14.2 Limits and Continuity in Higher Dimensions

Exp Find the following limits:

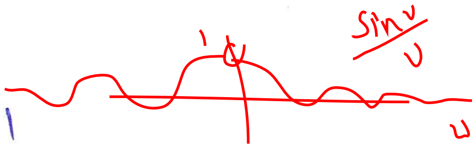
1. $\lim\limits_{(x,y)\to(\ln 2,0)} e^{x-y} = e^{\ln 2 - 0} = e^{\ln 2} = 2$

2. $\lim\limits_{(x,y)\to(0,0)} \frac{e^{y}\sin x}{x} = e^{0} \lim\limits_{x\to 0} \frac{\sin x}{x} = (1)(1) = 1$

3. $\lim\limits_{(x,y)\to(2,2)} \frac{x-y}{x^4-y^4} = \lim\limits_{(x,y)\to(2,2)} \frac{\cancel{x-y}}{(\cancel{x-y})(x+y)(x^2+y^2)} = \frac{1}{(4)(4+4)} = \frac{1}{32}$

4. $\lim\limits_{P\to(1,-1,-1)} \frac{2xy+yz}{x^2+z^2} = \frac{-2+1}{1+1} = \frac{-1}{2}$

5. $\lim\limits_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = \lim\limits_{u\to 0} \frac{\sin u}{u} = 1$



or $= \lim\limits_{r\to 0} \frac{\sin r^2}{r^2} = \lim\limits_{r\to 0} \frac{\cancel{2r}\cos r^2}{\cancel{2r}} = 1$

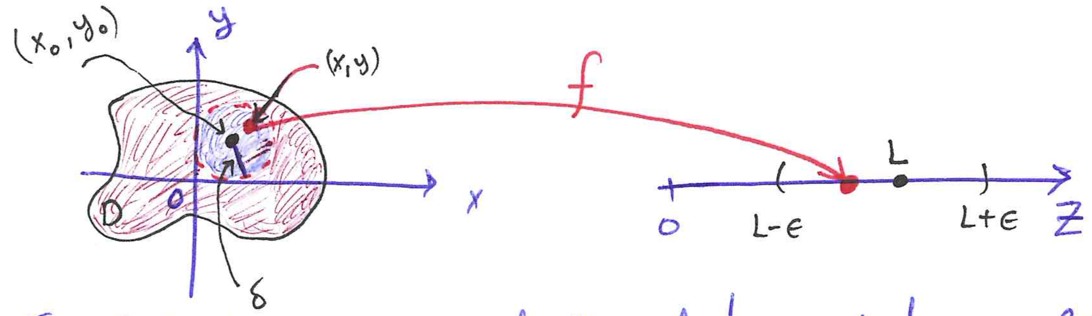
6. $\lim\limits_{(x,y)\to(0,0)} \frac{1-\cos(xy)}{xy} = \lim\limits_{u\to 0} \frac{1-\cos u}{u} = \lim\limits_{u\to 0} \frac{\sin u}{1} = 0$

Limits for Functions of Two Variables:

Def We say a function $f(x,y)$ approaches the Limit L as (x,y) approaches (x_0,y_0), and write $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = L$,

if for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all (x,y) in the domain of f:

If $\sqrt{(x-x_0)^2+(y-y_0)^2} < \delta$ then $|f(x,y) - L| < \epsilon$.



The definition says that the distance between $f(x,y)$ and L becomes small whenever the distance between (x,y) and (x_0,y_0) gets smaller.

Exp show that $\lim\limits_{(x,y)\to(0,0)} (x^2+y^2) = 0$ take $\delta = \sqrt{\epsilon}$

Let $\epsilon > 0$, we need to find $\delta > 0$ s.t if

$\sqrt{x^2+y^2} < \delta$ then $|f(x,y) - L| < \epsilon$

$$\Rightarrow |x^2+y^2-0| = |x^2+y^2| = x^2+y^2 < \delta^2 = \epsilon$$

Exp show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{y}{x^2+1} = 0$ take $\delta = \epsilon$

Let $\epsilon > 0$, we need to find $\delta > 0$ s.t if

$\sqrt{x^2+y^2} < \delta$ then $|f(x,y) - L| < \epsilon$

$$\Rightarrow \left|\frac{y}{x^2+1} - 0\right| = \frac{|y|}{x^2+1} \leq |y| = \sqrt{y^2} \leq \sqrt{y^2+x^2} < \delta = \epsilon$$

Exp show that $\lim\limits_{(x,y,z)\to(0,0,0)} xyz = 0$ take $\delta = \sqrt[3]{\epsilon}$

Let $\epsilon > 0$, we need to find $\delta > 0$ s.t if

$\sqrt{x^2+y^2+z^2} < \delta$ then $|f(x,y,z) - L| < \epsilon$

$$\Rightarrow |xyz - 0| = |xyz| = |x|\,|y|\,|z| = \sqrt{x^2}\,\sqrt{y^2}\,\sqrt{z^2} \leq \left(\sqrt{x^2+y^2+z^2}\right)^3 < \delta^3 = \epsilon$$

