Chapter 1 : The Real Number system .

1.2 : Ordered Field Axioms .

defined بواسطة

on $\mathbb{R}^2 := \mathbb{R}\times\mathbb{R}$

★ Postulate 1 [Field Axioms] :

$\forall$ a, b, c , there are functions + and $\cdot$ (ضرب عادي) on $\mathbb{R}^2$ defined with the properties :

1. $a+b\,,\ a\cdot b \in \mathbb{R}$ (Closure properties)

2. $a+(b+c) = (a+b)+c$ and $(a.b).c = a.(b.c)$. (Associative property)

3. $a+b = b+a$ and $a.b = b.a$. (Commutative property)

4. $a.(b+c) = ab+ac$. (Distributive Law) .

5. $\exists!\ 0\in\mathbb{R}$ s.t $a+0=a$, $\forall a\in\mathbb{R}$. (Existence of the Additive Identity)

6. $\exists!\ 1\in\mathbb{R}$ s.t $1\neq 0$ and $1.a = a$, $\forall a\in\mathbb{R}$ (Existence of the Multiplicative identity) .

7. $\forall a\in\mathbb{R}$, $\exists\ -a\in\mathbb{R}$ s.t $a+-a=0$. (Existence of Additive Inverses)

8. $\forall a\in\mathbb{R}\setminus\{0\}$, $\exists!\ a^{-1}\in\mathbb{R}$ s.t $aa^{-1}=1$ (Existence of Multiplicative Inverses)

H.W : prove , use postulate 1, to derive :

1. $(-1)^2 = 1$, $0.a = 0$. $\forall a \in \mathbb{R}$.

proof $0.a = 0$:

$0.a + a$	additive identity
$= 0a + 1.a$	Multiplicative identity .
$= a.0 + a.1$	Commutative property.
$= a(0+1)$	Distributive law .
$= a.1$	additive identity
$= 1.a$	commutative property
$= a$	multiplicative identity .

$\Rightarrow$ $0a = 0$, since the additive identity is unique and since we just proved that $a + 0a = a$, then $0a = 0$

proof $(-1)^2 = 1$:

$(-1)^2 a$

$= (-1)((-1)(a))$ Associative property

$= (-1)(-a)$, $-a = (-1)(a)$.

$= -(-a)$, $-a = (-1)(a)$

$= a$, $-(-a) = a$

$\Rightarrow$ $(-1)^2 = 1$

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2. $-a = (-1).a \quad , -(-a) = a \quad , -(a-b) = b-a \quad . \quad , \forall a,b \in \mathbb{R}$.

* proof $-a = (-1).a$.

$a + (-1)a$

$= (1)a + (-1)a$

$= a(1) + a(-1)$

$= a\,(1 + (-1))$

$= a\,(0) = 0$

$\Longrightarrow -a = (-1)a$. ■

* proof $-(-a) = a$.

$-(-a) + (-a)$

$= (-1)(-a) + (1)(-a)$, $-a = (-1)a$, Multiplicative identity

$= (-a)(-1) + (-a)(1)$, commutitative property

$= (-a)((-1) + (1))$, Distributive law

$= (-a) . 0$, Additive inverse

$= 0 . (-a)$, Commutative property .

$= 0$, $0a = 0$

$\Longrightarrow -(-a) = a$ ■

★. proof $-(a-b) = b-a$:

$(a-b) + (b-a)$

$= (a + (-1)b) + (b + (-1)a)$, $-a = (-1)a$

$= a + ((-1)b + b) + (-1)a$, Associative property

$= a + (b + (-1)b) + (-1)a$, Commutative property

$= a + (b + (-b)) + (-1)a$, $-a = (-1)a$

$= (a + 0) + (-1)a$, Additive inverse.

$= (0 + a) + (-1)a$, Commutative property

$= a + (-1)a$, Additive identity

$= a + (-a)$, $-a = (-1)a$.

$= 0$, Additive inverse.

$\Rightarrow -(a-b) = b-a$ ∎

3. If $a, b \in \mathbb{R}$ and $ab = 0$ then $a = 0$ or $b = 0$.

suppose $ab = 0$

If $a = 0$, we are done.

