Notes + 1.3.3

Exercise 1.3

1.3.0: T, F, prove the Trues ones and give counter examples to the False one.

a. If A and B are nonempty, bounded subset of $\mathbb{R}$ then $\sup(A\cap B) \leq \sup A$. True

proof: case 1) If $A\cap B = \emptyset$ then $\sup(A\cap B) = -\infty$ and there is nothing to prove.

case 2: If $A\cap B \neq \emptyset$ and bdd set

since $(A\cap B) \subset A$ then any upperbound of A is an upperbound of $(A\cap B)$

Therefore $\underline{\sup A \text{ is an upperbound of } (A\cap B)}$.

By completeness axiom the $\sup(A\cap B)$ exists.

By Def of supremum $\sup(A\cap B) \leq \sup A$ ∎

b. let ε be a positive real number. If A is a nonempty, bounded subset of $\mathbb{R}$ and $B = \{\varepsilon x : x\in A\}$ then $\sup B = \varepsilon \sup A$. True

If $x\in A$ Then $x \leq \sup A$

since $\varepsilon > 0$ we have $\boxed{\varepsilon x \leq \varepsilon \sup A}$

so the latter is an upperbound of B.

It follows that $\sup B \leq \varepsilon \sup A$

If $x\in A$ then $\varepsilon x \in B$ so $\varepsilon x \leq \sup B$ (i.e $\frac{\sup B}{\varepsilon}$ is an upperbound of A)

It follows that $\sup A \leq \frac{\sup B}{\varepsilon}$

$\varepsilon \sup A \leq \sup B$.

c. If $A+B$ defined $:= \{a+b,\ a\in A,\ b\in B\}$ where $A, B \neq \emptyset$, bounded subset of $\mathbb{R}$ then $\sup(A+B) = \sup A + \sup B$. True

If $x\in A$ and $y \in B$ Then $\underbrace{x \leq \sup A \text{ and } y \leq \sup B}_{+}$

$x+y \leq \sup A + \sup B$

So $\sup(A+y) \leq \sup A + \sup B$.

If this inequality is strict, then $\sup(A+B) - \sup B < \sup A$

By Approximation property there is an $a_0 \in A$ s.t $\sup(A+B) - \sup B < a_0$

This implies $\sup(A+B) - a_0 < \sup B$

By Approximation property there is an $b_0 \in B$ s.t $\sup(A+B) - a_0 < b_0$

We conclude $\sup(A+B) < a_0 + b_0$ $\times$

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d. If $A-B := \{a-b : a\in A,\ b\in B\}$ where $A, B \neq \emptyset$, bounded subset Then $\sup(A-B) = \sup A - \sup B$ False.

let $A = B = [0,1]$ then $A-B = [-1,1]$

$\sup(A-B) = 1 \neq 0 = \sup A - \sup B$.

1.3.1: Find the inf and sup of each of the following sets.

a. $E = \{x \in \mathbb{R} : x^2 + 2x = 3\}$. $x^2 + 2x - 3 = 0$

$(x-1)(x+3) = 0$

$x = -3,\ x = 1$

$x \in [-3, 1]$ So

$\inf E = -3$

$\sup E = 1$

b. $E := \{x \in \mathbb{R} : x^2 - 2x + 3 > x^2 \text{ and } x > 0\}$.

$$\cancel{x^2} - 2x + 3 > \cancel{x^2}$$

$$-2x + \cancel{3} > 0 - 3 \;\; \to \;\; \frac{-2x}{-2} > \frac{-3}{-2} \;\; \to \;\; x < \frac{3}{2}$$

$x < \frac{3}{2}$, $x > 0 \;\to\;$ $0 < x < \frac{3}{2}$ so $\inf E = 0$

$\sup E = \frac{3}{2}$.

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c. $E = \{\frac{p}{q} \in \mathbb{Q} : p^2 < 5q^2 \text{ and } p, q > 0\}$

$$p^2 < 5q^2$$

$$\frac{p^2}{q^2} < 5 \;\to\; \frac{p}{q} < \sqrt{5} \;\Rightarrow\; \inf E = 0 \;,\; \sup E = \sqrt{5}.$$

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d. $E = \{x \in \mathbb{R} : x = 1 + \frac{(-1)^n}{n} \text{ for } n \in \mathbb{N}\}$

since $1 + \frac{(-1)^n}{n} = 1 - \frac{1}{n}$ since n odd

and $1 + \frac{(-1)^n}{n} = 1 + \frac{1}{n}$ since n even

$\inf E = 0$, $\sup E = \frac{3}{2}$

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e. $E = \{x \in \mathbb{R} : x = \frac{1}{n} + (-1)^n \text{ for } n \in \mathbb{N}\}$

since $\frac{1}{n} + (-1)^n = \frac{1}{n} - 1$ since n odd. $\{0, -\frac{2}{3}, \dots -1\}$

and $\frac{1}{n} + (-1)^n = \frac{1}{n} + 1$ since $\underline{n}$ even.

