Homework 3 (chapter 4) Reem Ali

Abstract 1 .

Q2: suppose that $\langle a \rangle, \langle b \rangle, \langle c \rangle$ are cyclic groups of order 6 , 8 and 20 .
Find all generators of $\langle a \rangle, \langle b \rangle$ and $\langle c \rangle$.

$\rightarrow \langle a \rangle = \langle a^K \rangle$ s.t gcd $(n, K) = 1$

gcd $(6,1) = 1 \rightarrow K = 1,5$.

The generators of $\langle a \rangle = a^1 , a^5$.

$\rightarrow \langle b \rangle = \langle b^K \rangle$ s.t gcd $(8,1) = 1 \rightarrow K = 1,3,5,7$.

The generators of $\langle b \rangle = b^1 , b^3 , b^5 , b^7$

$\rightarrow \langle c \rangle = \langle c^K \rangle$ s.t gcd $(20,1) = 1 \rightarrow K = 1,3,7,9,11,13,17,19$

The generators of $\langle c \rangle = c^1 , c^3 , c^7 , c^9 , c^{11} , c^{13} , c^{17} , c^{19}$.

Q4: list the elements of the subgroups $\langle 3 \rangle$ and $\langle 15 \rangle$ in $\mathbb{Z}_{18}$. let a be agroup element of order 18 . list the elements of the subgroups $\langle a^3 \rangle$ and $\langle a^{15} \rangle$.

$\mathbb{Z}_{18} = \{0,1,2,3,4,5,6,7,8,9,10, \ldots ,17\}$

$\langle 3 \rangle = \{3,6,9,12,15,0\}$

$\langle 15 \rangle = \{15,12,9,6,3,0\}$

$\langle a^3 \rangle = \{a^3, a^6, a^9, a^{12}, a^{15}, a^0\}$

$\langle a^{15} \rangle = \{a^{15}, a^{12}, a^9, a^6, a^3, a^0\}$

Q8: let a be an element of a group and let $|a| = 15$, compute the orders of the following elements of G.

a. a^3, a^6, a^9, a^{12}.

$|a^3| = \frac{15}{\gcd(15,3)} = \frac{15}{3} = 5$

$|a^6| = \frac{15}{\gcd(15,6)} = \frac{15}{3} = 5$

$|a^9| = \frac{15}{\gcd(15,9)} = \frac{15}{3} = 5$

$|a^{12}| = \frac{15}{\gcd(15,12)} = \frac{15}{3} = 5$

15: 1, 3, 5, 15

3: 1, 3

6: 1, 2, 3, 6

9: 1, 3, 9

12: 1, 2, 3, 4, 6, 12.

b. a^5, a^{10}.

$|a^5| = \frac{15}{\gcd(15,5)} = \frac{15}{5} = 3$

$|a^{10}| = \frac{15}{\gcd(15,10)} = \frac{15}{5} = 3$.

5: 1, 5

10: 1, 2, 5, 10

c. a^2, a^4, a^8, a^{14}.

$|a^2| = \frac{15}{\gcd(15,2)} = \frac{15}{1} = 15$

$|a^4| = \frac{15}{\gcd(15,4)} = \frac{15}{1} = 15$

$|a^8| = \frac{15}{\gcd(15,8)} = \frac{15}{1} = 15$

$|a^{14}| = \frac{15}{\gcd(15,14)} = \frac{15}{1} = 15$.

2: 1, 2

4: 1, 2, 4

8: 1, 2, 4, 8

14: 1, 2, 7, 14.

Q9: How many subgroups does $\mathbb{Z}_{20}$ have? list a generator for each of these subgroups. suppose that $G = \langle a \rangle$ and $|a| = 20$. How many subgroups does G have? list the generator for each of these subgroups.

$\mathbb{Z}_{20} = \{0, 1, 2, 3, \ldots, 19\}$

# of subgroups of $\mathbb{Z}_{20}$ = # of positive devisor K of 20.

$K = 1, 2, 4, 5, 10, 20 \rightarrow \mathbb{Z}_{20}$ have 6 subgroups.

