1.4 Matrix Algebra

Theorem 1.4.1 *Each of the following statements is valid for any scalars α and β and for any matrices A, B, and C for which the indicated operations are defined.*

1. $A + B = B + A$
2. $(A + B) + C = A + (B + C)$
3. $(AB)C = A(BC)$
4. $A(B + C) = AB + AC$
5. $(A + B)C = AC + BC$
6. $(\alpha\beta)A = \alpha(\beta A)$
7. $\alpha(AB) = (\alpha A)B = A(\alpha B)$
8. $(\alpha + \beta)A = \alpha A + \beta A$
9. $\alpha(A + B) = \alpha A + \alpha B$

EXAMPLE 1 If

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}, \qquad \text{and} \qquad C = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

verify that $A(BC) = (AB)C$ and $A(B + C) = AB + AC$.

Solution

$$A(BC) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 16 & 11 \end{bmatrix}$$

$$(AB)C = \begin{bmatrix} -4 & 5 \\ -6 & 11 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 16 & 11 \end{bmatrix}$$

$$A(B + C) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 7 \\ 5 & 15 \end{bmatrix}$$

$$AB + AC = \begin{bmatrix} -4 & 5 \\ -6 & 11 \end{bmatrix} + \begin{bmatrix} 5 & 2 \\ 11 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 7 \\ 5 & 15 \end{bmatrix}$$

Notation

Since $(AB)C = A(BC)$, we may simply omit the parentheses and write ABC. The same is true for a product of four or more matrices. In the case where an $n \times n$ matrix is multiplied by itself a number of times, it is convenient to use exponential notation. Thus, if k is a positive integer, then

$$A^k = \underbrace{AA \cdots A}_{k \text{ times}}$$

EXAMPLE 2 If

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

then

$$A^2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

$$A^3 = AAA = AA^2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}$$

and in general

$$A^n = \begin{bmatrix} 2^{n-1} & 2^{n-1} \\ 2^{n-1} & 2^{n-1} \end{bmatrix}$$

The Identity Matrix

Just as the number 1 acts as an identity for the multiplication of real numbers, there is a special matrix I that acts as an identity for matrix multiplication; that is,

$$IA = AI = A \tag{4}$$

for any $n \times n$ matrix A. It is easy to verify that, if we define I to be an $n \times n$ matrix with 1's on the main diagonal and 0's elsewhere, then I satisfies equation (4) for any $n \times n$ matrix A. More formally, we have the following definition:

Definition

The $n \times n$ **identity matrix** is the matrix $I = (\delta_{ij})$, where

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Remark. (The role of identity matrices in matrix arithmetic)

If A is any $m \times n$ matrix, then

$$AI_n = A \qquad \text{and} \qquad I_mA = A$$

As an example, let us verify equation (4) in the case $n = 3$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 & 1 \\ 2 & 6 & 3 \\ 0 & 1 & 8 \end{bmatrix} = \begin{bmatrix} 3 & 4 & 1 \\ 2 & 6 & 3 \\ 0 & 1 & 8 \end{bmatrix}$$

and

$$\begin{bmatrix} 3 & 4 & 1 \\ 2 & 6 & 3 \\ 0 & 1 & 8 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 4 & 1 \\ 2 & 6 & 3 \\ 0 & 1 & 8 \end{bmatrix}$$

The column vectors of the $n \times n$ identity matrix I are the standard vectors used to define a coordinate system in Euclidean n-space. The standard notation for the jth column vector of I is $\mathbf{e}_j$, rather than the usual $\mathbf{i}_j$. Thus, the $n \times n$ identity matrix can be written

$$I = (\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n)$$

Matrix Inversion

Definition

An $n \times n$ matrix A is said to be **nonsingular** or **invertible** if there exists a matrix B such that $AB = BA = I$. The matrix B is said to be a **multiplicative inverse** of A.

If B and C are both multiplicative inverses of A, then

$$B = BI = B(AC) = (BA)C = IC = C$$

Thus, a matrix can have at most one multiplicative inverse. We will refer to the multiplicative inverse of a nonsingular matrix A as simply the *inverse* of A and denote it by A^{-1}.

Definition

An $n \times n$ matrix is said to be **singular** if it does not have a multiplicative inverse.

