8.4. Integration of Rational Functions by Partial Fractions.

Example(1): $\int \frac{5x-3}{x^2-2x-3}\,dx = \int \frac{5x-3}{(x+1)(x-3)}\,dx$

$$\frac{5x-3}{(x+1)(x-3)} = \frac{A}{(x+1)} + \frac{B}{(x-3)}$$

$$\frac{5x-3}{(x+1)(x-3)} = \frac{A(x-3)+B(x+1)}{(x+1)(x-3)}$$

$\Rightarrow \quad 5x-3 = A(x-3)+B(x+1) = (A+B)x - 3A+B$

$\Rightarrow \quad 5x-3 = (A+B)x + (B-3A).$

$\therefore$ x-term : $A+B = 5$

(ثوابت) x^0-term : $B-3A = -3$

$\Rightarrow \quad \boxed{A = 2}\ , \ \boxed{B = 3}$

$$\Rightarrow \int \frac{(5x-3)}{(x+1)(x-3)}\,dx = \int \left(\frac{2}{x+1} + \frac{3}{x-3} \right) dx$$

$$= 2\ln|x+1| + 3\ln|x-3| + C$$

$$= \ln\left|(x+1)^2(x-3)^3\right| + C$$

Example (2): $\displaystyle\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}\,dx.$

$$\frac{x^2+4x+1}{(x-1)(x+1)(x-3)} = \frac{A}{(x-1)} + \frac{B}{(x+1)} + \frac{C}{(x+3)} \quad \dots (*)$$

$$\Rightarrow \; x^2+4x+1 = A(x+1)(x+3)+B(x-1)(x+3)+C(x-1)(x+1) \quad \dots (**)$$

$$\Rightarrow \; x^2+4x+1 = (A+B+C)x^2+(4A+2B)x+(3A-3B-C)$$

$\Rightarrow$ x^2- term : $A+B+C=1$

x- term : $4A+2B=4$

Constant term : $3A-3B-C=1$

$\Rightarrow$ Solve the three equations for A, B, C.

OR, we can use "Coverup" method for "Linear factors":

Let $x=1$, then $(**) \Rightarrow 1+4+1 = A(2)(4) \Rightarrow \boxed{A=\frac{3}{4}}$

Let $x=-1$, then $(**) \Rightarrow (-1)^2+4(-1)+1 = B(-2)(2) \Rightarrow \boxed{B=\frac{1}{2}}$

Let $x=-3$, then $(**) \Rightarrow 9-12+1 = C(-4)(-2) \Rightarrow \boxed{C=\frac{-1}{4}}$

Back to $(*)$, then:

$$\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}\,dx = \int \frac{3/4}{(x-1)} + \frac{1/2}{(x+1)} + \frac{-1/4}{(x+3)}\,dx$$

$$= \frac{3}{4}\ln|x-1| + \frac{1}{2}\ln|x+1| - \frac{1}{4}\ln|x+3| + C$$

Example (3): $\int \frac{6x+7}{(x+2)^2}\,dx$

$$\frac{6x+7}{(x+2)^2} = \frac{A}{(x+2)} + \frac{B}{(x+2)^2} = \frac{A(x+2)+B}{(x+2)^2}.$$

$\Rightarrow \quad 6x+7 = A(x+2)+B$

$\Rightarrow \quad 6x+7 = Ax + (2A+B)$

Therefore, Constant-term : $2A+B = 7$

x - term : $A = 6$

$\Rightarrow \quad \boxed{A=6}$ and $\boxed{B=-5}$

$-5(x+2)^{-2}$

$$\Rightarrow \int \frac{6x+7}{(x+2)^2}\,dx = \int \frac{6}{(x+2)} + \frac{-5}{(x+2)^2}\,dx$$

$$= 6\ln|x+2| + \frac{5}{(x+2)} + C.$$

Remark(1): $\frac{1}{(x-a)^n} = \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_n}{(x-a)^n}.$

Remark (2): For $\int \frac{f(x)}{g(x)}\,dx$, to start the partial fraction the degree of $f(x)$ should be less than the degree of $g(x)$. If NOT, use long division.

