16.1 Line Integrals

- How to calculate total mass of a wire (سلك) lying along a curve C in space.
- How to find the work done by a variable force acting along such curve C.

- We need to integrate a given function $f(x,y,z)$ over a curve C rather than over an interval $[a,b]$.

- Suppose that we need to integrate a real-valued function $f(x,y,z)$ over the curve C (f is defined on C).
- Suppose also that the curve C is parametrically given by $\vec{r}(t) = g(t)\,\vec{i} + h(t)\,\vec{j} + k(t)\,\vec{k}$ where $a \le t \le b$.
- Then, the line "path" integral of f over C is

$$* \; \cdots \quad \int_C f(x,y,z)\,ds = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k, y_k, z_k)\,\Delta s_k$$

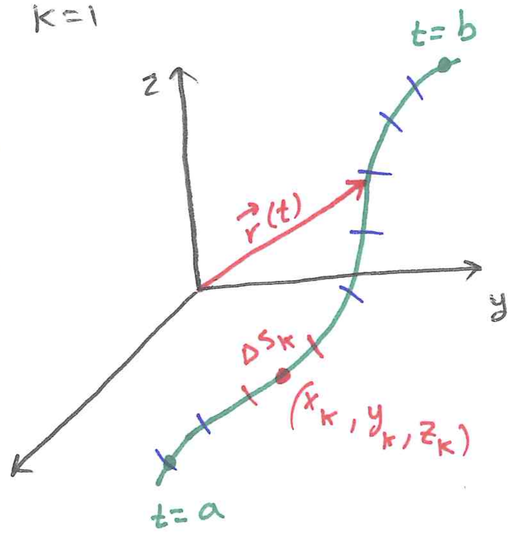
provided the limit exists.

- We partion C into finite number n of subarcs Δs_k and form the sum where

$$S_n = \sum_{k=1}^{n} f(x_k, y_k, z_k)\,\Delta s_k$$

The point $(x_k, y_k, z_k) \in \Delta s_k$

- $\Delta s_k \to 0$ as $n \to \infty$



The curve C

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- If the curve C is smooth for $a \le t \le b$

  then $\vec{v} = \frac{d\vec{r}}{dt}$ is continuous and never $\vec{0}$,
  and so f is cont. on C,
  and thus, the limit in $*$ exists.

- From section $\boxed{13.3}$ we know that the arc length:

$$L = s(t) = \int_a^t |\vec{v}(u)|\, du$$

Hence, $\boxed{ds = |\vec{v}(t)|\, dt}$ where

$$\vec{r}(t) = g(t)\,\vec{i} + h(t)\,\vec{j} + k(t)\,\vec{k}$$

and $\vec{v}(t) = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}$

with $|\vec{v}(t)| = \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$

$x = g(t)$
$y = h(t)$
$z = k(t)$

So $*^1$ becomes $\int_C f(x,y,z)\, ds = \int_a^b f(g(t), h(t), k(t))\, |\vec{v}(t)|\, dt \quad *^2$

Thus, to evaluate a line integral of a continuous function $f(x,y,z)$ over a curve C:

① Find a smooth parametrization of C by
$\vec{r}(t) = g(t)\,\vec{i} + h(t)\,\vec{j} + k(t)\,\vec{k}$, $a \le t \le b$

② Evaluate $*^2$ above.

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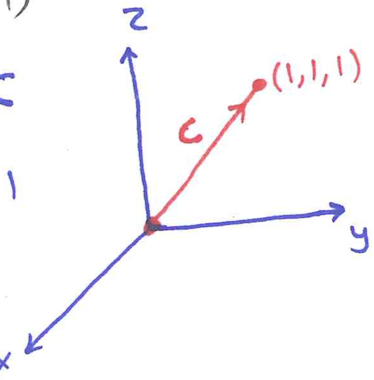
<u>Exp'</u> Integrate $f(x,y,z) = x - 3y^2 + z$ over the line segment C joining the origin to the point $(1,1,1)$



- Choose the simplest parametrization of C

$$\vec{r}(t) = t\,\vec{i} + t\,\vec{j} + t\,\vec{k}, \quad 0 \le t \le 1$$

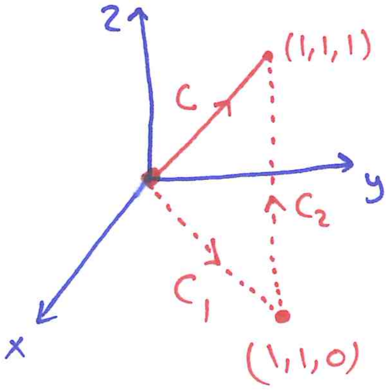
- $\vec{v}(t) = \vec{i} + \vec{j} + \vec{k}$
- $|\vec{v}(t)| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$ (never $\vec{0}$)
- $\displaystyle\int_C f(x,y,z)\,ds = \int_0^1 f(t,t,t)\,\sqrt{3}\,dt$

$$= \int_0^1 (t - 3t^2 + t)\sqrt{3}\,dt = 0$$

<u>Remark'</u>: If f is defined on a piecewise smooth curve C made by a finite number of smooth curves $C_1, C_2, \dots, C_n$, then

$$\int_C f\,ds = \int_{C_1} f\,ds + \int_{C_2} f\,ds + \dots + \int_{C_n} f\,ds \qquad \text{(Additivity)}$$

<u>In Exp'</u> we can choose another path from origin to $(1,1,1)$, which is the union of C_1 and C_2.



Now, integrate $f(x,y,z) = x - 3y^2 + z$ over $C_1 \cup C_2$.

