CH4

Applications of derivatives

4.1 Increasing and decreasing functions

Def: Let $f(x)$ be a function defined on an interval I then

(a) f is increasing on I if whenever

$x_2 > x_1 \Rightarrow f(x_2) > f(x_1)$

(b) f is decreasing on I if whenever

$x_2 > x_1 \Rightarrow f(x_2) < f(x_1)$

To determine the intervals where the function is increasing or decreasing, we use the derivatives

Th Suppose that f is continuous on $[a,b]$ and differentiable on (a,b), then

(a) If $f'(x) > 0$, $\forall x \in (a,b)$, then f is increasing on $[a,b]$

(b) If $f'(x) < 0$, $\forall x \in (a,b)$ then f is decreasing on $[a,b]$

Ex's

Ex's Find the intervals where the function is increasing or decreasing.

① $f(x) = x^2 - 4x - 5$

$f'(x) = 2x - 4 = 0$

$2x = 4$

$x = 2$

f': $- - - - - -$ $\;2\;$ $+ + + +$

decreasing on $(-\infty, 2]$

Increasing on $[2, \infty)$

② $f(x) = x^3 - 12x - 5$

$f'(x) = 3x^2 - 12$

$f'(x) = 0 = 3x^2 - 12 = 0$

$x^2 = 4$

$x = \pm 2$

$+ + +$ $\;-2\;$ $- - - - - -$ $\;2\;$ $+ + +$

Decreasing on $[-2, 2]$

Increasing on $(-\infty, -2] \cup [2, \infty)$

4.2 Extreme values of functions:

<u>Def:</u> Let f be a function with domain D, then

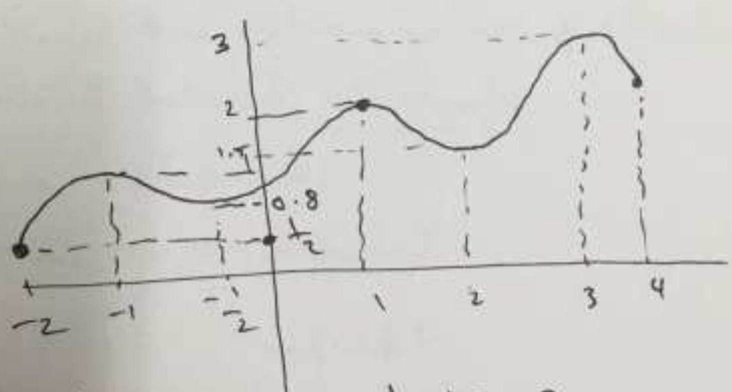
(a) f has absolute maximum value on D at a point c if $f(x) \le f(c)$, $\forall x \in D$

(b) f has absolute minimum value on D at a point c if $f(x) \ge f(c)$, $\forall x \in D$

(c) f has a local max. value at c if $f(x) \le f(c)$ in some interval around c

(d) f has a local minimum value at c if $f(x) \ge f(c)$ in some interval around c

Ex:



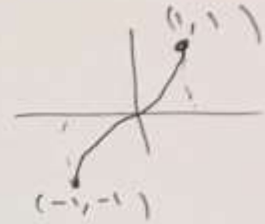
absolute minimum is $\frac{1}{2}$ at $x = -2$

maximum is 3 at $x = 3$

local minimums at $x = -\frac{1}{2}, 2$

local maximum at $x = -1, 1$

Ex 2 $f(x) = x^3$ on $[-1,1]$



Absolute max is 1 at $x = 1$

Absolute min is -1 at $x = -1$

Theorem: If $f(x)$ is continous on a closed interval $[a,b]$, then f has both an absolute max value and an absolute minimum value on $[a,b]$

$\Rightarrow \exists x_1 \in [a,b]$ such that $f(x_1) \leq f(x), \forall x \in [a,b]$

$\Rightarrow \exists x_2 \in [a,b]$ such that $f(x_2) \geq f(x), \forall x \in [a,b]$

Def: An interior point where f' equals zero or undefined is called a critical point of f.

Note: To find the extreme values of a function on a closed interval, We check

(1) The endpoints

(2) the critical points

Ex 1 Consider $f(x) = x^{2/3}$ on $[-1,8]$

critical points: $f'(x) = \frac{2}{3} x^{-1/3} = \frac{2}{3x^{1/3}}$,

$f'(0)$ is undefined

$\Rightarrow$ So we need to evaluate f only at $x = 0, -1, 8$ (why!)

