Ch.5 :- Integration

1 (a) $\int \sin(5x)\,dx$

$$= \frac{-\cos(5x)}{5} + C = \boxed{\frac{-1}{5}\cos(5x) + C}$$

(b) $\int \tan^2 x\,dx$

$$= \int (\sec^2 x - 1)\,dx$$

$$= \boxed{\tan x - x + C}$$

(c) $\int (1 + \cot^2\theta)\,d\theta$

$$\int (1 + \csc^2\theta - 1)\,d\theta = \int \csc^2\theta\,d\theta$$

$$= \boxed{-\cot\theta + C}$$

(d) $\int \frac{\csc\theta\,d\theta}{\csc\theta - \sin\theta}$

$$= \int \frac{\frac{1}{\sin\theta}}{\frac{1}{\sin\theta} - \sin\theta}\,d\theta = \int \frac{\frac{1}{\sin\theta}}{\frac{1 - \sin^2\theta}{\sin\theta}}\,d\theta$$

$$= \int \frac{1}{\cos^2\theta}\,d\theta = \int \sec^2\theta\,d\theta$$

$$= \boxed{\tan\theta + C}$$

$\boxed{2}$ Find the derivative of the following functions:-

(a) $y = \int_1^x \frac{dt}{t}$

$y' = \frac{1}{x} - 1.(0) = \frac{1}{x}$.

(b) $y = \int_0^{\sqrt{x}} \cos t \; dt$

$y' = \cos(\sqrt{x}) \cdot \left(\frac{1}{2\sqrt{x}}\right)$.

$= \frac{\cos\sqrt{x}}{2\sqrt{x}}$

(c) $y = \int_{\tan x}^{0} \frac{dt}{1+t^2}$

Note: $1 + \tan^2 x = \sec^2 x$
$1 + \cot^2 x = \csc^2 x$

$y = -\int_0^{\tan x} \frac{dt}{1+t^2}$

$y' = -\frac{1}{1+\tan^2 x} \cdot \sec^2 x = \frac{-\sec^2 x}{\sec^2 x} = \boxed{-1}$

$\boxed{3}$ Find the Linearization of $g(x) = 3 + \int_1^{x^2} \sec(t-1)\,dt$

at $\boxed{x=-1}$

The linearization of $f(x)$ at $x=a$

$$\boxed{\ell(x) = f(a) + f'(a)(x-a)}$$

$$g(-1) = 3 + \int_1^1 \sec(t-1)\,dt = 3$$

$$g'(x) = 0 + \sec(x^2-1)\cdot 2x = 2x\sec(x^2-1)$$

$$g'(-1) = -2\sec(0) = -2$$

$\longrightarrow$ Then the linearization of $g(x)$ at $x=-1$

$$\ell(x) = 3 + -2(x+1)$$

$$\ell(x) = 3 - 2x - 2 = 1 - 2x$$

Extra question Estimate the $g(x)$ value if $x=-0.9$

$$\ell(0.9) \approx g(0.9)$$

$$\ell(0.9) = 1 - 2(0.9) = -0.8$$

4 Solve the following definite Integrals:

ⓐ $\int_1^{\sqrt{2}} \frac{s^2+\sqrt{s}}{s^2}\, ds$

$$\int_1^{\sqrt{2}} 1 + \frac{\sqrt{s}}{s^2}\, ds = \int_1^{\sqrt{2}} 1 + s^{-\frac{3}{2}}\, ds$$

$$= s + -2 \cdot s^{-\frac{1}{2}} \Big|_1^{\sqrt{2}}$$

$$= \sqrt{2} - 2\left(\sqrt{2}\right)^{-1/2} - [1-2]$$

$$= \sqrt{2} - 2\,(2)^{-\frac{1}{4}} + 1$$

$$= \sqrt{2} - 2^{3/4} + 1$$

$$= \boxed{\sqrt{2} - \sqrt[4]{8} + 1}$$

ⓑ $\int_0^{\pi/6} (\sec x + \tan x)^2\, dx$

$$= \int_0^{\pi/6} \sec^2 x + 2\sec x \tan x + \tan^2 x \quad dx$$

$$= \int_0^{\pi/6} \left(\sec^2 x + 2\sec x \tan x + \sec^2 x - 1\right) dx$$

$$= \int_0^{\pi/6} \left(2\sec^2 x + 2\sec x \tan x - 1\right) dx$$

$$= 2\tan x + 2\sec x - x \Big|_0^{\pi/6}$$

$$= 2\cdot\left(\frac{1}{\sqrt{3}}\right) + 2\left(\frac{2}{\sqrt{3}}\right) - \frac{\pi}{6} - \left(2(0) + 2(1) - 0\right)$$

$$= \frac{2}{\sqrt{3}} + \frac{4}{\sqrt{3}} - \frac{\pi}{6} - 2 = \cancel{\frac{6}{\sqrt{3}} + \frac{4}{\sqrt{3}}} \quad \boxed{\frac{6}{\sqrt{3}} - \frac{\pi}{6} - 2}$$

