Exercises :

3.4.0 : True or False.

a. If f is uniformly continuous on $(0,\infty)$ and g is positive and bounded on $(0,\infty)$, then fg is uniformly continuous on $(0,\infty)$. False

let $f(x) = x$ and $g(x) = 1$

if $x < 2$ and 2 if $x \geq 2$

Then f is uniformly continuous on $(0,\infty)$ and g is positive and bounded

But $f(x)g(x)$ is not continuous at $x = 2$ so cannot be uniformly cont.

b. The function $x \log(\frac{1}{x})$ is uniformly continuous on $(0,1)$. True

By lobital Rule :

$x \log(\frac{1}{x}) \to 0$ as $x \to 0^+$.

By Thm 2 $\rightarrow$ $x \log(\frac{1}{x})$ is uniformly continuous.

c. The function $\frac{\cos x}{mx + b}$ is uniformly continuous on $(0,1)$ for all nonzero $m, b \in \mathbb{R}$.

False, let $m = -b = 1$ Then $\frac{\cos x}{(mx+b)} \to \frac{\cos 1}{0^-} = -\infty$ as $x \to 1^-$.

So By Thm 3.2, this function cannot possibly be uniformly cont. on $(0,1)$.

d. If f, g are uniformly continuous on an interval $[a,b]$ and $g(x) \neq 0$ for $x \in [a,b]$, then $\frac{f}{g}$ is uniformly continuous on $[a,b]$. True.

f and g are bounded on $[a,b]$ By Extreme value Thm.

since $g(x) \neq 0$ it follows By the Intermediate Value Thm. that either $g(x) > 0$ or $g(x) < 0$ $\forall x \in [a,b]$.

We may suppose $g(x) > 0$ By extreme value Thm.

$g(x) \geq \varepsilon_0 > 0$ for $x \in [a,b]$.

Therefor $\frac{f}{g}$ is uniformly continuous on $[a,b]$ By 3.4.5(d).

3.4.1: using def. prove that each of the following functions is uniformly continuous on $(0,1)$:

a. $f(x) = x^2 + x$.

given $\varepsilon > 0$ and set $\delta = \frac{\varepsilon}{3}$

If $x, a \in (0,1)$ and $|x-a| < \delta$ then $|f(x) - f(a)|$

$$
\begin{aligned}
&= |x^2 + x - a^2 - a| \\
&= |(x^2 - a^2) + x - a| \\
&= |(x-a)(x+a) + (x-a)| \\
&= |x-a|\,|x+a+1| \\
&= |x-a|\,|1+1+1| \\
&< \delta \; 3 \\
&< \frac{\varepsilon}{3} (3) \\
&< \varepsilon
\end{aligned}
$$

b. $f(x) = x^3 - x + 2$.

given $\varepsilon > 0$ and set $\delta = \varepsilon \backslash 4$

If $x, a \in (0,1)$ and $|x-a| < \delta$ then $|x^3 - x + 2 - (a^3 - a + 2)|$

$$= |x^3 - a^3 - x + a + 2 - 2|$$

$$= |(x-a)(x^2 + xa + a^2) - x + a|.$$

$$= |x-a|\,|x^2 + xa + a^2 - 1|$$

$$\leq |x-a|\,\left(|x^2 + xa + a^2| + |-1|\right)$$

$$\leq |x-a|\,\left(|1+1+1| + 1\right)$$

$$< \delta\ 4$$

$$< \frac{\varepsilon}{4}\ 4$$

$$< \varepsilon$$

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c. $f(x) = x \sin 2x$.

given $\varepsilon > 0$ and set $\delta = \varepsilon \backslash 3$.