EXAMPLE 3 The matrices

$$\begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -\frac{1}{10} & \frac{2}{5} \\ \frac{3}{10} & -\frac{1}{5} \end{bmatrix}$$

are inverses of each other, since

$$\begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{10} & \frac{2}{5} \\ \frac{3}{10} & -\frac{1}{5} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} -\frac{1}{10} & \frac{2}{5} \\ \frac{3}{10} & -\frac{1}{5} \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

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EXAMPLE 5 The matrix

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

has no inverse. Indeed, if B is any 2×2 matrix, then

$$BA = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} b_{11} & 0 \\ b_{21} & 0 \end{bmatrix}$$

Thus, BA cannot equal I. ■

Theorem. The matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible if and only if

$$ad - bc \neq 0,$$

in which case the inverse is given by the formula

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example. Determine whether the following matrix is invertible. If so, find its inverse

$$A = \begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix}$$

Theorem 1.4.2 *If A and B are nonsingular $n \times n$ matrices, then AB is also nonsingular and $(AB)^{-1} = B^{-1}A^{-1}$.*

Proof

$$(B^{-1}A^{-1})AB = B^{-1}(A^{-1}A)B = B^{-1}B = I$$
$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AA^{-1} = I$$

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Remark.

A product of any number of invertible matrices is invertible, and the inverse of the product is the product of the inverses in the reverse order.

Powers of a Matrix.

If A is a square matrix, then we define the nonnegative integer powers of A to be

$$A^0 = I$$

and

$$A^n = \underbrace{A\,A\,\cdots\,A}_{n \text{ factors}}$$

and if A is invertible, then we define the negative integer powers of A to be

$$A^{-n} = (A^{-1})^n = \underbrace{A^{-1}\,A^{-1}\,\cdots\,A^{-1}}_{n \text{ factors}}$$

Remark

If A is a square matrix, then for nonnegative integers r and s,

$$A^r A^s = A^{r+s} \qquad \textit{and} \qquad (A^r)^s = A^{rs}$$

The Square of a Matrix Sum.

$(x+y)^2 = x^2 + 2xy + y^2$

$$(A+B)^2 = A^2 + AB + BA + B^2$$

$(A+B)^2 = (A+B)(A+B) = A^2 + AB + BA + B^2$

Example. Simplify $(2I + A)^2$.

$(2I+A)^2 = (2I+A)(2I+A) = 4I^2 + 2IA + 2AI + A^2$

$= 4I + 2A + 2A + A^2$

$= 4I + 4A + A^2$

Example. If $A^2 = A$, show that $2A - I$ is invertible and is its own inverse.

$A^2 = A \implies (2A-I)^{-1} = 2A - I$??

$(2A-I)(2A-I) = 4A^2 - 2AI - 2IA + I^2$

$= 4A - 2A - 2A + I$

$= I$

Theorem. If A is invertible and n is a nonnegative integer, then:

a) A^{-1} is invertible and

$$(A^{-1})^{-1} = A$$

b) A^n is invertible and

$$(A^n)^{-1} = A^{-n} = (A^{-1})^n$$

c) For any nonzero scalar k, kA is invertible and

$$(kA)^{-1} = k^{-1}A^{-1} = \frac{1}{k}A^{-1}$$

Example. If A is an invertible matrix and $A^3 = I$. Find A^{-3}.

Example. If A is invertible and $AB = O$. Show that $B = O$

Example. If A is invertible and $AB = AC$. Show that $B = C$

Theorem. If A is an invertible matrix, then A^T is also invertible and

$$(A^T)^{-1} = (A^{-1})^T$$

Proof.

EXERCISES

3. Find nonzero 2×2 matrices A and B such that $AB = O$.

4. Find nonzero matrices A, B, and C such that

$$AC = BC \qquad \text{and} \qquad A \neq B$$

14. Let A and B are $n \times n$ matrices. Show that if

$$AB = A \quad \text{and} \quad B \neq I$$

then A must be singular.

19. Let A be an $n \times n$ matrix. Show that if $A^2 = O$, then $I - A$ is nonsingular and $(I - A)^{-1} = I + A$.

24. A matrix A is said to be *idempotent* if $A^2 = A$. Show that each of the following matrices are idempotent.

(a) $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ **(b)** $\begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$