Th (Properties of Limits of functions of two Variables)

If $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = L$ and $\lim\limits_{(x,y)\to(x_0,y_0)} g(x,y) = M$, $M, L \in \mathbb{R}$

then:

[1] $\lim\limits_{(x,y)\to(x_0,y_0)} [f(x,y) \pm g(x,y)] = L \pm M$

[2] $\lim\limits_{(x,y)\to(x_0,y_0)} K f(x,y) = K L$, $K \in \mathbb{R}$

[3] $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)\, g(x,y) = L M$

[4] $\lim\limits_{(x,y)\to(x_0,y_0)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M}$, $M \neq 0$

[5] $\lim\limits_{(x,y)\to(x_0,y_0)} [f(x,y)]^n = L^n$, n is positive integer

[6] $\lim\limits_{(x,y)\to(x_0,y_0)} \sqrt[n]{f(x,y)} = \sqrt[n]{L}$, n is positive integer and if n is even, then must be: $L > 0$

Two-Path Test for Nonexistance of a Limit:

If a function $f(x,y)$ has different limits along two different paths in the domain of f as $(x,y) \to (x_0,y_0)$, then

$\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ DNE.

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* In most cases: we use the path $y = kx$ or the path $y = kx^2$, $k \in \mathbb{R}$.

Exp show that the function $f(x,y) = \frac{2x^2y}{x^4+y^2}$ has no limit as $(x,y) \to (0,0)$.

The path $y = kx^2$

$$\lim_{\substack{(x,y)\to(0,0) \\ \text{along } y=kx^2}} f(x,y) = \lim_{x\to 0} \frac{2x^2(kx^2)}{x^4+(kx^2)^2} = \lim_{x\to 0} \frac{2kx^4}{x^4+k^2x^4} = \frac{2k}{1+k^2}$$

different limit for different values of k, i.e. $= \begin{cases} 0 & \text{if } k=0 \\ 1 & \text{if } k=1 \end{cases}$

Remark: Having the same limit along all straight lines approaching (x_0,y_0) $\not\Rightarrow$ limit exists at (x_0,y_0).

In the Exp above, the limit is 0 along every path $y=kx$

$$\lim_{\substack{(x,y)\to(0,0) \\ \text{along } y=kx}} f(x,y) = \lim_{x\to 0} \frac{2x^2(kx)}{x^4+(kx)^2} = \lim_{x\to 0} \frac{2kx^3}{x^4+k^2x^2} = \lim_{x\to 0} \frac{2kx}{x^2+k^2} = 0$$

Exp show that $\lim_{(x,y)\to(0,0)} \frac{x^4}{x^4+y^2} = DNE$

The path $y = kx^2$

$$\lim_{\substack{(x,y)\to(0,0) \\ \text{alnog } y=kx^2}} f(x,y) = \lim_{x\to 0} \frac{x^4}{x^4+k^2x^4} = \lim_{x\to 0} \frac{1}{1+k^2} = \frac{1}{1+k^2} = \begin{cases} 1 & \text{if } k=0 \\ \frac{1}{2} & \text{if } k=1 \end{cases}$$

Exp show that $\lim_{(x,y)\to(0,0)} \frac{xy}{|xy|} = DNE$

The path $y=kx$, $k\neq 0$

$$\lim_{\substack{(x,y)\to(0,0) \\ \text{along } y=kx}} f(x,y) = \lim_{x\to 0} \frac{kx^2}{|kx^2|} = \lim_{x\to 0} \frac{kx^2}{|k|x^2} = \frac{k}{|k|} = \begin{cases} 1 & \text{if } k>0 \\ -1 & \text{if } k<0 \end{cases}$$

Def * A function $f(x,y)$ is continuous at the point (x_0,y_0) if

1. f is defined at (x_0,y_0)
2. $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists
3. $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$

* f is continuous if it is continuous at every point of its domain.

Exp [1] $f(x,y) = \dfrac{x+y}{x-y}$ is continuous at all (x,y) s.t $x \neq y$

[2] $f(x,y) = \dfrac{x+y}{2+\cos x}$ is continuous at all (x,y)

[3] $f(x,y,z) = \ln(z - x^2 - y^2 - 1)$ is continuous at all (x,y,z) s.t $x^2+y^2+1 < z$

Continuity of Composites:

If f is continuous at (x_0, y_0) and g is a single variable function continuous at $f(x_0,y_0)$, then $h = g \circ f = g(f(x,y))$ is continuous at (x_0, y_0).

Functions of More Than Two Variables

* The definitions of limit and continuity for functions of two variables and the conclusions about limits, continuity for sums, products, ... all extend to functions of three or more variables.