If $a \neq 0$, we will show that $b = 0$

$\underline{b} = b \cdot 1 = b \cdot (a \cdot \frac{1}{a}) = (ba)\frac{1}{a} = (ab)\frac{1}{a} = 0 \cdot \frac{1}{a}$ By hypothes

$= 0 \cdot \frac{1}{a} = 0$ since $0 \cdot c = 0$, $c \in \mathbb{R}$.

$\Rightarrow b = 0$ ∎

★ postulate 2 : [order Axioms].

$\exists$ a relation $<$ on $\mathbb{R}\times\mathbb{R}$ s.t :

(i) $\forall a,b \in \mathbb{R}$, exactly one of the following is true : $a<b$, $a>b$ or $a=b$ (Trichotomy)

(ii) $\forall a,b,c \in \mathbb{R}$, $a<b$ and $b<c \Rightarrow a<c$. (Transitive)

(iii) $\forall a,b,c \in \mathbb{R}$, $a<b$ and $c \in \mathbb{R}$ then $a+c < b+c$ (Additive).

(iv) $\forall a,b,c \in \mathbb{R}$ → $a<b$, $c>0 \Rightarrow ac<bc$

→ $a<b$, $c<0 \Rightarrow ac > bc$. (multiplicative)

RMK :

- $a \leq b$ means $a<b$ or $a=b$.
- $a<b<c$ means $a<b$ and $b<c$.
  exp : $2<x<1$, makes no sense .
- $a \in \mathbb{R}$ nonnegative if $a \geq 0$

  $a \in \mathbb{R}$ positive if $a>0$.

RMK :

① The set of Natural numbers , $\mathbb{N} := \{1, 2, 3, 4, \dots\}$.

② The set of integers , $\mathbb{Z} = \{0, \pm1, \pm2, \pm3, \dots\}$

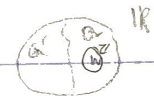
③ The set of Rationals , $\mathbb{Q} = \{ \frac{a}{b} , b \neq 0 , a, b \in \mathbb{Z} \}$.

④ The set of irrationals , $\mathbb{Q}^c = \mathbb{R} \setminus \mathbb{Q}$ exp: $\sqrt{2}, e, \pi$.

Notes :

- Equality in Q is defined ; $\frac{m}{n} \cancel{=} \frac{p}{q} \iff mq = np$.

- $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \overset{\text{subset}}{\subsetneq} \mathbb{R}$.



RMK : $\mathbb{N}$ and $\mathbb{Z}$ satisfy :

(i) $n, m \in \mathbb{Z}$, then $n+m$, $n-m$ and $m.n \in \mathbb{Z}$. close under additio
// // Multiplica

(ii) if $n \in \mathbb{Z}$, then $n \in \mathbb{N} \iff n \geq 1$.

(iii) $\nexists$ (there) is no $n \in \mathbb{Z}$ satisfies $0 < n < 1$.

exp : Check post. 1 for $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{Q}^c$.

$\mathbb{Q}$ satisfy postulate 1 .

exp: If $a \in \mathbb{R}$ prove that, $a \neq 0 \Rightarrow a^2 > 0$, In particular $-1 < 0 < 1$.

prove.

suppose that $a \neq 0$, Then $a > 0$ or $a < 0$

case 1: $a > 0$

$a.a > 0.a$ (multiplicative)

$a^2 > 0$ (Faild axioms).

case 2: $a < 0$ → a negative

$a.a \leq a.0$ → تنعكس لأنه a

$a^2 > 0$

This proves $a^2 > 0$, for $a \neq 0$.

In particular, take $a = 1 \neq 0$

حسب الي أثبتناه فوق انه $a^2 > 0$ بطلع يعني $1^2 = 1$ $1 > 0$

$1^2 = 1 > 0$ ... (i)

and -1 to both side $-1+1 > -1+0$

$0 > -1$ ... (ii)

(i) and (ii) give $-1 < 0 < 1$ #.

