Chapter 9: Normal subgroups and Factor groups.

Def: Normal subgroup.

A subgroup H of a group G is called a normal subgroup of G if $aH = Ha$ for all a in G. We denote this by $H \lhd G$.

left coset   right cosets

exp: $G = Z_6$, $H = \{0, 3\}$

$0 + H = H = H + 0$.

$1 + H = \{1, 4\} = H + 1$

$2 + H = \{2, 5\} = H + 2$

So H is Normal subgroup.

Note: every subgroup of An Abelian group is Normal.

exp: $G = S_3$, $H = \{\rho_0, \rho_1, \rho_2\}$ S_3: Not Abelian

2 cosets

$H = \{e, (123), (132)\}$

$(12)H = \{(12), (23), (13)\}$ $(12)(123) = (1)(23)$

$(12)(132) = (13)(2)$.

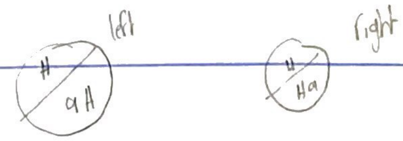
$H(12) = \{(12), (13), (23)\}$

So H is Normal subgroup.

→ if $H \leq G$ and $[G:H] = 2$ then $H \lhd G$.

زي الحكي إلي فوق

cosets = 2 بس



exp: $G = S_3$, $H = \{e, (12)\}$ $H \ntrianglelefteq S_3$.

left cosets:

- H: $e, (12)$
- $(13)H$: $(13), (123)$
- $(32)H$: $(32), (132)$

Right cosets:

- H: $e, (12)$
- $H(13)$: $(13)\ (132)$
- $H(23)$: $(23), (123)$

$(13)H \neq H(13)$.

$(32)H \neq H(32)$

So H is Not Normal.

Note: every subgroup of any group of index $= 2$ is Normal in G.

exp: $A_n \triangleleft S_n$.

Thm 9.1: Normal subgroup test:

مش مطلوب Proof

A subgroup H of G is Normal in G iff $xHx^{-1} \subseteq H$ for all x in G.

$\rightsquigarrow$ $H \triangleleft G$ iff $\forall h \in H, g \in G \rightarrow ghg^{-1} \in H$

Thm 9.2: Factor groups.

let G be a group and let H be a Normal subgroup of G. The set $G \backslash H = \{aH : a \in G\}$ is a group under the operation $(aH)(bH) = abH$

exp: $G = (\mathbb{Z}, +)$

$H = (4\mathbb{Z}, +)$

exp 7 on Book.

$\rightarrow$ cosets $= \{0+H, 1+H, 2+H, 3+H\}$

table

Thm 9.3: The $G \backslash Z$ Theorem.

حفظ let G be a group and let be $Z(G)$ be a center of G. If $G \backslash Z(G)$ is cyclic then G is Abelian.

Thm 9.4: $G \backslash Z(G) \approx \mathrm{Inn}(G)$.

For any group G, $G \backslash Z(G)$ is isomorphic to $\mathrm{Inn}(G)$.