$2\sqrt{3} - \pi/6 - 2$

$\boxed{C}$ $\int_0^{\pi} \cos x + |\cos x| \quad dx$

$$|\cos x| = \begin{cases} \cos x & 0 \leq x \leq \frac{\pi}{2} \\ -\cos x & \frac{\pi}{2} \leq x \leq \pi \end{cases}$$

$$\int_0^{\pi/2} 2\cos x \quad dx \quad + \int_{\pi/2}^{\pi} (\cos x - \cos x) \; dx$$

$$\int_0^{\pi/2} 2\cos x \quad dx \quad + \int_{\pi/2}^{\pi} 0 \quad dx$$

$$= 2 . \sin x \Big|_0^{\pi/2} = 2(1) - 2(0) = 2$$

$\boxed{Q5}$ Use Subshitution to Solve the following integral :-

ⓐ $\int \frac{dx}{\sqrt{x}\,(1+\sqrt{x})^2}$

Assume $U = 1 + \sqrt{x}$

$dU = \frac{1}{2\sqrt{x}} \, dx$

$$\int \frac{2\cancel{\sqrt{x}} \, dU}{\cancel{\sqrt{x}} \, (U)^2}$$

$$\frac{2 . U^{-1}}{-1} + C = \frac{-2}{U} + C$$

last step $= \boxed{\frac{-2}{1+\sqrt{x}} + C}$

b $\int \frac{\sec z \tan z}{\sqrt{\sec z}} \, dz$

let $w = \sec z$

$dw = \sec z \tan z \, dz$

$\int \frac{\cancel{\sec z \tan z}}{\sqrt{w}} \quad \frac{dw}{\cancel{\sec z \tan z}}$