- Choose the simplest parametrization for C_1 and C_2:

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$C_1: \vec{r}_1(t) = t\,\vec{i} + t\,\vec{j}, \quad 0 \le t \le 1$

$\vec{v}_1(t) = \vec{i} + \vec{j} \Rightarrow |\vec{v}_1| = \sqrt{1+1} = \sqrt{2}$

$C_2: \vec{r}_2(t) = \vec{i} + \vec{j} + t\,\vec{k}, \quad 0 \le t \le 1$

$\vec{v}_2(t) = \vec{k} \Rightarrow |\vec{v}_2| = \sqrt{1} = 1$

- $$\int_{C_1 \cup C_2} f(x,y,z)\,ds = \int_{C_1} f(x,y,z)\,ds + \int_{C_2} f(x,y,z)\,ds$$

$$= \int_0^1 f(t,t,0)\sqrt{2}\,dt + \int_0^1 f(1,1,t)\,dt$$

$$= \int_0^1 (t-3t^2+0)\sqrt{2}\,dt + \int_0^1 (1-3+t)\,dt$$

$$= -\frac{1}{\sqrt{2}} - \frac{3}{2}$$

Remark²: The value of the line integral along a path joining two points can change if we change the path between them. (see exp above)

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Mass and Moment Calculations

- We treat coil springs (نوابض لولبية) and wires (أسلاك) as masses distributed along smooth curves in space.
- The distribution is described by a continuous density function $\delta(x, y, z)$ representing mass per unit length.
- Given a parametrization of a curve C:

$$\vec{r}(t) = x(t)\,\vec{i} + y(t)\,\vec{j} + z(t)\,\vec{k}, \quad a \le t \le b$$

- The density is the function $\delta(x(t), y(t), z(t))$.
- The arc length differential is then given by

$$ds = |\vec{v}(t)|\,dt = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$

- The spring's mass (or wire's mass) becomes

$$M = \int_C \delta\, ds$$

$dm = \delta\, ds$ is element of mass

- First moments about the planes coordinate:

$$M_{yz} = \int_C x\,\delta\, ds, \quad M_{xz} = \int_C y\,\delta\, ds, \quad M_{xy} = \int_C z\,\delta\, ds$$

- Coordinates of the center of mass:

$$\bar{x} = \frac{M_{yz}}{M}, \quad \bar{y} = \frac{M_{xz}}{M}, \quad \bar{z} = \frac{M_{xy}}{M}$$

$: The moment of inertia is defined as $I = m r^2$ where m is mass and r is distance to the rotation axis.

- $Moments of inertia (القصور جمود) about axes:

$$I_x = \int_C (y^2 + z^2)\,\delta\, ds, \quad I_y = \int_C (x^2 + z^2)\,\delta\, ds, \quad I_z = \int_C (x^2 + y^2)\,\delta\, ds$$

- $Moment of inertia about lines:

$$I_L = \int_C r^2\,\delta\, ds$$

where $r(x, y, z)$ is distance from point (x, y, z) to line L.

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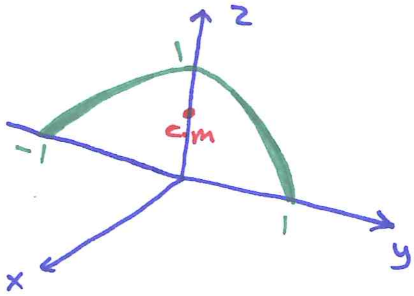
Exp A slender metal arch (قوس معدني نحيل) with denser at the bottom than top, (أكثر كثافة في الجزء السفلي) lies along the semicircle $y^2+z^2=1$, $z\ge 0$ in the yz-plane. Suppose the density at point (x,y,z) on the arch is $\delta(x,y,z)=2-z$. Find



1. Mass

• The parametrization of the semicircle is

$$\vec{r}(t)=(\cos t)\,\vec{j}+(\sin t)\,\vec{k} \quad \text{where } 0\le t\le \pi$$

with

$$\vec{v}(t)=(-\sin t)\,\vec{j}+(\cos t)\,\vec{k}$$

$$|\vec{v}(t)|=\sqrt{\sin^2 t+\cos^2 t}=1$$

Hence, $ds=|\vec{v}|\,dt$
$=dt$

• $$M=\int_C \delta\, ds=\int_C (2-z)\,ds=\int_0^{\pi}(2-\sin t)\,dt=2\pi-2.$$

2. First moments M_{xy}, M_{yz}, M_{xz}

$$M_{xy}=\int_C z\,\delta\, ds=\int_C z(2-z)\,ds=\int_0^{\pi}\sin t\,(2-\sin t)\,dt=\frac{8-\pi}{2}$$

$$M_{yz}=\int_C x\,\delta\, ds=\int_C 0\,\delta\, ds=0$$

$$M_{xz}=\int_C y\,\delta\, ds=\int_C \cos t\,(2-z)\,ds=\int_0^{\pi}\cos t\,(2-\sin t)\,dt=\Big[2\sin t+\tfrac{1}{2}\cos 2t\Big]_0^{\pi}$$
$$=0$$

3. Center of the arch's mass

$$\bar{x}=\frac{M_{yz}}{M}=0 \qquad \bar{y}=\frac{M_{xz}}{M}=0 \qquad \bar{z}=\frac{M_{xy}}{M}=\frac{4-\frac{\pi}{2}}{2\pi-2}\approx 0.57$$

4. Moment of inertia I_x for the arch.

$$I_x=\int_C (y^2+z^2)\,\delta\, ds=\int_0^{\pi}(\cos^2 t+\sin^2 t)(2-\sin t)\,dt=\int_0^{\pi}(2-\sin t)\,dt$$
$$=2\pi-2$$

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