$f(0) = 0$

$f(1) = 1$ $\Rightarrow$ Absolute max is 4 at $x = 8$

$f(8) = 4$ Absolute min is 0 at $x = 0$

Th If f is differentiable and has an extreme value at an interior point c then $f'(c)=0$

(Note that if $f'(c)=0$, it <u>does not</u> imply that $f(c)$ is an extrem value

For example if $f(x)=x^3$, $f'(x)=3x^2=0$
$x=0$

$\Rightarrow f'(0)=0$

But $f(0)$ is not a max nor a min.

To classify the critical as a max or min, we use the 1^{st} derivative test <u>or</u> the 2^{nd} derivative test.

Th (1^{st} derivative test)

$\boxed{\text{Th}}$ **1^{st} derivative test**

Suppose that have a critical point at $x=c$ and that $f'(x)$ exists in an open interval containing c, then

ⓐ If $f'(x)$ changes from positive to negative at $x=c$ then $f(c)$ is a local maximum f': $+ + + + \mid - - -$ at c, $\nearrow c \searrow$

ⓑ If $f'(x)$ change from negative to positive at $x=c$ the $f(c)$ is a local mini f': $- - - - \mid + + +$ at c, $\searrow c \nearrow$

ⓒ If $f'(x)$ does not change sign then f does not have an extreme value

$\boxed{\text{Th8}}$ **2^{nd} derivative test:**

Suppose that $f'(c)=0$ and that $f''(x)$ is continuous in an open interval containing c, then

ⓐ If $f''(c)<0$ then $f(c)$ is a local max.

ⓑ If $f''(c)>0$ the $f(c)$ is a local min.

ⓒ If $f''(c)=0$ then the test fails.

Remark

ⓐ If $f''(x)>0$ on interval I then $f(x)$ is concave up on I

$+ + + + + + + +$ $f''(x)$ $\smile$

ⓑ If $f''(x)<0$ on an interval I, the $f(x)$ is concave down on I $- - - - -$ $\frown$

6)

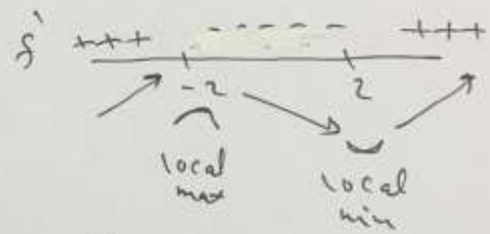
Def. A point where f has tangent line and changes concavity is called an inflection point

Ex's Find the intervals in which the following functions are increasing, decreasing, concave up or down. The find the extrem values and inflection points and sketch their graphs.

Ex1 $f(x) = x^3 - 12x - 5$

$f'(x) = 3x^2 - 12 = 0$

$x = \pm 2$



$f(2) = 8 - 24 - 5 = -21$

$(2, -21)$ local min

$f(-2) = -8 + 24 - 5 = 11$

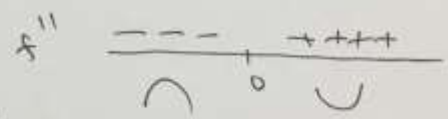
$(-2, 11)$ is local max

decreasing $[-2, 2]$

increasing $[-\infty, -2] \cup [2, \infty]$

$f''(x) = 6x = 0$

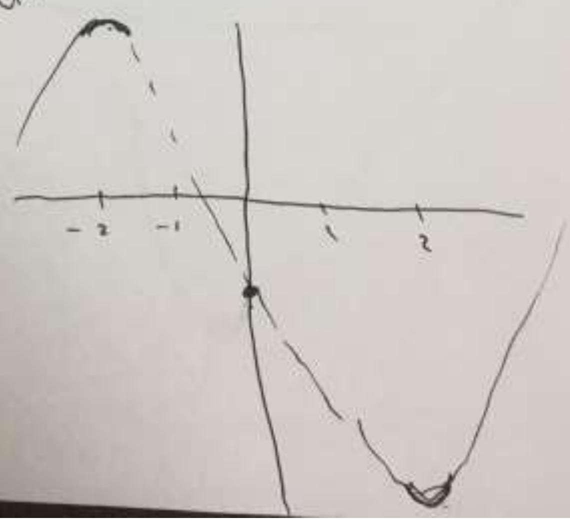
$x = 0$



$(0, -5)$ is a critical point.