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exp: If $a \in \mathbb{R}$, prove that:

(i) $0 < a < 1 \Rightarrow 0 < a^2 < a$

spse $0 < a < 1$

Then $0.a < a.a < 1.a$

$\Rightarrow 0 < a^2 < a$

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(ii) $a > 1 \Rightarrow a^2 > a$.

$a > 1$ and $1 > 0$

then $a > 0$

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The $a.a > a.1$

$\Rightarrow a^2 > a$.

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Q1.2.2 (Exercises)

H.W : Prove the following.

1. $0 \leq a < b$ and $0 \leq c < d \Rightarrow ac < bd$,

suppose $(0 \leq a < b) \cdot c$ and $(0 \leq c < d) \cdot b$

$\Rightarrow 0 \leq ac < bc$ and $0 \leq cb < db$

$\Rightarrow ac < bc$ and $bc < db$

$\Rightarrow ac < bd$ (By transitive property (order axiom))

2. $0 \leq a < b \Rightarrow 0 \leq a^2 < b^2$ and $0 \leq \sqrt{a} \leq \sqrt{b}$.

$0 \leq a < b \rightarrow 0 \leq a^2 < b^2$

$0 \leq a < b \Rightarrow 0 \leq \sqrt{a} \leq \sqrt{b}$

By contradiction.

let $\sqrt{a} \geq \sqrt{b}$

$(\sqrt{a})^2 \geq (\sqrt{b})^2$

But $a \geq b$ X

contradicts By $a < b$

3. $0 < a < b$ then $\frac{1}{a} > \frac{1}{b} > 0$.

part 1 spse $a < b$ By contradiction

and let $\frac{1}{a} \leq \frac{1}{b}$

By Multiplicative property implies

$$\left(\frac{1}{a}\right) \cdot ab \leq \left(\frac{1}{b}\right) \cdot ab$$

$$\frac{ab}{\cancel{a}} \leq \frac{a\cancel{b}}{\cancel{b}} \rightarrow b \leq a \quad \times. \text{ with } a < b.$$

part 2:

If $\left(\frac{1}{b} \leq 0\right) \cdot b^2 \rightarrow b \leq 0 \quad \times.$

4. show each of these statements is false if the hypothesis $a \geq 0$ or $a > 0$ is removed.

To show it not hold when $a < 0$

→ let $a = -2$, $b = -1$, $c = 2$, $d = 5$ Then

$a < b$ and $c < d$ but $ac > bd$

→ $a^2 = 4$, $b^2 = 1$ → $a^2 > b^2$

→ $\frac{1}{a} = \frac{-1}{2}$, $\frac{1}{b} = -1$ → $\frac{1}{a} > \frac{1}{b}$.

Def: The absolute value of $a \in \mathbb{R}$ is $|a| := \begin{cases} a, & a \geq 0 \\ -a, & a < 0. \end{cases}$

RMK: The absolute value is multiplicative, i.e

$|ab| = |a||b|$, $\forall a, b \in \mathbb{R}$.

Proof: We consider 4 casses.

case 1: $a = 0$ or $b = 0$ then $ab = 0$

So By def $|ab| = 0 = |a||b|$.

case 2: $a > 0$ and $b > 0$, Then $a.b > 0.b \rightarrow ab > 0$

Hence, By def $|ab| = ab = |a||b|$.

case 3: $(a > 0$ and $b < 0)$ or $(b > 0$ and $a < 0)$

suppose $a > 0$ and $b < 0$ then $ab < 0.b \rightarrow ab < 0$

Hence, By def, H.w 1, Associativity and commutativity:

$|ab| = -(ab) = (-1)(ab) = a((-1)b) = a(-b) = |a||b|$.

case 4: $a < 0$ and $b < 0$. Then $a.b > 0.b \rightarrow ab > 0$

Hence, by def $|ab| = ab = (-1)^2(ab) = (-a)(-b) = |a||b|$.

