2.3 Bolzano - Weierstrass Theorem.

Notice that the seq $\{(-1)^n\}$ doesn't converge but it has convergent subsequence. In this section, we will prove that this is a general principle. That is, every bounded seq. has a convergent subsequence.

Df: ① Let $\{x_n\}_{n\in\mathbb{N}}$ be a seq. of real numbers

(i) $\{x_n\}$ is said to be increasing (resp, strictly increasing) iff $x_1 \le x_2 \le \dots$ (resp. $x_1 < x_2 < \dots$).

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(ii) $\{x_n\}$ is said to be decreasing (resp., strictly decreasing) iff $x_1 \ge x_2 \ge \dots$ (resp., $x_1 > x_2 > \dots$).

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(iii) $\{x_n\}$ is said monotone iff it is either increasing or decreasing.

RMK:

1. Some times, we call decreasing seq. nonincreasing and increasing seq. nondecreasing.
   يعني في مساواة — برضو في مساواة

2. If $\{x_n\}$ is increasing (resp., decreasing) and $x_n \to a$ as $n \to \infty$, we shall write $x_n \uparrow a$ (resp. $x_n \downarrow a$), as $n \to \infty$.

3. every strictly increasing seq. is increasing and every strictly decreasing seq. is decreasing

4. $\{x_n\}$ is increasing iff the sequence $\{-x_n\}$ is decreasing.

Proof $x_n \uparrow \rightarrow x_n \le x_{n+1}$ $\quad -x_n = y_n$ $\quad$ والعكس صحيح

$-x_{n+1} \le -x_n$

$y_{n+1} \le y_n \rightarrow$ so it's decreasing.

- We Know that any convergent seq. is bounded.

We Know establish the converse for Monotone sequences.

bdd $\not\Rightarrow$ conv.   decr. $\not\Rightarrow$ conv.

incr. $\not\Rightarrow$ conv.   Monotone $\not\Rightarrow$ conv.

bdd above + incr. $\Rightarrow$ conv.

bdd below + decr. $\Rightarrow$ conv.

مهم

Thm 1 : Monotone convergence Thm (MCT)   مهمة كثير كثير !

If $\{x_n\}$ is increasing and bounded above, or $\{x_n\}$ is decreasing and bounded below, or $\{x_n\}$ is decreasing and bounded below, then $\{x_n\}$ converges to a finite limit.

proof : supose first that $\{x_n\}$ is increasing and bounded above.

By completeness Axiom, the supremom $\beta := \sup\{x_n : n \in \mathbb{N}\}$ exists and finite.

let $\varepsilon > 0$. By the approximation property for suprema choose $N \in \mathbb{N}$ s.t

$$\beta - \varepsilon < x_N \leq \beta.$$

since $\{x_n\}$ is increasing $\Rightarrow x_N \leq x_n \quad \forall n \geq N$.

since $\beta = \sup\{x_n : n \in \mathbb{N}\} \Rightarrow x_n \leq \beta, \quad \forall n \in \mathbb{N}$

$\Big\}\ x_N \leq \beta, \ \forall n \geq N$

It follows that $\beta - \varepsilon < x_n \leq \beta, \quad \forall n \geq N$.

β عكس $\rightarrow -\varepsilon < x_n - \beta \leq 0 < \varepsilon$

$\rightarrow |x_n - \beta| < \varepsilon, \ \forall n \geq N$. $\boxed{\text{Thus } x_n \uparrow \beta \text{ as } n \to \infty}$ ☒

part 2 : suppose $\{x\}$ decreasing and bdd below.

By completeness Axiom ; $\alpha := \inf\{x_n : n \in \mathbb{N}\}$ exists and finite.

$-\alpha = \sup\{-x_n\}$

By part 1, $\alpha = -(-\alpha) = -\Big(\overset{\text{lim معلوم}}{\lim_{n\to\infty}(-x_n)}\Big) = \lim_{n\to\infty} x_n$

So $x_n \downarrow \alpha$

☒

بدنا نثبت شغلتين ← **exp:** If $|x|<1$, then $\lim_{n\to\infty} x^n = 0$ by Monton

pf.