Example (4): $\displaystyle\int \frac{2x^3-4x^2-x-3}{x^2-2x-3}\,dx$

Using Long division :

$$\begin{array}{r|l} & \quad 2x \\ \hline x^2-2x-3 & 2x^3-4x^2-x-3 \\ & -2x^3 \overset{+}{-} 4x^2 \overset{+}{-} 6x \\ \hline & \qquad\qquad 5x-3 \end{array}$$

$$\therefore \frac{2x^3-4x^2-x-3}{x^2-2x-3} = 2x + \frac{5x-3}{x^2-2x-3}$$

$$\Rightarrow \int \frac{2x^3-4x^2-x-3}{x^2-2x-3}\,dx = \int 2x\,dx + \underbrace{\int \frac{5x-3}{(x-3)(x+1)}\,dx}_{\text{Example (1)}}$$

$$= x^2 + 2\ln|x+1| + 3\ln|x-3| + C.$$

Example (5): $\displaystyle\int \frac{-2x+4}{(x^2+1)(x-1)^2}\,dx.$

$$\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1} + \frac{D}{(x-1)^2}$$

$$\Rightarrow -2x+4 = (Ax+B)(x-1)^2 + C(x^2+1)(x-1) + D(x^2+1)$$

$$\Rightarrow -2x+4 = (Ax+B)(x^2-2x+1) + C(x^3-x^2+x-1) + Dx^2 + D$$

$$-2x+4 = (A+C)x^3 + (B-2A-C+D)x^2$$
$$+ (A+-2B+C)x + (B-C+D).$$

$$\Rightarrow \left.\begin{aligned} A+C &= 0 \\ B-2A-C+D &= 0 \\ A-2B+C &= -2 \\ B-C+D &= 4 \end{aligned}\right\} \Rightarrow \begin{aligned} A &= 2 \\ B &= 1 \\ C &= -2 \\ D &= 1 \end{aligned}$$

check !!

$$\therefore \int \frac{-2x+4}{(x^2+1)(x-1)^2}\,dx = \int \left[\frac{2x+1}{x^2+1} + \frac{-2}{x-1} + \frac{1}{(x-1)^2}\right] dx$$

$$= \int \frac{2x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx - 2\int \frac{1}{x-1}\,dx + \int (x-1)^{-2}\,dx$$

$$= \ln|x^2+1| + \tan^{-1}x - 2\ln|x-1| - \frac{1}{x-1} + K.$$

Example (6) : $\displaystyle\int \frac{1}{x(x^2+1)^2}\,dx$

$$\frac{1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}.$$

$$\Rightarrow 1 = A(x^2+1)^2 + (Bx+C)(x)(x^2+1) + (Dx+E)x.$$

$$1 = A(x^4+2x^2+1) + (Bx^2+Cx)(x^2+1) + Dx^2 + Ex$$

$$1 = Ax^4 + 2Ax^2 + A + Bx^4 + Bx^2 + Cx^3 + Cx + Dx^2 + Ex$$

x^4 - term : $A + B = 0$

x^3 - term : $C = 0$

x^2 - term : $2A + B + D = 0$

x - term : $C + E = 0$

Constant - term : $A = 1$

$\Rightarrow$ $A = 1$, $B = -1$, $C = 0$, $D = -1$, $E = 0$

check !!

$$\therefore \int \frac{1}{x(x^2+1)^2}\,dx = \int \frac{1}{x}\,dx + \frac{1}{2}\int \frac{(2)-1x}{x^2+1}\,dx + \underbrace{\int \frac{-1x}{(x^2+1)^2}\,dx}_{\text{I : substitution}}$$

For I: Let $u = x^2+1 \Rightarrow du = 2x\,dx \Rightarrow \displaystyle\int \frac{-du}{2u^2} = \frac{1}{2(x^2+1)} + K$

$$\therefore \int \frac{1}{x(x^2+1)^2}\,dx = \ln|x| - \frac{1}{2}\ln|x^2+1| + \frac{1}{2(x^2+1)} + K$$

$$= \ln\left(\frac{|x|}{\sqrt{x^2+1}}\right) + \frac{1}{2(x^2+1)} + K.$$

The Heaviside "Cover up" Method for Linear Factors

When the degree of the polynomial $f(x)$ is Less than the degree of $g(x)$ and

$$g(x) = (x-r_1)(x-r_2)\cdots(x-r_n)$$

is a product of n distinct linear factors, each raised to the first power, there is a quick way to expand $\frac{f(x)}{g(x)}$ by partial fractions:

Example (7): Find A and B and C in the following:

$$\frac{x^2+1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$$

To find A, set $x=1$, then:

$$A = \frac{1^2+1}{\boxed{(x-1)}(1-2)(1-3)} = \frac{2}{(-1)(-2)} = \boxed{1}$$

Cover

To find B, set $x = 2$, then

$$B = \frac{2^2+1}{(2-1)\boxed{(x-2)}(2-3)} = \frac{5}{(1)(-1)} = \boxed{-5}$$

Cover

To find C, set $x=3$, then $C = \frac{3^2+1}{(3-1)(3-2)\boxed{(x-3)}} = \boxed{5}$

Cover

Example (8): Evaluate $\int \frac{x+4}{x^3+3x^2-10x}\,dx$

$$\frac{x+4}{x^3+3x^2-10x} = \frac{x+4}{x(x+5)(x-2)} = \frac{A}{x} + \frac{B}{x+5} + \frac{C}{x-2}$$

To find A, set $x=0$:

$$A = \frac{0+4}{\underset{\text{Cover}}{x}(0+5)(0-2)} = \frac{4}{5(-2)} = -\frac{2}{5}.$$

[Multiply both sides by x]

To find B, set $x=-5$:

$$B = \frac{-5+4}{(-5)\underset{\text{Cover}}{(x+5)}(-5-2)} = \frac{-1}{(-5)(-7)} = \frac{-1}{35}.$$

[Multiply both sides by $x+5$]

To find C, set $x=2$:

$$C = \frac{2+4}{(2)(2+5)\underset{\text{Cover}}{(x-2)}} = \frac{6}{(2)(7)} = \frac{3}{7}.$$

[Multiply both sides by $x-2$]

$$\therefore \int \frac{x+4}{x^3+3x^2-10x}\,dx = \int \frac{-2/5}{x} + \frac{-1/35}{x+5} + \frac{3/7}{x-2}\,dx$$

$$= -\frac{2}{5}\ln|x| - \frac{1}{35}\ln|x+5| + \frac{3}{7}\ln|x-2| + K$$

Another Way to find the Coefficients :

Example (9) : $\int \frac{x-1}{(x+1)^3}\,dx$

$$\frac{x-1}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}$$

$$\Rightarrow \quad x-1 = A(x+1)^2 + B(x+1) + C \quad \ldots (*)$$

- Let $x=-1$, then

$$-1-1 = A(0) + B(0) + C$$

$$\Rightarrow \boxed{C=-2}$$

- Differentiate both sides of $(*)$ with respect to x :

$$1 = 2A(x+1) + B \quad \ldots (**)$$

substitute $x=-1$, then $\boxed{B=1}$

- Differentiate both sides of $(**)$ w.r.t x :

$$0 = 2A \quad \Rightarrow \quad A = 0$$

Let $u = x+1$
$du = dx$

$$\therefore \int \frac{x-1}{(x+1)^3}\,dx = \int \frac{1}{(x+1)^2} - \frac{2}{(x+1)^3}\,dx = \int u^{-2} - 2u^{-3}\,du$$

$$= \frac{u^{-1}}{-1} - 2\frac{u^{-2}}{-2} + k = \frac{-1}{x+1} + \frac{1}{(x+1)^2} + k.$$

Q18) $\displaystyle\int_{-1}^{0} \frac{x^3}{x^2-2x+1}\,dx$

$$\begin{array}{r|l} & x+2 \\ \hline x^2-2x+1 & x^3 \\ & -x^3 \pm 2x^2 \mp x \\ \hline & 2x^2 - x \\ & -2x^2 \pm 4x \mp 2 \\ \hline & 3x-2 \end{array}$$

$$= \int_{-1}^{0} (x+2)\,dx + \int_{-1}^{0} \frac{3x-2}{(x-1)^2}\,dx \quad \cdots(*)$$

Now, $\displaystyle\frac{3x-2}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2}$.