If $x, a \in (0,1)$ and $|x-a| < \delta$ then $|x \sin 2x - a \sin 2a|$

$$= |x \sin 2x - a \sin 2x + a \sin 2x - a \sin 2a|$$

$$\leq |x-a|\,|\sin 2x| + |a|\,|\sin 2x - \sin 2a|$$

$$\leq |x-a| \cdot 1 + 1\,|\sin 2x - \sin 2a|$$

$\boxed{|\sin x| \leq |x|}$ متطابقة

$$\leq |x-a| + 2\left|\sin\left(\frac{2x-2a}{2}\right)\cos\left(\frac{2x+2a}{2}\right)\right|$$

$$\leq |x-a| + 2\,|\sin(x-a)|\,|\cos(x+a)| \quad \leq 1$$

$|\sin x| \leq |x|$ وهاي حسب

$|\sin(x-a)| \leq |x-a|$

$$\leq |x-a| + 2\,|x-a| \cdot 1$$

$$< 3\,|x-a|$$

$$< 3\ \delta$$

$$< \varepsilon$$

3.4.2 : prove that each of the following functions is uniformly continuous on $(0,1)$.

a. $f(x) = \frac{\sin x}{x}$

By Lopital Rule : $\frac{\sin x}{x} \to 1$ as $x \to 0$ So

f is uniformly continuous on $(0,1)$ By Thm 1.

b. $f(x) = x \cos \frac{1}{x^2}$.

By squeez Thm, $x \cos \frac{1}{x^2} \to 0$ as $x \to 0$ So

f is uniformly continuous on $(0,1)$ By Thm 1.

c. $f(x) = x \log x$.

By Lopital Rule, $x \log x \to 0$ as $x \to 0$ so f is uniformly conti. on $(0,1)$ By Thm 1.

d. $f(x) = (1-x^2)^{\frac{1}{x}}$.

By Lopital Rule, $(1-x^2)^{\frac{1}{x}} \to 1$ as $x \to 0^+$ So, f is uniformly cont. on $(0,1)$ By Thm 1.

3.4.3: Assuming $\sin x$ is continuous on $\mathbb{R}$, find all real α s.t $x^{\alpha} \sin\left(\frac{1}{x}\right)$ is uniformly continuous on the open interval :

If $\alpha > 0$ then $\left| x^{\alpha} \sin \frac{1}{x} \right| \leq x^{\alpha} \to 0$ as $x \to 0^+$, if $\alpha > 0$.

Thus, $x^{\alpha} \sin \frac{1}{x}$ is uniformly continuous on $(0,1)$ for all $\alpha > 0$.

If $\alpha \leq 0$ and $x_n = \frac{2}{(2n+1)\pi}$ then $x_n^{\alpha} \sin\left(\frac{1}{x_n}\right) = (-1)^n x_n^{\alpha}$ DNC as $n \to \infty$.

i.e, $x^{\alpha} \sin\left(\frac{1}{x}\right)$ has No limit as $x \to 0^+$.

$\Longrightarrow$ Therefor, $x^{\alpha} \sin\left(\frac{1}{x}\right)$ uniformly continuous on $(0,1)$ since $\alpha > 0$

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3.4.4 : a. suppose that $f : [0,\infty) \to \mathbb{R}$ is continuous and that there is an $L \in \mathbb{R}$ s.t $f(x) \to L$ as $x \to \infty$. prove that f is uniformly continuous on $[0,\infty)$.

given $\varepsilon > 0$ and choose N so large that $x \geq N$ implies $|f(x) - L| < \frac{\varepsilon}{3}$

By Thm , f is uniformly continuous on $[0, N]$.

Thus, there is a $\delta > 0$ s.t $|x - y| < \delta$ and $x, y \in [0, N]$ implies

$$|f(x) - f(y)| < \frac{\varepsilon}{3}.$$

let $x, y \in [0,\infty)$ and suppose $|x - y| < \delta$. If $x, y \in [0, N]$, then $|f(x) - f(y)| < \varepsilon$.

If both $x, y \notin [0, N]$ then $|f(x) - f(y)| \leq |f(x) - L| + |f(y) - L|$

$$< \frac{2\varepsilon}{3} < \varepsilon.$$

If one of the pair x, y belongs to $[0, N]$ and the other does not

ex → If $x \in [0, N]$ and $y \notin [0, N]$, then $|x - N| \leq |x - y| < \delta$

Thus, $|f(x) - f(y)| \leq |f(x) - f(N)| + |f(N) - L| + |f(y) - L|$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \quad \blacksquare$$

b. prove that $f(x) = \frac{1}{x^2 + 1}$ is uniformly continuous on $\mathbb{R}$.

since $f(x)$ is continuous on $\mathbb{R}$ and $f(x) \to 0$ as $x \to \infty$

so f is uniformly continuous on $[0,\infty)$ by part a.