$\int w^{-1/2} \, dw = 2w^{1/2} + C$

$\boxed{= 2\sqrt{\sec z} + C}$

c $\int \sqrt{\frac{x-1}{x^5}} \, dx$

$\int \sqrt{\frac{x-1}{x} \cdot \frac{1}{x^4}} \, dx$

$\int \sqrt{\frac{x-1}{x}} \cdot \frac{dx}{x^2}$

let $u = \frac{x-1}{x}$

$u = 1 - \frac{1}{x}$

$du = (0 - - x^{-2}) \, dx$

$du = \frac{1}{x^2} \, dx$

$\int \sqrt{u} \, \frac{x^2 \, du}{x^2}$

$\int \sqrt{u} \, du$

$= \frac{2}{3} u^{3/2} + C$

$\boxed{= \frac{2}{3} \left(\frac{x-1}{x} \right)^{3/2} + C}$

$\boxed{d}$ $\int x^3 \sqrt{x^2+1}\, dx$

let $z = x^2+1$

$dz = 2x\, dx$

$\int x^3 \sqrt{z} \frac{dz}{2x}$

$\frac{1}{2} \int x^2 \sqrt{z}\, dz$

$\frac{1}{2} \int (z-1)\, z^{1/2}\, dz$

$\frac{1}{2} \int z^{3/2} - z^{1/2}\, dz$

$\frac{1}{2} \left[\frac{2}{5} z^{5/2} - \frac{2}{3} z^{3/2}\right] + C$

$\frac{1}{5} z^{5/2} - \frac{1}{3} z^{3/2} + C$

$= \frac{1}{5}(x^2+1)^{5/2} - \frac{1}{3}(x^2+1)^{3/2} + C$

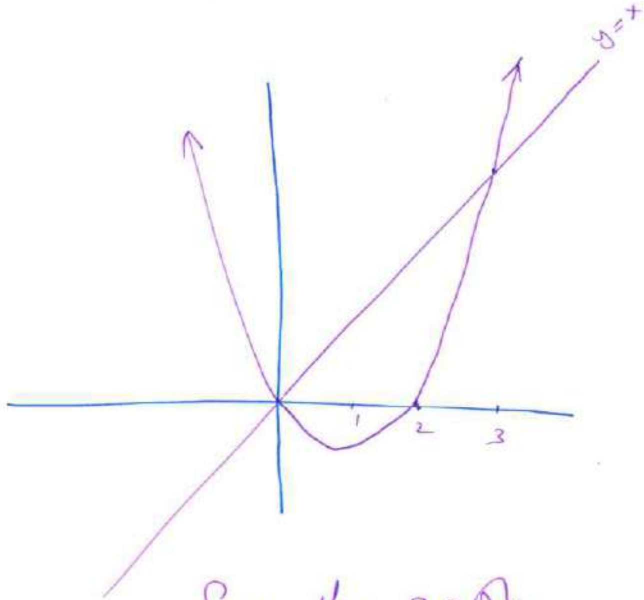
Q_6 Find the area enclosed between the given functions [in the x-y plane]

يمكننا ايجاد Area بدون الرسم باستخدام القيمه المطلقه

$$\text{Area} = \left| \int_a^b f(x) - g(x) \, dx \right|$$ Since a, b are the intersection points

$$\text{Area} = \left| \int_c^d f(y) - g(y) \, dy \right|$$ c, d are the intersection points

ⓐ $y = x^2 - 2x$, $y = x$



Intersection point :-

$$x^2 - 2x = x$$

$$x^2 - 3x = 0$$

$$\boxed{x=0} \quad \boxed{x=3}$$

from the graph

$x > x^2 - 2x$ in $[0, 3]$

$$A = \int_0^3 x - (x^2 - 2x) \, dx$$

$$= \int_0^3 x - x^2 + 2x \, dx = \int_0^3 3x - x^2 \, dx$$

$$= \left[\frac{3x^2}{2} - \frac{x^3}{3} \right]_0^3 = \frac{27}{2} - \frac{27}{3} - 0$$

$$= \frac{27}{2} - 9 = \frac{9}{2}$$

(b) $y = x^2$, $y = -x^2+4x$

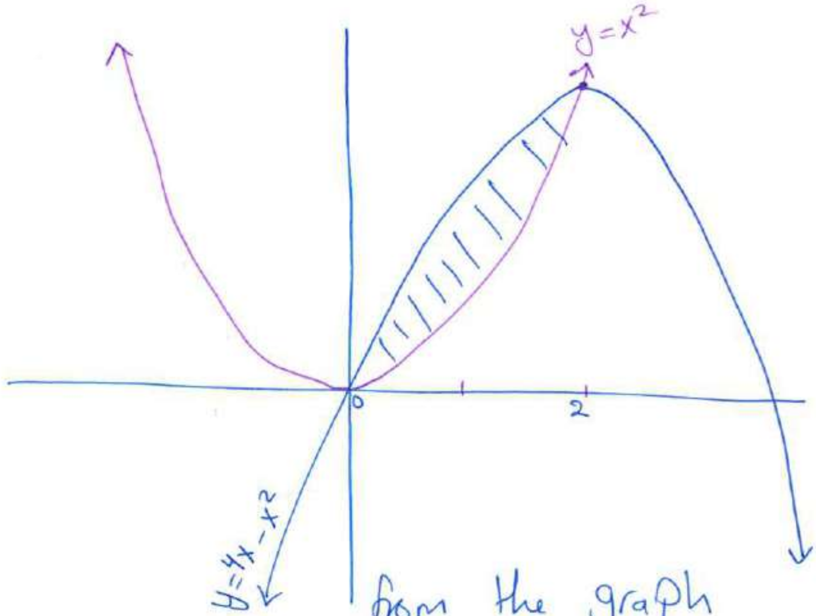
Intersection point:

$x^2 = -x^2 + 4x$

$2x^2 - 4x = 0$

$2x[x-2] = 0$

$\boxed{x=0}$ $\boxed{x=2}$



from the graph

$4x - x^2 > x^2$

$$A = \int_0^2 4x - x^2 - x^2 \, dx$$

$$\int_0^2 4x - 2x^2 \, dx$$

$$\frac{\overset{2}{\cancel{4}}x^2}{\cancel{2}} - \frac{2x^3}{3}\Big]_0^2 = 8 - \frac{16}{3} = \boxed{\frac{8}{3}}$$