$\langle \frac{20}{1} \rangle = \langle 20 \rangle = \{0\}$

$\langle \frac{20}{5} \rangle = \langle 4 \rangle = \{4, 8, 12, 16, 0\}$

$\langle \frac{20}{2} \rangle = \langle 10 \rangle = \{10, 0\}$

$\langle \frac{20}{2} \rangle = \langle 2 \rangle = \{2, 4, 6, 8, 10, 12, 14, 16, 18, 0\}$

$\langle \frac{20}{4} \rangle = \langle 5 \rangle = \{5, 10, 15, 0\}$

$\langle \frac{20}{20} \rangle = \langle 1 \rangle = \{1\}$

# of subgroups of $G = 6$.

$\langle a^{20} \rangle = \{a^0\}$

$\langle a^4 \rangle = \{a^4, a^8, a^{12}, a^{16}, a^0\}$

$\langle a^{10} \rangle = \{a^0, a^{10}\}$

$\langle a^2 \rangle = \{a^2, a^4, a^6, a^8, a^{10}, a^{12}, a^{14}, a^{16}, a^{18}, a^0\}$

$\langle a^5 \rangle = \{a^5, a^{10}, a^{15}, a^0\}$

$\langle a^1 \rangle = \{a^1\}$.

Q14: suppose that a cyclic group G has exactly three subgroups: G itself, $\{e\}$, and a subgroup of order 7. What is $|G|$? What can you say if 7 is replaced with p where p is prime?

let $|G| = n$ $\rightarrow$ 7 and $\frac{n}{7}$ divisors of n.

If $\frac{n}{7} \neq 7$ then G has at least 4 divisors. X No.

So must $\frac{n}{7} = 7$ $\rightarrow$ $n = 7^2 = 49$

$\Longrightarrow$ $|G| = 49$.

In general $\rightarrow$ $|G| = p^2$.

Q20: suppose that G is an Abelian group of order 35 and every element of G satisfies the equation $x^{35} = e$. prove that G is cyclic. Does your argument work if 35 is replaced with 33? From google.

To show that if G is an abelian group of order 35 and $x^{35} = 1$ $\forall x \in G$.

then G is cyclic: First $x^{35} = 1$ implies $|x| = 1, 5, 7$ or 35.

If G has an element of order 35 so it is cyclic.

If not, all nonidentity elements have order 5 or 7.

In fact it must have elements of both orders, for if, say any $x \neq 1$ in G had order 5. G would have 34 elements of order 5 with is impossible since $4 = |5|$ does not divide 34. Likewise $6 = |7|$ does not divide 34. So it can't be the case that $|x| = 7$ for all $x \neq 1$.

Thus, if G has No element of order 35, there are $x, y \in G$ with $|x| = 5$, $|y| = 7$. Now xy must have order 1, 5 or 7, but none of these are possible: if $|xy| = 1$ $x = y^{-1}$ which is imposible since x and y^{-1} have differents order.

If $|xy| = 5$; $1 = (xy)^5 = y^5$ (her we use that G is Abelian) which contradicts $|y| = 7$; similarly $|xy| = 7$ yields $1 = (xy)^7 = x^7 = x^2$ contradicting $|x| = 5$

Thus, G has an element of order 35, and is cyclic.

This argument does not work for groups of order 33

since $2 = |3| \backslash 33 - 1 = 32$.

cont. Q20:

This argument does not work for groups of order 33 since $2 = |3| \mid 33-1 = 32$.

Q22 : prove that a group of order 3 must be cyclic.

$G = \{e, a, b\}$ with $a \neq e$, $b \neq e$, $a \neq b$

$a, b \in G \rightarrow$ $ab = b \rightarrow a = e$ ※. with $a \neq e$

$ab = a \rightarrow b = e$ ※. with $b \neq e$

$ab = e \rightarrow b = a^{-1}$

Then the group table of G is :

$\cdot$	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

that means :

$G = \langle a \rangle = \langle b \rangle$

So G is cyclic.