But $f(-x) = f(x)$, Hence, f is uniformly continuous on $\mathbb{R}$ by symmetry.

3.4.5: suppose that $\alpha \in \mathbb{R}$, that E is a nonempty subset of $\mathbb{R}$, and that $f, g : E \to \mathbb{R}$ are uniformly continuous on E.

a. prove that $f+g$ and αf are uniformly continuous on E.

Given $\varepsilon > 0$ and $\delta > 0$ s.t $x, y \in E$ and

$|x-y| < \delta$ imply $|f(x)-f(y)|$ and $|g(x)-g(y)| < \frac{\varepsilon}{2}$

$$\Rightarrow |(f+g)(x) - (f+g)(y)| \leq |f(x) - f(y)| + |g(x) - g(y)|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$
$$< \varepsilon \quad \blacksquare$$

αf similar.

b. suppose that f, g bounded on E. prove that fg is uniformly continuous on E.

let $M = \sup\{1 + |f(x)| + |g(x)| : x \in E\}$, Then $M > 0$ and both $|f(x)|$ and $|g(x)|$ are less than M for $x \in E$.

given $\varepsilon > 0$ choose $\delta > 0$ s.t $x, y \in E$ and $|x-y| < \delta$ imply $|f(x) - f(y)|$ and $|g(x) - g(y)| < \frac{\varepsilon}{2M}$.

if $|(fg)(x) - (fg)(y)| \leq |g(y)||f(x) - f(y)| + |f(x)||g(x) - g(y)|$

$$< M\frac{\varepsilon}{2M} + M\frac{\varepsilon}{2M}$$
$$= \varepsilon \quad \blacksquare$$

3.4.5. c. Show that there exists functions f, g uniformly continuous on $\mathbb{R}$ s.t fg is not uniformly continuous.

Let $f(x) = x = g(x)$ Then f and g are uniformly continuous on $\mathbb{R}$

But $(fg)(x) = x^2$ is not uniformly continuous on $\mathbb{R}$.

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d. suppose that f is bounded on E and that there is a positive constant ε_0 s.t $g(x) \geq \varepsilon_0$ for all $x \in E$. prove that $\frac{f}{g}$ is uniformly continuous on E.

If $g(x) \geq \varepsilon_0 > 0$, then $\frac{1}{g}$ is continuous on E and bounded by $\frac{1}{\varepsilon_0}$.

Moreover, since g is uniformly continuous:

$$\left| \frac{1}{g(x)} - \frac{1}{g(y)} \right| = \frac{|g(x) - g(y)|}{|g(x)||g(y)|} \leq \varepsilon_0^{-2} |g(x) - g(y)|,$$

$\frac{1}{g}$ is uniformly continuous on E. Hence By b. $\frac{f}{g} := f\left(\frac{1}{g}\right)$ is uniformly cont on E

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e. Show that there exist function f, g uniformly continuous on the intervalue $(0,1)$, with $g(x) > 0$ for all $x \in (0,1)$, s.t $\frac{f}{g}$ is not uniformly continuous on $(0,1)$.

Let $f(x) = x$ and $g(x) = x^2$, Then f and g are uniformly continuous on $(0,1)$.

But $\left(\frac{f}{g}\right)(x) = \frac{1}{x}$ is not uniformly continuous on $(0,1)$.

3.4.6: a. let I be a bounded interval. prove that if $f: I \to \mathbb{R}$ is uniformly continuous on I then f is bounded on I.

suppose I has endpoints a, b. By Thm

there is a continuous function g on $[a,b]$ s.t $f(x) = g(x)$ for all $x \in I$

By extreme value thm, g is bounded on $[a,b]$.

There for, f is bounded on $I \subseteq [a,b]$.

3.4.6. b. prove that a maybe false if I is unbounded or if f is merely continuous.

$f(x) = x$ is uniformly continuous on $[0, \infty)$ But not bounded there.

$f(x) = \frac{1}{x}$ is continuous on $(0,1)$ But not bounded there either.