(c) $x = y^2$ $\qquad x = 3 - 2y^2$

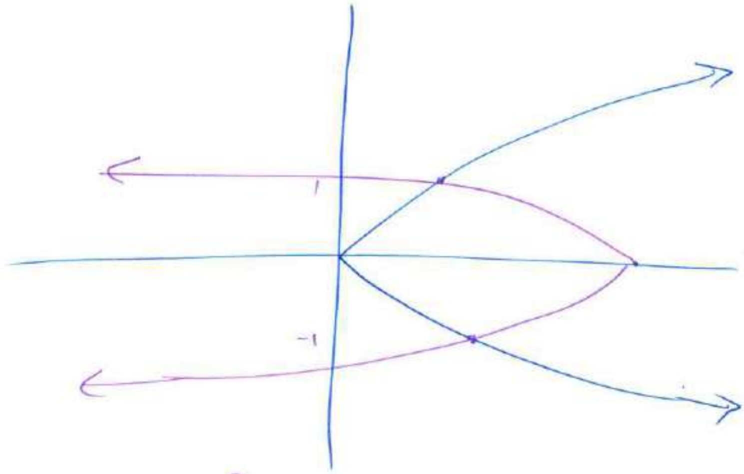
Intersection point

$y^2 = 3 - 2y^2$

$3y^2 = 3$

$y^2 = 1$

$\boxed{y = \pm 1}$



from the graph.

$\boxed{3 - 2y^2 > y^2}$ in the interval $y = [-1, 1]$

$$A = \int_{-1}^{1} 3 - 2y^2 - y^2 \, dy$$

$$= \int_{-1}^{1} 3 - 3y^2 \, dy = 3y - \frac{3y^3}{3} \Big]_{-1}^{1}$$

$$= 3 - 1 - [-3 + 1] = \boxed{4}$$

(d) $X = y^3 - y^2$ $\quad\quad$ $X = 2y$

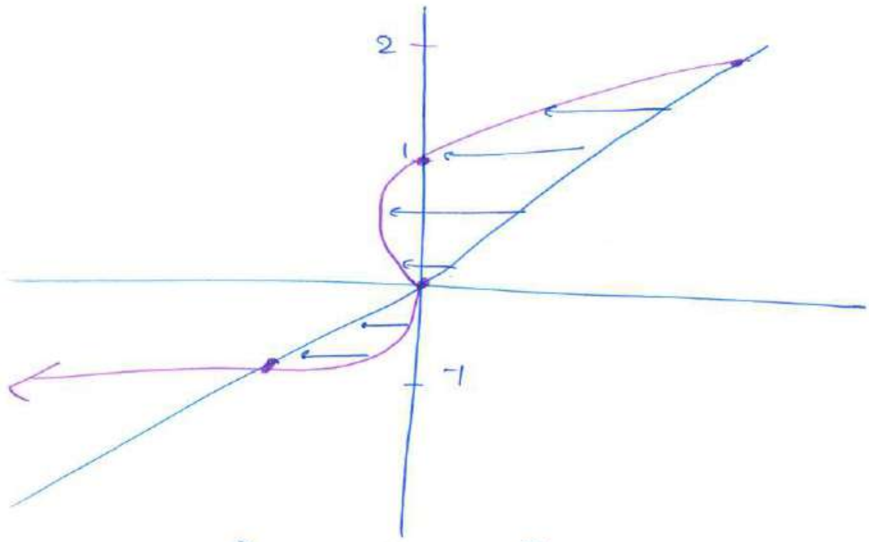
Intersection point

$$y^3 - y^2 = 2y$$

$$y^3 - y^2 - 2y = 0$$

$$y(y^2 - y - 2) = 0$$

$\boxed{y=0}$ $\boxed{y=-1}$ $\boxed{y=2}$



from the graph

$2y > y^3 - y^2$ $\quad y \in [0, 2]$

$y^3 - y^2 > 2y$ $\quad y \in [-1, 0]$

$$A = \int_{-1}^{0} (\quad) dy + \int_{0}^{2} (\quad) dy$$

$$A = \int_{-1}^{0} (y^3 - y^2 - 2y)\, dy + \int_{0}^{2} (2y - y^3 + y^2)\, dy$$

$$= \left[\frac{y^4}{4} - \frac{y^3}{3} - \frac{2y^2}{2}\right]_{-1}^{0} + \left[\frac{\cancel{2}y^2}{\cancel{2}} - \frac{y^4}{4} + \frac{y^3}{3}\right]_{0}^{2}$$

$$= 0 - \left[\frac{1}{4} + \frac{1}{3} - \frac{1}{1}\right] + \left[4 - 4 + \frac{8}{3}\right]$$

$$\frac{5}{12} + \frac{8}{3} = \boxed{\frac{37}{12}}$$