Q24: For any element a in any group G, prove that $\langle a \rangle$ is a subgroup of $C(a)$ (the centralizer of a).

$\langle a \rangle$ is already a group under multiplication in G

So we just need to show it is a subset of $C(a)$.

This is easy : $a.a = a.a$, so $a \in C(a)$.

As $C(a)$ is a group (in particular, closed under multiplication and inversion)

We must have that any a^n is also in $C(a)$

This is precisely what it means for $\langle a \rangle \subset C(a)$

Q38: Consider the set $\{4, 8, 12, 16\}$. show that this set is a group under multiplication Modulo 20. what is the identity element? Is the group cyclic? If so, find all of its generators.

$\otimes_{20}$	4	8	12	16
4	16	12	8	4
8	12	4	16	8
12	8	16	4	12
16	4	8	12	16

$\Rightarrow$ Identity $= 16$

$\Rightarrow$ Inverse $=$ $4 : 4$ $\quad 12 : 8$

$8 : 12$ $\quad 16 : 16$.

So $\{4, 8, 12, 16\}$ its a group under multiplication.

$\rightarrow$ Now, we check that the group is cyclic or not.

$\rightarrow$ calculate order of each element of the group.

$4^2 = 16 \rightarrow |4| = 2$

$8^4 = 16 \rightarrow |8| = 4$

$12^4 = 16 \rightarrow |12| = 4$

$\Rightarrow |8| = |12| = |G|$ By using theorem for a group G If the order of the group and the order of the element is same the G is cyclic.

$\langle 8 \rangle = \langle 12 \rangle = \{4, 8, 12, 16\}$ are the generators of the group G.

Q44: suppose that G is cyclic group and that 6 divides $|G|$. How many elements of order 6 does G have? If 8 divides $|G|$, How many elements of order 8 does G have? If a is one element of order 8, list the other elements of order 8.

explaine : since G is cyclic, for any divisor d of $|G|$ we have a single subgroup H of order d. The generators of H are therefor the only elements of order d, and are of the form h^K where $\gcd(K,d) = 1$ (K is positive, $K<d$) for any generator of h of H.

* For $d = 6$ (K: 1, 2, 3, 4, 5)

$\gcd(1,6) = 1$

$\gcd(5,6) = 1$ So $K=1$, $K=5$

$\Longrightarrow$ there are two elements of order 6.

* For $d = 8$ (K: 1, 2, 3, 4, 5, 6, 7)

$\gcd(1,8) = 1$

$\gcd(3,8) = 1$

$\gcd(5,8) = 1$

$\gcd(7,8) = 1$ So $K=1$, $K=3$, $K=5$, $K=7$

$\Longrightarrow$ there are 4 elements of order 6.

* $\langle a \rangle = \{a, a^3, a^5, a^7\}$

Q45: list all the elements of $\mathbb{Z}_{40}$ that have order 10. let $|x| = 40$ list all the elements of $\langle x \rangle$ that have order 40.

* let x be a generator of $\mathbb{Z}_n$ $\rightarrow$ $|x^a| = \frac{n}{\gcd(n,a)}$.

Given $n = 40$, $|x^a| = 10$ $\rightarrow$ $\frac{10}{1} = \frac{40}{\gcd(40,a)}$

$\Longrightarrow$ $\gcd(40, a) = 4$

$\gcd(10, \frac{a}{4}) = 1$

$\rightarrow$ $\frac{a}{4} = 1, 3, 7, 9$

$\Longrightarrow$ $a = 4, 12, 28, 36$.

* $\langle x \rangle = \langle x^4, x^{12}, x^{28}, x^{36} \rangle$.

Q64: prove that $H = \left\{ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} / n \in \mathbb{Z} \right\}$ is a cyclic subgroup of $GL(2, \mathbb{R})$.

we claim that $H = \langle A \rangle$ where $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Indeed, because $A \in H$, $\langle A \rangle \subseteq H$.

* ممكن باستخدام induction اثباتها لكل positive integer ,