It suffices to prove that $|x|^n \to 0$ as $n \to \infty$

First, we notice that $|x|^n$ is monotone <u>decreasing</u> $|x|<1$

since, $|x|<1$ implies $|x|^{n+1} < |x|^n$, $\forall n \in \mathbb{N}$. (def)

Next, notice that $|x|^n$ is <u>bounded</u> below (by 0)

Hence, by the <u>Monotone</u> convergence theorem → $|x|^n$ converge to a finite limit say L

$$|x|^n \to L \text{ as } n \to \infty$$

Next find L.

$$L = \lim_{n\to\infty} |x|^{n+1} = |x| \lim_{n\to\infty} |x|^n = |x| . L$$

$$\to L = |x| L$$

$$\Longrightarrow L(1-|x|) = 0$$

$$\Longrightarrow L = 0 \text{ or } |x| = 1 \text{ reject since } |x|<1$$

$\therefore L = 0$

$\therefore |x|^n \to 0$ as $n \to \infty$

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exp: If $x > 0$, then $\lim_{n\to\infty} x^{\frac{1}{n}} = 1$

pf: We consider 3 casses. $(1)^{\frac{1}{n}} = 1$

Case 1: $x = 1$ Then $x^{\frac{1}{n}} = 1$, $\forall n \in \mathbb{N}$

and its follows that $\lim_{n\to\infty} x^{\frac{1}{n}} = \lim_{n\to\infty} 1 = 1$

Trivial

cont.

case 2 : $x > 1$, we shall apply the MCT . سوف

we shall show that ~~show that~~ $\{x^{\frac{1}{n}}\}$ is decreasing and bounded below .

Indeed, since $x>1$ then $(x^{n+1})^{\frac{1}{n(n+1)}} > (x^{n})^{\frac{1}{n(n+1)}}$. Taking the $n(n+1)$ st root of this inequality .

We obtain $x^{\frac{1}{n}} > x^{\frac{1}{n+1}}$, ie $\{x^{\frac{1}{n}}\}$ is decreasing.

since $x>1$ implies $x^{\frac{1}{n}} > 1$ it follows that $\{x^{\frac{1}{n}}\}$ is bounded below.

Hence, by the MCT , $L := \lim_{n\to\infty} x^{\frac{1}{n}}$ exists .

To find its value L , we have

$$L = \lim_{n\to\infty} x^{\frac{1}{n}} = \lim_{n\to\infty} (x^{\frac{1}{2n}})^2 = \left(\lim_{n\to\infty} \frac{1}{x^{2n}}\right)^2 = L^2$$

$\Longrightarrow L = L^2$ ie $L=0$ or $L=1$.

since $x^{\frac{1}{n}} > 1$, the comparison Thm shows that

$\lim_{n\to\infty} x^{\frac{1}{n}} \geq \lim_{n\to\infty} 1 \rightarrow$ ie $L \geq 1$

Hence $L = 1$ ✓ So reject $L=0$.

case 3 : $0<x<1$, Then $\frac{1}{x} > 1$

It follows from case 2 that $\lim_{n\to\infty} x^{\frac{1}{n}} = \dfrac{1}{\lim_{n\to\infty} (\frac{1}{x})^{\frac{1}{n}}} = 1$

$=1$ by case 2

$x>1 \rightarrow \lim = 1$.

done .

Df 2 : A sequence of <u>sets</u> $\{I_n\}_{n\in\mathbb{N}}$ is said to be **nested** iff

$I_1 \supseteq I_2 \supseteq \ldots\ldots$ (contains, contains) $\qquad \{I_1, I_2, \ldots\}$

exp: $(0, \frac{1}{n})$

$I_1 = (0,1)$

$I_2 = (0, \frac{1}{2})$

(exp: $(0,n)$ not nested.)

RMK : This is monotone property for sequence of sets.

$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$

So $\{I_n\} = \{(0, \frac{1}{n})\}$ is nested.

