

8.7 Improper Integrals (Type I and Type II)

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Def: Improper Integrals of Type I are integrals with infinite limits of integration:

[1] $\int_a^\infty f(x) dx = \lim_{b \rightarrow \infty} \int_a^b f(x) dx$ where f is continuous on $[a, \infty)$

[2] If f is continuous on $(-\infty, a)$, then

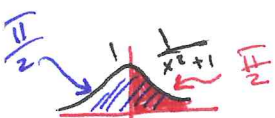
$$\int_{-\infty}^a f(x) dx = \lim_{b \rightarrow -\infty} \int_b^a f(x) dx$$

[3] If f is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^c f(x) dx + \int_c^{\infty} f(x) dx, \quad c \in \mathbb{R}$$

In each case, if the limit is finite, then the improper integral converges and it is equal to this limit "Area".

otherwise the improper integral diverges "Infinite Area".

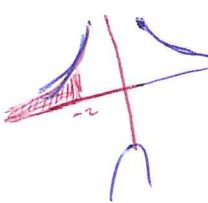


Exp ① $\int_0^\infty \frac{dx}{x^2+1} = \lim_{b \rightarrow \infty} \int_0^b \frac{dx}{x^2+1} = \lim_{b \rightarrow \infty} \tan^{-1} x \Big|_0^b = \lim_{b \rightarrow \infty} \tan^{-1} b = \frac{\pi}{2}$

② $\int_{-\infty}^{-2} \frac{2 dx}{x^2-1} = \int_{-\infty}^{-2} \frac{2 dx}{(x-1)(x+1)} = \frac{2}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$ $A=1$
 $B=-1$

$$= \int_{-\infty}^{-2} \left(\frac{1}{x-1} - \frac{1}{x+1} \right) dx$$

$$= \lim_{b \rightarrow -\infty} \int_b^{-2} \left(\frac{1}{x-1} - \frac{1}{x+1} \right) dx$$



$$= \lim_{b \rightarrow -\infty} \left(\ln|x-1| - \ln|x+1| \right) \Big|_b^{-2}$$

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$$= \lim_{b \rightarrow -\infty} \left[\ln \left| \frac{x-1}{x+1} \right| \right] \Big|_b^{-2} = \lim_{b \rightarrow -\infty} \left[\ln \left| \frac{-3}{-1} \right| - \ln \left| \frac{b-1}{b+1} \right| \right]$$

$$= \ln 3 - \ln \lim_{b \rightarrow -\infty} \frac{b-1}{b+1} = \ln 3 - \ln 1 = \ln 3$$

Note that $\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^0 \frac{dx}{1+x^2} + \int_0^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi$

Exp $\int_0^{\infty} \frac{16 \tan^{-1} x}{1+x^2} dx = \lim_{b \rightarrow \infty} \int_0^b \frac{16 \tan^{-1} x}{1+x^2} dx$ $u = \tan^{-1} x$
 $du = \frac{1}{1+x^2} dx$

$$= \lim_{b \rightarrow \infty} \int_{\tan^{-1} 0}^{\tan^{-1} b} 16 u du = \lim_{b \rightarrow \infty} 8 u^2 \Big|_0^{\tan^{-1} b}$$

$$= \lim_{b \rightarrow \infty} 8 \left[(\tan^{-1} b)^2 - (\tan^{-1} 0)^2 \right] = 8 \left(\frac{\pi}{2} \right)^2 = 2\pi^2$$

Def Improper Integrals of Type II are integrals of functions that become infinite at a point within the interval of integration (Vertical Asymptotes)

① If f is discontinuous at a , then $\int_a^b f(x) dx = \lim_{c \rightarrow a^+} \int_c^b f(x) dx$

② If f is discontinuous at b , then $\int_a^b f(x) dx = \lim_{c \rightarrow b^-} \int_a^c f(x) dx$

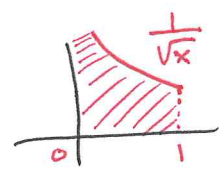
③ If f is discontinuous at c , where $a < c < b$, then $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$

In each case, if the limit is finite, then the improper integral converges and it is equal to this limit "Area".

