ and $(3, -3)$

* The First normal is $y + 1 = -2(x+1)$

$$\boxed{y = -2x - 3}$$

* The second normal is $y + 3 = -2(x - 3)$

$$\boxed{y = -2x + 3}$$

$\boxed{Q6}$ $L(x) = f(a) + f'(a)(x-a)$ linearization of $f(x)$ $\boxed{x=a}$

(a) $f(x) = \tan x$ $\qquad x = \frac{\pi}{4}$

$f\left(\frac{\pi}{4}\right) = 1$

$f'(x) = \sec^2 x$

$f'\left(\frac{\pi}{4}\right) = 2$ $\qquad \boxed{L(x) = 1 + 2\left(x - \frac{\pi}{4}\right) = 2x - \frac{\pi}{2} + 1}$

(b) $g(x) = \frac{1}{x}$, $x = 1$

$g(1) = 1$

$g'(x) = \frac{-1}{x^2}$

$g'(1) = -1$

$L(x) = 1 + -1(x-1)$

$\boxed{L(x) = 1 - x + 1 = 2 - x}$

(c) $h(x) = \frac{x^2}{x^2+1}$, $x = 0$

$h(0) = 0$

$h'(x) = \frac{2x}{(x^2+1)^2}$

$h'(0) = 0$

$\boxed{L(x) = 0 + 0(x-0) = 0}$

(d) $f(x) = 1 + \cos\theta$, $\theta = \frac{\pi}{3}$

$f(\frac{\pi}{3}) = \frac{3}{2}$

$f'(x) = -\sin\theta$

$f'(\frac{\pi}{3}) = \frac{-\sqrt{3}}{2}$

$\boxed{L(\theta) = \frac{3}{2} + \frac{\sqrt{3}}{2}(\theta - \frac{\pi}{3}) = \frac{3}{2} - \frac{\sqrt{3}}{2}\theta + \frac{\sqrt{3}\,\pi}{6}}$

Q7 The radius of a circle increased from 2 → 2.02 m

$A(r) = \pi r^2$

True change in Area $= \Delta A$

(a) Estimate the resulting change in Area.

Estimated Change in Area $= dA$

$= A' \, dr$

$= (4\pi)(0.02)$

$= 0.08 \pi \, m^2$

$$\boxed{\begin{aligned} dr &= \Delta r \\ &= 0.02 \\ A' &= 2\pi r \\ A'\big|_{r=2} &= 4\pi \end{aligned}}$$

(b) Find the True Change in Area.

True value of Area $= \Delta A$

$= A_2 - A_1$

$= A(2.02) - A(2)$

$= 4.0804\pi - 4\pi$

$= 0.0804 \pi \, m^2$

(c) Find the $|\Delta A - dA|$

Error $= |\Delta A - dA| = |0.0804\pi - 0.08\pi|$

$= 0.0004 \pi \, m^2$

(d) Express dA as a percentage of the circle original area

$$\frac{dA}{A} * 100\% = \frac{0.08\pi}{4\pi} * 100\% = 2\%$$