Thm 1 : Let $a \in \mathbb{R}$ and $M \geq 0$ Then $|a| \leq M \Leftrightarrow -M \leq a \leq M$.

proof :

$\boxed{\Longrightarrow}$ spse $|a| \leq M$

Then $-|a| \geq -M$

case 1 : $a \geq 0$

$|a| = a$, By Def of absolute value

Those, $-M \leq 0$ hypothesis $M \geq 0$

$0 \leq a$ assumption $a \geq 0$

$a = |a|$

$|a| \leq M$ By assumming

$\Rightarrow -M \leq a \leq M$

case 2 : spse, $a < 0$

Thus, $|a| = -a$ By Def

Thus, $-M \leq -|a|$ By assumming

$-|a| = -(-a)$

$-(-a) = a$

$a < 0$

$0 \leq M$

This prove $-M \leq a \leq M$ in either case

$\boxed{\Longleftarrow}$ Conversely, spse that $-M \leq a \leq M$

Then $a \leq M$ and $-M \leq a$

multiply the second inequality by -1

we get $-a \leq M$

case 1 : $a \geq 0$

$|a| = a \leq M$

we are done.

case 2 : $a < 0$

$|a| = -a \leq M$.

This prove $|a| \leq M$ in either case.

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H.W: one can prove that : $|a| < M \iff -M < a < M$.

Thm 2 :

(i) (positive Definite) : $\forall a \in \mathbb{R}$, $|a| \geq 0$ with $|a| = 0 \iff a = 0$.

(ii) (symmetric) : $\forall a, b \in \mathbb{R}$, $|a-b| = |b-a|$.

imp (iii) (Triangle inequality) : $|a+b| \leq |a| + |b|$ and $\big||a| - |b|\big| \leq |a-b|$.

★ prove (i), (ii) exercise (textbook p.11).

proof (iii) : To prove the first ineq $|a+b| \leq |a| + |b|$:

notice that $|x| \leq |x|$, $\forall x \in \mathbb{R}$

Thus, by Thm 1 $\Rightarrow$ $\left.\begin{array}{l} -|a| \leq a \leq |a| \\ \text{and } -|b| \leq b \leq |b| \end{array}\right\} +$

Then $-(|a| + |b|) \leq \underbrace{(a+b)}_{a} \leq |a| + |b|$ =

Thm 1 again $|a+b| \leq |a| + |b|$.

To prove the second ineq $\big||a| - |b|\big| < |a-b|$

Apply the first ineq to $(a-b) + b$

$$|a| - |b| = |(a-b) + b| - |b|$$
$$\leq |a-b| + |b| - |b|$$
$$\therefore |a| - |b| \leq |a-b| \quad \text{.... (I)}$$

conti. $|a|-|b| \leq |a-b|$

reverse the roles of a and b.

$\Rightarrow |b|-|a| \leq |b-a| = |a-b| \rightarrow$ By II

$\Longrightarrow -(|a|-|b|) \leq |a-b|$ .... (II) $\Rightarrow -|a-b| \leq |a|-|b|$

(I) and (II) give $-|a-b| \overset{\text{By (II)}}{\leq} |a|-|b| \overset{\text{By (I)}}{\leq} |a-b|$

$\Rightarrow \quad \big||a|-|b|\big| \leq |a-b|$ ∎

Warning: $b < c \not\Rightarrow |a+b| < |a+c|$

exp: $-5 < 1$ But $|-5+1| \not< |1+1|$

T/F

① If $b<c$, then $a+b < a+c$ T

② If $b<c$, then $|a+b| < |a+c|$ F

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exp: Prove that if $-2 < x < 1$, then $|x^2 - x| \leq 6$.

pf: since $-2<x<1$ and $1<2$ then $-2<x<2$.

i.e, $|x| < 2$.

Now: $|x^2 - x| \leq |x^2| + |-x|$ Triangle equality

$= |x|^2 + |x|$

$\leq (2)^2 + 2 = 6$.

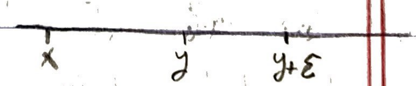
مهم كثير

Thm 3: let $x, y, a \in \mathbb{R}$ Then:

(i) $x < y + \varepsilon, \ \forall \varepsilon > 0 \iff x \leq y$.