$\Rightarrow \quad 3x-2 = A(x-1)+B = Ax + (B-1)$

$\Rightarrow \quad \boxed{A=3}$ and $\boxed{B=1}$

$$(*) = \int_{-1}^{0} (x+2)\,dx + \int_{-1}^{0} \frac{3}{x-1} + \frac{1}{(x-1)^2}\,dx$$

$$= \left(\frac{x^2}{2}+2x\right)\Big|_{-1}^{0} + 3\ln|x-1|\Big|_{-1}^{0} - \frac{1}{(x-1)}\Big|_{-1}^{0}$$

$$= \left[\frac{0}{2}+2(0)\right] - \left[\frac{(-1)^2}{2}+2(-1)\right] + \left[3\ln|-1| - 3\ln|-2|\right] - \left[\frac{1}{(0-1)} - \frac{1}{(-1-1)}\right]$$

$$= -\frac{1}{2} + 2 - 3\ln 2 + 1 - \frac{1}{2}$$

$$= 2 - 3\ln 2.$$

Q30) $\displaystyle\int \frac{x^2+x}{x^4-3x^2-4}\,dx$

$$\frac{x^2+x}{x^4-3x^2-4} = \frac{x^2+x}{(x-2)(x+2)(x^2+1)}$$

$$= \frac{A}{x-2} + \frac{B}{x+2} + \frac{Cx+D}{x^2+1}$$

$$\Rightarrow \quad x^2+x = A(x+2)(x^2+1) + B(x-2)(x^2+1) + (Cx+D)(x^2-4)$$

$$= (A+B+C)x^3 + (2A-2B+D)x^2 + (A+B-4C)x + (2A-2B-4D)$$

$$\left.\begin{array}{ll} x^3\text{-term} : & A+B+C=0 \\ x^2\text{-term} : & 2A-2B+D=1 \\ x\text{-term} : & A+B-4C=1 \\ \text{Constant-term} : & 2A-2B-4D=0 \end{array}\right\} \Rightarrow \quad \begin{array}{l} A=\frac{3}{10} \\ B=-\frac{1}{10} \\ C=-\frac{1}{5} \\ D=\frac{1}{5} \end{array}$$

$$\therefore \int \frac{x^2+x}{x^4-3x^2-4}\,dx = \int \frac{3/10}{x-2} + \frac{-1/10}{x+2} + \frac{-\frac{1}{5}x}{x^2+1} + \frac{\frac{1}{5}}{x^2+1}\,dx$$

$$= \frac{3}{10}\ln|x-2| - \frac{1}{10}\ln|x+2| - \frac{1}{10}\ln|x^2+1| + \frac{1}{5}\tan^{-1}x + K.$$

Q47) $\displaystyle\int \frac{\sqrt{x+1}}{x}\,dx$ (Hint : Let $x+1=u^2$)

$$= \int \frac{\sqrt{u^2}}{u^2-1}\cdot 2u\,du = \int \frac{2u^2}{u^2-1}\,du \quad , \qquad \begin{array}{r|l} u^2-1 & \overline{\;2u^2\;} \\ & \underline{-2u^2 \pm 2} \\ & \quad 2 \end{array} \quad \text{quotient } 2$$

$$= \int 2\,du + \int \frac{2}{(u-1)(u+1)}\,du \quad \cdots (*)$$

Now, $\dfrac{2}{(u-1)(u+1)} = \dfrac{A}{u-1} + \dfrac{B}{u+1}$

$\Rightarrow \quad 2 = A(u+1) + B(u-1) = (A+B)u + (A-B)$

$\Rightarrow \quad \left.\begin{array}{l} A+B=0 \\ A-B=2 \end{array}\right\} \Rightarrow \boxed{A=1}$ and $\boxed{B=-1}$

$$(*) = \int 2\,du + \int\left(\frac{1}{u-1} + \frac{-1}{u+1}\right)du$$

$$= 2u + \ln|u-1| - \ln|u+1| + K.$$

$$= 2\sqrt{x+1} + \ln\left|\sqrt{x+1}-1\right| - \ln\left|\sqrt{x+1}+1\right| + K.$$

$$= 2\sqrt{x+1} + \ln\left|\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right| + K$$