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$\inf E = -1$, $\sup E = \frac{3}{2}$

F. $E = \{ 2 - \frac{(-1)^n}{n^2} : n \in \mathbb{N} \}$

since $2 - \frac{(-1)^n}{n^2} = 2 + \frac{1}{n^2}$ since n ~~even~~ odd $\{3, \dots\}$ $n-1$

And $2 - \frac{(-1)^n}{n^2} = 2 - \frac{1}{n^2}$ since n ~~odd~~ even $\{1, \dots\}$

$\inf E = \frac{7}{4}$, $\sup E = 3$.

1.3.2: prove that for each $a \in \mathbb{R}$ and each $n \in \mathbb{N}$ there exist a rational r_n such that $|a - r_n| < \frac{1}{n}$.

since $\frac{a-1}{n} < \frac{a+1}{n}$ choose $r_n \in \mathbb{Q}$ s.t $\frac{a-1}{n} < r_n < \frac{a+1}{n}$

i.e $|a - r_n| < \frac{1}{n}$.

1.3.4: prove that a lower bound of a set need not be unique but the infimum of E is unique:

If m is a lower bound of E Then so is any $M \leq m$.

let M, m two infimum of E then M, m are lower bound of E

Hence, By def $m \leq M$ and $M \leq m$

By Trichotomy property conclude $m = M$

so the inf is unique.

1.3.5:

suppose that E is bounded and nonempty subset of $\mathbb{Z}$.

since $-E$ is a bounded and nonempty subset of $\mathbb{Z}$ it has a supremum

By the completeness Axiom and that supremum belongs to $-E$ by Thm2

Hence by reflection principle $\inf E = -\sup(-E) \in -(-E) = E$

$\boxtimes$

1.3.7 :

a. prove that if x is an upperbound of a set $E \subseteq \mathbb{R}$ and $x \in E$ then ~~x~~ x is the supremum of E.

① let x be an upperbound of E and $x \in E$.

② If M is any upperbound of E then $M \geq x$

Hence, By definition $\sup E = x$

b. Make and prove an analogous statement for the infimum of E.

If x is a lowerbound of E and $x \in E$ then $x = \inf E$.

proof: $-x$ is an upperbound of $-E$ and $-x \in -E$ So $-x = \sup(-E)$

Thus $x = -\sup(-E) \rightarrow x = \inf E$.

c. show by example that the converse of each of these statements is false.

If E is the set of points x_n such that $x_n = 1 - \frac{1}{n}$ for odd n and $x_n = \frac{1}{n}$ for even

then $\sup E = 1$, $\inf E = 0$ but neither 0 nor 1 belong to E.

1.3.8 : suppose that $E, A, B \subset \mathbb{R}$ and $E = A \cup B$, prove that if E has a supremum and both A and B are nonempty, then $\sup A$ and $\sup B$ both exist. and $\sup E$ is one of the numbers $\sup A$ or $\sup B$.

since $A \subseteq E$ any upperbound of E is an upperbound of A.

since A is nonempty By completness axiom that A has a supremum.

simmilarly $B \rightarrow B$ has supremum.

By Monotone property $\sup A, \sup B \leq \sup E$.

set $M := \max\{\sup A, \sup B\}$ and observe that M is an upperbound of both A and B.

If $M < \sup E$ then there is an $x \in E$ s.t $M < x < \sup E$

But $x \in E$ implies $x \in A$ or $x \in B$

Thus M is not an upperbound for one of the sets A or B

$\times$

1.3.9 : A dyadic rational is a number of the form $\frac{K}{2^n}$ for some $K, n \in \mathbb{Z}$, prove that if a and b are real numbers and $a < b$ then there exists a dyadic rational q s.t $a < q < b$.

By induction $2^n > n$. Hence By Archemedian principle there is an $n \in \mathbb{N}$ s.t $2^n > \frac{1}{b-a}$

let $E := \{K \in \mathbb{N} : 2^n b \leq K\}$. By Archemedian principle, E is nonempty

Hence, let M_0 be the least element in E, $q = \frac{M_0 - 1}{2^n}$.

since $b > 0$, $M_0 \geq 1$

since M_0 least element in E it follows that $M_0 - 1 < 2^n b$

i.e $q < b$ on other hand $M_0 \in E$ implies $2^n b \leq M_0$ so

$$a = b - (b + a) < \frac{M_0}{2^n} - \frac{1}{2^n} = \frac{M_0 - 1}{2^n} = q$$

$\rightarrow \boxed{a < q}$

$a < q < b$ ⊠

1.3.11: If $a, b \in \mathbb{R}$ and $b-a>1$ then there is at least one $K \in \mathbb{Z}$ s.t $a<K<b$.

let $E := \{n \in \mathbb{Z} ; n \leq a\}$

If $a \geq 0$ then $0 \in E$. then By Archemeadean principle there is an $m \in N$ s.t $m > -a$ i.e $n := -m \in E$.

Thus E is nonempty.

since E is bounded above (by a), By completness axiom and Thm 2 it follows that $n_0 = \sup E$ exists and belong to E.

set $K = n_0 + 1$, since $K > \sup E$, K cannot belong to E.

i.e $a<K$

since $n_0 \in E$ and $b-a>1$

$$K = \underline{n_0} + 1 \leq a+1$$ $n_0 = \sup E$, a upper bdd E.

$$< a + (b-a)$$

$$K < b$$

$a<K<b$ ☒