1. Suppose that $a, b \in R$, $a < b$ and that $f : [a, b] \to R$ is bounded. What does it mean to say that f is (Riemann) integrable on $[a, b]$?

f is Riemann integrable on $[a,b]$ if for every $\varepsilon > 0$, there exists a partition P on $[a,b]$ such that:

$$U(f,P) - L(f,P) < \varepsilon.$$

2. Let $f(x) = \begin{cases} 2 & 0 \leq x < 1 \\ 5 & 1 \leq x \leq 2 \end{cases}$

Prove, using the definition, that f is integrable on $[0, 2]$.

Let $\epsilon > 0$.

Take the partition P on $[0,2]$ s.t.

$P = \{0, x_1, x_2, 2\}$ where $0 < x_1 < 1 < x_2 < 2$, and $x_2 - x_1 < \frac{\varepsilon}{3}$, since:

$0 < \frac{\varepsilon}{3} + x_1$, so there is a real number x_2 s.t. $0 < x_2 < \frac{\varepsilon}{3} + x_1 \Rightarrow x_2 - x_1 < \frac{\varepsilon}{3}$.

$$U(f,P) - L(f,P) = (M_1 - m_1)(x_1 - 0) + (M_2 - m_2)(x_2 - x_1) + (M_3 - m_3)(2 - x_2)$$

Where:

$M_1 = \sup_{[0,x_1]}(f) = 2$, $M_2 = \sup_{[x_1,x_2]}(f) = 5$, $M_3 = \sup_{[x_2,2]}(f) = 5$.

$m_1 = \inf_{[0,x_1]}(f) = 2$, $m_2 = \inf_{[x_1,x_2]}(f) = 2$, $m_3 = \inf_{[x_2,2]}(f) = 5$.

$$\rightarrow U(f,P) - L(f,P) = \underbrace{\cancel{(2-2)x_1}}_{0} + (5-2)(x_2 - x_1) + \underbrace{\cancel{(5-5)(2-x_2)}}_{0}$$

$$= 3(x_2 - x_1)$$

$$< 3 \cdot \frac{\varepsilon}{3}$$

$$= \varepsilon.$$

So $\forall \varepsilon > 0$, $P = \{0, x_1, x_2, 2\}$ on $[0,2]$ satisfyes $U(f,P) - L(f,P) < \varepsilon$.

So $f(x)$ is Riemann integrable on $[0,2]$. #