$x - y < \varepsilon, \ \forall \varepsilon > 0$

$\Rightarrow x - y < 0 \rightarrow (x \leq y)$.



(ii) $x > y - \varepsilon, \ \forall \varepsilon > 0 \iff x \geq y$

(iii) $|a| < \varepsilon, \ \forall \varepsilon > 0 \iff a = 0$

نصاً + برهاناً

بستخدمها إذا طلب اثبات a=0

prove:

(i) $\Rightarrow$ spse $x < y + \varepsilon \quad \forall \varepsilon > 0$ but $x > y$ (contradiction).

set $\varepsilon_0 = x - y > 0 \rightarrow$ $\left(\begin{matrix} x > y \\ -y \;\; -y \end{matrix} \rightsquigarrow x - y > 0 \right)$ كيف اجت

and observe that $x = y + \varepsilon_0$

Hence, by the Trichotomy property, $x \nless y + \varepsilon_0$

This is contradicts the hypothesis for $\varepsilon = \varepsilon_0$ Henc $x \leq y$

$\Leftarrow$ conversly, spse that $x \leq y$ and $0 < \varepsilon$

Then $x < y$ or $x = y$

if $x = y$ then $x < y + \varepsilon$

if $x < y$ and $0 < \varepsilon$, then $x + 0 < y + \varepsilon$

$x < y + \varepsilon \quad \forall \varepsilon > 0$ in either case.

#

(ii) spse that $x > y - \varepsilon, \quad \forall \varepsilon > 0$

Then $-x < -y + \varepsilon$

use (i), $-x < -y + \varepsilon \Leftrightarrow -x \leq -y \Leftrightarrow x \geq y$.

(iii) $|a| < \varepsilon = 0 + \varepsilon$ — By (i) $x < y + \varepsilon \Leftrightarrow x \leq y$

by part (i), $|a| \leq 0 \rightarrow$ $\underbrace{|a|}_{x} < \varepsilon = \underbrace{0}_{y} + \varepsilon$

and we know, $|a| \geq 0$ — $|a| < 0 + \varepsilon \Leftrightarrow |a| \leq 0$

so we have $|a| \leq 0$ and $|a| \geq 0$

Trichotomy property $\Rightarrow |a| = 0$

$|a| = 0 \Leftrightarrow a = 0$ (we know)

So $a = 0$

conversely, spse $a = 0$, $\varepsilon > 0$, we need to prove $|a| < \varepsilon$

$a = 0 \Rightarrow |a| = |0| = 0 < \underbrace{\varepsilon}_{\text{assumption}}$.

Def: Let $a, b \in \mathbb{R}$, A closed interval is of the form:

$$[a,b] := \{x : a \le x \le b\}.$$

$$[a,\infty) := \{x : x \ge a\}$$

مغلقات

$$(-\infty, b] := \{x : x \le b\}.$$

$$(-\infty,\infty) := \{x : x \in \mathbb{R}\}.$$

★ open interval:

$$(a,b) := \{x \in \mathbb{R} : a < x < b\}$$
$$(a,\infty) := \{x \in \mathbb{R} : x > a\}$$
$$(-\infty,b) := \{x \in \mathbb{R} : x < b\}$$
$$(-\infty,\infty) := \{x \in \mathbb{R}\}.$$

★
- $[a,b) := \{x \in \mathbb{R} : a \le x < b\}$
- $(a,b] := \{x \in \mathbb{R} : a < x \le b\}.$

★ • An interval I is bounded iff it has the form $[a,b]$, (a,b), $[a,b)$ or $(a,b]$ for $-\infty < a \le b < \infty$.

• If $a=b$, then an interval I is degenerate and nondegenerate if $a<b$.

• The length of a bounded interval I with endpts a and b is $|I| := |b-a|$.

• RMK: by Thm 1: $|a| \le M \Leftrightarrow a \in [-M, M]$.