concave down $(-\infty, 0)$

concave up $(0, \infty)$



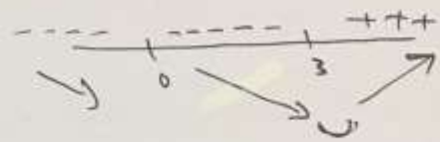
Ex2 $f(x) = x^4 - 4x^3 + 10$

$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$

$\Rightarrow$ critical points $x=0$, $x=3$

local min at $(3, f(3))$

$\rightarrow (3, -17)$



$f''(x) = 12x^2 - 24x = 12x(x-2)$

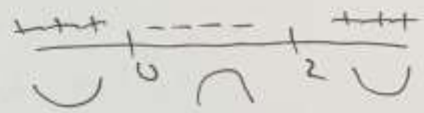
Inflection points $(0, 10)$, $(2, -6)$

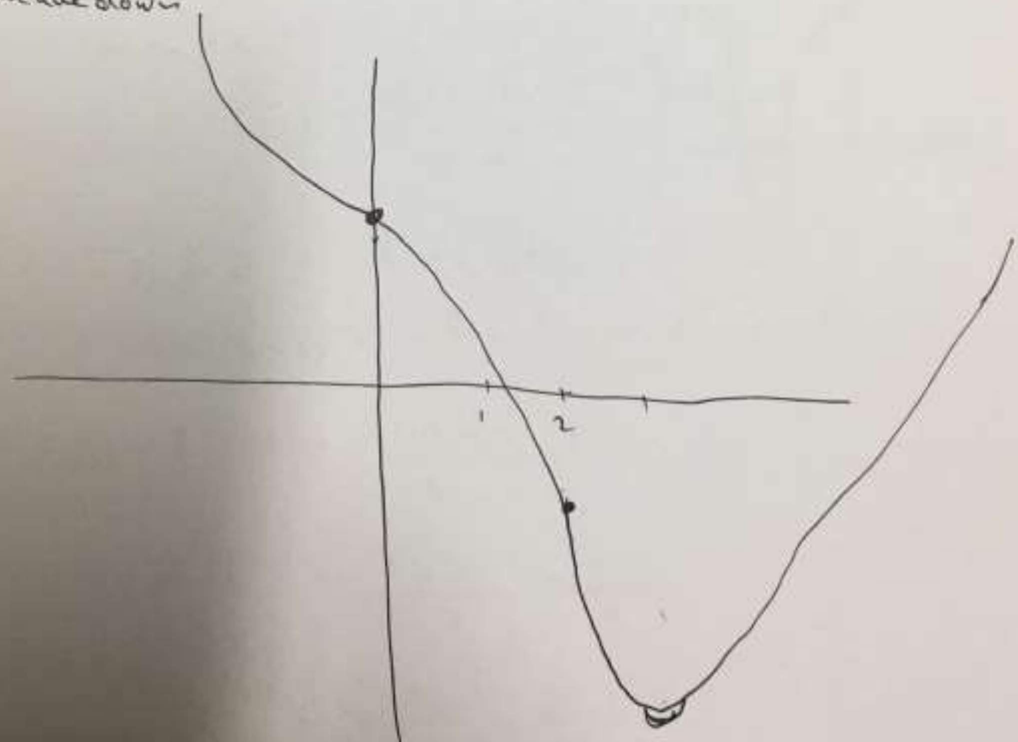
Increasing:

Decreasing:

Concave up:

concave down





4.3 The Mean Value theorem:

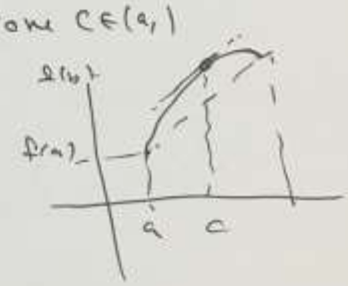
Th Roll's theorem

If $f(x)$ is continuous on $[a,b]$
and differentiable on (a,b)
and $f(a)=f(b)$
then there exists at least one point $c\in(a,b)$
such that $f'(c)=0$

Th The mean value theorem.

If $f(x)$ is cont. on $[a,b]$
and differentiable on (a,b)
then there exists at least one $c\in(a,b)$
such that

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$



Ex Find c that satisfies the MVT for $f(x)=x^2$, $x\in[1,4]$

$$\Rightarrow f'(c)=\frac{f(4)-f(1)}{4-1}$$

$$2c=\frac{16-1}{3}=\frac{15}{3}$$

$$\rightarrow c=\frac{15}{6}=\frac{5}{2}$$