$\bigcap_{n=1}^{\infty} I_n = \emptyset$

Thm 2 : Nested Interval property.

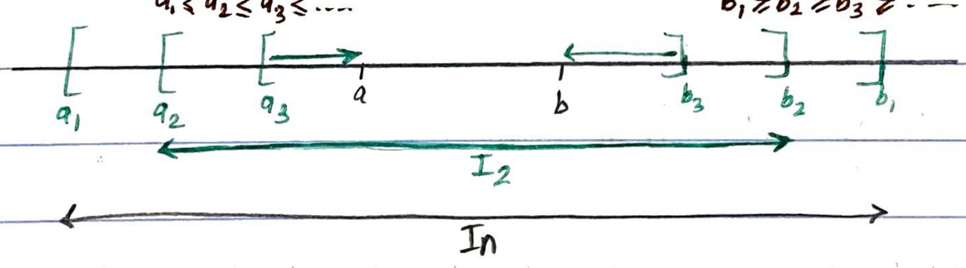
$I_n = [a_n, b_n]$

If $\{I_n\}_{n\in\mathbb{N}}$ is a nested sequence of nonempty close bdd intervals, then $\boxed{E := \bigcap_{n=1}^{\infty} I_n \neq \emptyset}$. Moreover, if the lengths of these intervals satisfy $|I_n| \to 0$ as $n \to \infty$, then E is a single point.

$I_n = b_n - a_n$ (length)

proof : let $I_n = [a_n, b_n]$. since I_n is nested, then $\{a_n\}$ is increasing seq. and bdd above by b_1 and $\{b_n\}$ is decreasing and bdd below by a_1

$a_1 \leq a_2 \leq a_3 \leq \cdots \qquad b_1 \geq b_2 \geq b_3 \geq \cdots$



Thus By MCT, $\exists\, a, b \in \mathbb{R}$ s.t $a_n \uparrow a$ and $b_n \downarrow b$ as $n \to \infty$.

(def of interval) since $a_n \leq b_n$, $\forall n \in \mathbb{N}$ it follows that

(comparison Thm) $\leftarrow \qquad a = \lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n = b$

Thus, $\quad a_n \leq a \leq b \leq b_n$.

Hence, $x \in I_n$, $\forall n \in \mathbb{N}$ iff $x \in [a,b]$

Inparticular, any $x \in [a,b]$, $x \in \bigcap_{n=1}^{\infty} I_n$ i.e $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Next If $|I_n| \to 0$ as $n \to \infty$ Then

$$(b_n - a_n) \to 0 \text{ as } n \to \infty$$

$$b - a = \lim_{n\to\infty} (b_n - a_n) = 0 \Rightarrow b - a = 0 \Rightarrow b = a$$

Hence, E is a single point if $|I_n| \to 0$ as $n \to \infty$

RMK 1: The Nested Interval property (Thm 2) might not hold if "closed" is omitted.

closed

proof: $I_n = (0, \frac{1}{n})$, $n \in \mathbb{N}$ are bdd and nested

counter example

$(I_1 = (0,1) \supseteq (0, \frac{1}{2}) = I_2 \supset \dots)$ But Not closed.

claim: $\bigcap_{n=1}^{\infty} I_n = \emptyset$

spse not, $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$, $\exists x \in I_n$, $\forall n \in \mathbb{N}$

i.e $0 < x < \frac{1}{n}$, $\forall n \in \mathbb{N}$

i.e $\boxed{n < \frac{1}{x}, \forall n \in \mathbb{N}}$ this contradicts the Archimedian principle

It follows that $\bigcap_{n=1}^{\infty} I_n = \emptyset$

RMK 2: The Nested Interval property (Thm 2) might not hold If "bounded" is omitted.

proof: The Interval $I_n = [n, \infty)$, $n \in \mathbb{N}$ are closed and nested but not bdd.

Again $\bigcap_{n=1}^{\infty} I_n = \emptyset$

Thm 3 : Bolzano - Weierstrass Theorem

every bounded sequence of real numbers has a convergent subsequence.

Proof : يؤجل لبعدين