Otherwise the improper integral diverges "infinite area"

Exp $\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{c \rightarrow 0^+} \int_c^1 \frac{dx}{\sqrt{x}} = \lim_{c \rightarrow 0^+} 2\sqrt{x} \Big|_c^1$

$$= \lim_{c \rightarrow 0^+} (2 - 2\sqrt{c}) = 2 - 0 = 2$$



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Exp $\int_0^4 \frac{dx}{\sqrt{4-x}} = \lim_{b \rightarrow 4^-} \int_0^b \frac{dx}{\sqrt{4-x}} = \lim_{b \rightarrow 4^-} -2\sqrt{4-x} \Big|_0^b$

$$= \lim_{b \rightarrow 4^-} [-2\sqrt{4-b} + 4] = 4$$

Exp* For what values of p does the integral $\int \frac{dx}{x^p}$ converge?

$p \neq 1 \Rightarrow \int_1^{\infty} \frac{dx}{x^p} = \lim_{c \rightarrow \infty} \int_1^c x^{-p} dx = \lim_{c \rightarrow \infty} \frac{x^{1-p}}{1-p} \Big|_1^c$

$$= \lim_{c \rightarrow \infty} \left[\frac{c^{1-p}}{1-p} - \frac{1}{1-p} \right] = \begin{cases} \frac{1}{p-1}, & p > 1 \\ \infty, & p < 1 \end{cases}$$

Diverges $p=1 \Rightarrow \int_1^{\infty} \frac{dx}{x} = \lim_{b \rightarrow \infty} \int_1^b \frac{dx}{x} = \lim_{b \rightarrow \infty} \ln|x| \Big|_1^b = \lim_{b \rightarrow \infty} \ln|b| = \infty$

Exp $\int_1^{\infty} \frac{dx}{x\sqrt{x^2-1}} = \int_1^2 \frac{dx}{x\sqrt{x^2-1}} + \int_2^{\infty} \frac{dx}{x\sqrt{x^2-1}}$

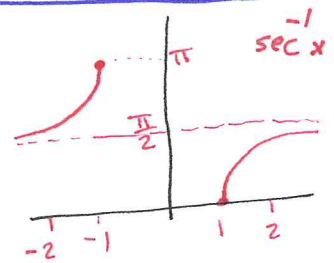
$$= \lim_{b \rightarrow 1^+} \int_b^2 \frac{dx}{x\sqrt{x^2-1}} + \lim_{c \rightarrow \infty} \int_2^c \frac{dx}{x\sqrt{x^2-1}}$$

$$= \lim_{b \rightarrow 1^+} \left[\sec^{-1}|x| \right]_b^2 + \lim_{c \rightarrow \infty} \left[\sec^{-1}|x| \right]_2^c$$

$$= \lim_{b \rightarrow 1^+} \left[\cancel{\sec^{-1} 2} - \cancel{\sec^{-1} b} \right] + \lim_{c \rightarrow \infty} \left[\cancel{\sec^{-1} c} - \cancel{\sec^{-1} 2} \right]$$

$$= \lim_{b \rightarrow 1^+} -\sec^{-1} b + \lim_{c \rightarrow \infty} \sec^{-1} c$$

$$= 0 + \frac{\pi}{2} = \frac{\pi}{2}$$



Tests for Convergence and Divergence

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Th (Direct Comparison Test)

Let f and g be continuous on $[a, \infty)$ with
 $0 \leq f(x) \leq g(x)$ for all $x \geq a$.

* If $\int_a^{\infty} g(x) dx$ converges, then $\int_a^{\infty} f(x) dx$ converges.

* If $\int_a^{\infty} f(x) dx$ diverges, then $\int_a^{\infty} g(x) dx$ diverges.

