Quiz #2 (Key)

Exercise#1 [2 marks]. If $\alpha > 0$, $\beta > 0$, determine the following limit:

$$\lim_{n\to\infty}\left(\sqrt{(n+\alpha)(n+\beta)} - n\right).$$

0.5

$$= \lim_{n\to\infty} \frac{(n+\alpha)(n+\beta) - n^2}{\sqrt{(n+\alpha)(n+\beta)} + n}$$

0.5 0.5

$$= \lim_{n\to\infty} \frac{(\alpha+\beta)n + \alpha\beta}{\sqrt{(n+\alpha)(n+\beta)} + n} = \lim_{n\to\infty} \frac{\alpha+\beta+\frac{\alpha\beta}{n}}{\sqrt{(1+\frac{\alpha}{n})(1+\frac{\beta}{n})} + 1}$$

0.5

$$= \frac{\alpha+\beta}{2}.$$

Exercise#2 [4 marks]. Use the **definition of a limit** to prove:

$$\lim_{n\to\infty} \frac{1}{\ln(n^2+n)} = 0.$$

Let $\varepsilon > 0$ be given. By the Archimedean Principle, there is an $N \in \mathbb{N}$ s.t. $N > e^{1/\varepsilon}$. 2

Thus, $n \geq N$ implies

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$$\left|\frac{1}{\ln(n^2+n)} - 0\right| = \frac{1}{\ln(n^2+n)} < \frac{1}{\ln n} < \frac{1}{\ln N} < \varepsilon$$

Exercise#3 [9 marks]. Prove or disprove.

(a) If the sequence $\{x_n\}$ converges to $a > 0$, then there exists a natural number K such that if $n \geq K$, then $\frac{1}{2}a < x_n < 2a$.

(1 True.

pf. If $\varepsilon := \frac{a}{2} > 0$, then $n \geq N$ implies that

(2 $|x_n - a| < \frac{a}{2}$ which is equivalent to $\frac{a}{2} < x_n < \frac{3a}{2} < 2a$.

(b) If $y_n := \sqrt{n+1} - \sqrt{n}$, for $n \in \mathbb{N}$, then $\sqrt{n}\, y_n$ converges.

(1 True.

pf. (2 $\sqrt{n}\, y_n = \dfrac{\sqrt{n}}{\sqrt{n+1} + \sqrt{n}} = \dfrac{1}{\sqrt{1+\frac{1}{n}} + 1} \longrightarrow \frac{1}{2}$ as $n \to \infty$.

(c) Let $\{x_n\}$ and $\{y_n\}$ be sequences of positive numbers such that

$$\lim_{n\to\infty} \frac{x_n}{y_n} = +\infty.$$

If $\lim_{n\to\infty} y_n = +\infty$, then $\lim_{n\to\infty} x_n = +\infty$.

(1 True.

pf. Since $\lim \frac{x_n}{y_n} = +\infty$, $\exists\, K_1 \in \mathbb{N}$ s.t $x_n \geq y_n$.

(2 Since $\lim_{n\to\infty} y_n = +\infty$, then if $M \in \mathbb{R}$ is given then $\exists\, K_2 \in \mathbb{N}$ s.t if $n \geq K_2$, then $y_n > M$. Since $x_n \geq y_n$, it follows that $x_n > M$ for all $n > K_2$. Since M is an arbitrary, it follows that $\lim_{n\to\infty} x_n = +\infty$