Exp $\int_1^{\infty} \frac{\sin^2 x}{x^2} dx$ converges because $0 \leq \frac{\sin^2 x}{x^2} \leq \frac{1}{x^2}$ on $[1, \infty)$
 and $\int_1^{\infty} \frac{1}{x^2} dx$ converges "see exp *"
 Thus, by the Direct Comparison Test $\int_1^{\infty} \frac{\sin^2 x}{x^2} dx$ converges

Exp $\int_1^{\infty} \frac{1}{\sqrt{x^2 - 0.1}} dx$ diverges because $\frac{1}{\sqrt{x^2 - 0.1}} \geq \frac{1}{x}$ on $[1, \infty)$
 and $\int_1^{\infty} \frac{1}{x} dx$ diverges "see exp *"
 Thus, by the Direct Comparison Test, $\int_1^{\infty} \frac{dx}{\sqrt{x^2 - 0.1}}$ diverges

Exp $\int_0^{\pi} \frac{dt}{\sqrt{t} + \sin t}$ converges because $0 \leq \frac{1}{\sqrt{t} + \sin t} \leq \frac{1}{\sqrt{t}}$ on $[0, \pi]$
 and $\int_0^{\pi} \frac{dt}{\sqrt{t}} = \lim_{b \rightarrow 0^+} \int_b^{\pi} t^{-\frac{1}{2}} dt = \left[2\sqrt{t} \right]_b^{\pi} = \lim_{b \rightarrow 0^+} (2\sqrt{\pi} - 2\sqrt{b}) = 2\sqrt{\pi}$
 which converges, Thus, by the Direct Comparison test, $\int_0^{\pi} \frac{dt}{\sqrt{t} + \sin t}$ converges.

2. Th (limit Comparison Test)

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If • f and g are positive and continuous functions on $[a, \infty)$

• $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = L$ where $0 < L < \infty$,

then $\int_a^{\infty} f(x) dx$ and $\int_a^{\infty} g(x) dx$ both converge or both diverge.

Exp $\int_1^{\infty} \frac{dx}{1+x^2}$

• let $f(x) = \frac{1}{x^2}$, $g(x) = \frac{1}{1+x^2}$

• f and g are positive continuous on $[1, \infty)$

• $\lim_{x \rightarrow \infty} \frac{f}{g} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x^2}}{\frac{1}{1+x^2}} = \lim_{x \rightarrow \infty} \frac{1+x^2}{x^2} = 1$ "finite +"

and $\int_1^{\infty} \frac{dx}{x^2}$ converges "see exp*" . Thus, by the Limit Comparison Test $\Rightarrow \int_1^{\infty} \frac{dx}{1+x^2}$ converges.

Exp $\int_2^{\infty} \frac{dx}{\sqrt{x-1}}$

• Let $f(x) = \frac{1}{\sqrt{x-1}}$, $g(x) = \frac{1}{\sqrt{x}}$

• f and g are positive continuous on $[2, \infty)$

• $\lim_{x \rightarrow \infty} \frac{f}{g} = \lim_{x \rightarrow \infty} \frac{\frac{1}{\sqrt{x-1}}}{\frac{1}{\sqrt{x}}} = \lim_{x \rightarrow \infty} \frac{\sqrt{x}}{\sqrt{x-1}} = \lim_{x \rightarrow \infty} \frac{1}{\sqrt{1-\frac{1}{x}}} = 1$

and $\int_2^{\infty} \frac{dx}{\sqrt{x}} = \lim_{b \rightarrow \infty} \int_2^b x^{-\frac{1}{2}} dx = \lim_{b \rightarrow \infty} 2\sqrt{x} \Big|_2^b = \lim_{b \rightarrow \infty} [2\sqrt{b} - 2\sqrt{2}] = +\infty$

which diverges $\Rightarrow \int_2^{\infty} \frac{dx}{\sqrt{x-1}}$ diverges by the limit Comparison Test.