8.4 Integration of Rational Function By Partial Fractions.

12 $\int \frac{2x+1}{x^2-7x+12}$

$$\int \frac{2x+1}{(x-4)(x-3)} = \frac{A}{x-4} + \frac{B}{x-3}$$

$$2x+1 = A(x-3) + B(x-4)$$

$\boxed{x=4} \longrightarrow A = 9$

$\boxed{x=3} \longrightarrow B = 7$

$$I = \int \frac{9}{x-4}\,dx + \int \frac{7}{x-3}\,dx$$

$$= 9\ln|x-4| + 7\ln|x-3| + C$$

$\boxed{14}$ $\displaystyle\int_{1/2}^{1} \frac{y+4}{y^2+y}\, dy$

$$\frac{y+4}{y(y+1)} = \frac{A}{y} + \frac{B}{y+1}$$

$$y+4 = A(y+1) + By$$

$y=0 \longrightarrow \boxed{A=4}$

$y=-1 \longrightarrow \boxed{B=-3}$

$$I = \int \frac{4}{y}\, dy + \int \frac{-3}{y+1}\, dy$$

$$= 4 \ln|y| - 3 \ln|y+1| + c$$

$$\int_{1/2}^{1} \frac{y+4}{y^2+y} = 4 \ln|y| - 3 \ln|y+1| \Big]_{1/2}^{1}$$

$$= -3 \ln 2 - 4 \ln\left(\tfrac{1}{2}\right) + 3 \ln\left(\tfrac{3}{2}\right)$$

$$= -3 \ln 2 + 4 \ln 2 + 3 \ln 3 - 3 \ln 2$$

$$= -2 \ln 2 + 3 \ln 3 = \ln \frac{27}{4}$$

$\boxed{15}$ $\displaystyle\int \frac{dt}{t^3+t^2-2t}$

$$\int \frac{dt}{t(t+2)(t-1)} = \frac{A}{t} + \frac{B}{t+2} + \frac{C}{t-1}$$

$$1 = A(t+2)(t-1) + Bt(t-1) + Ct(t+2)$$

$\boxed{t=0} \longrightarrow \boxed{A = \frac{-1}{2}}$

$\boxed{t=-2} \longrightarrow \boxed{B = \frac{1}{6}}$

$\boxed{t=1} \longrightarrow \boxed{C = \frac{1}{3}}$

$$I = \int \frac{-1/2}{t}\,dt + \int \frac{1/6}{t+2}\,dt + \int \frac{1/3}{t-1}\,dt$$

$$= \frac{-1}{2}\ln|t| + \frac{1}{6}\ln|t+2| + \frac{1}{3}\ln|t-1| + C.$$

$\boxed{20}$ $I = \int \frac{x^2}{(x-1)(x^2+2x+1)} dx$

$$\frac{x^2}{(x-1)(x^2+2x+1)} = \frac{x^2}{(x-1)(x+1)^2} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$x^2 = A(x+1)^2 + B(x-1)(x+1) + C(x-1)$$

$\boxed{x=1} \longrightarrow 4A = 1 \longrightarrow \boxed{A = \frac{1}{4}}$

$\boxed{x=-1} \longrightarrow -2C = 1 \longrightarrow \boxed{C = \frac{-1}{2}}$

$\boxed{x=0} \longrightarrow A + -B - C = 0$

$\frac{1}{4} - B + \frac{1}{2} = 0 \longrightarrow \boxed{B = \frac{3}{4}}$

$$I = \int \frac{1/4}{x-1} + \int \frac{3/4}{x+1} + \int \frac{-1/2}{(x+1)^2}$$

$\hookrightarrow U = x+1$

$$= \frac{1}{4} \ln|x-1| + \frac{3}{4} \ln|x+1| + \frac{1}{2} \frac{}{(1+x)} + C$$

$\boxed{23}$

$$I = \int \frac{y^2 + 2y + 1}{(y^2+1)^2}\, dy$$

$$\frac{y^2+2y+1}{(y^2+1)^2} = \frac{Ay+B}{(y^2+1)} + \frac{Cy+D}{(y^2+1)^2}$$

$$y^2+2y+1 = (Ay+B)(y^2+1) + Cy + D$$

$$y^2+2y+1 = Ay^3 + Ay + By^2 + B + Cy + D$$

$$y^2+2y+1 = Ay^3 + By^2 + (A+C)y + (B+D)$$

$y^3 :$ $\boxed{A=0}$

$y^2 :$ $\boxed{B=1}$

$y :$ $\boxed{C=2}$

$y^0 :$ $\boxed{D=0}$

$$I = \int \frac{1}{y^2+1}\, dy + \int \frac{2y}{(y^2+1)^2}\, dy$$

let $U = y^2+1$
$dU = 2y\, dy$

$$I = \tan^{-1}(y) + \int \frac{1}{U^2}\, dU$$

$$I = \tan^{-1}(y) + \frac{-1}{y^2+1} + C = \tan^{-1}(y) - \frac{1}{y^2+1} + C$$

$\boxed{29}$ $\displaystyle\int \frac{x^2}{x^4-1}\,dx$

$x^4-1=(x^2-1)(x^2+1)$

$(x-1)(x+1)(x^2+1)$

$$\frac{x^2}{x^4-1}=\frac{A}{x-1}+\frac{B}{x+1}+\frac{Cx+D}{x^2+1}$$

$$x^2=A(x+1)(x^2+1)+B(x-1)(x^2+1)+(Cx+D)(x^2-1)$$

$\boxed{x=1} \longrightarrow 1=4A \longrightarrow A=\frac{1}{4}$

$\boxed{x=-1} \longrightarrow 1=-4B \longrightarrow B=\frac{-1}{4}$

$\boxed{x=0} \longrightarrow 0=A-B-D$

$0=\frac{1}{4}-\frac{1}{4}-D \longrightarrow D=\frac{1}{2}$

$\boxed{x=2} \longrightarrow 4=15A+5B+(2C+D)(3)$

$4=\frac{15}{4}+\frac{-5}{4}+6C+\frac{3}{2}$

$4=4+6C \longrightarrow \boxed{C=0}$

$$I=\int\frac{1/4}{x-1}\,dx-\int\frac{1/4}{x+1}\,dx+\frac{1}{2}\int\frac{1}{x^2+1}\,dx$$

$$=\frac{1}{4}\ln|x-1|-\frac{1}{4}\ln|x+1|+\frac{1}{2}\tan^{-1}x+C=\frac{1}{4}\ln\left|\frac{x-1}{x+1}\right|+\frac{1}{2}\tan^{-1}x+C$$

$\boxed{37}$ $\int \frac{y^4+y^2-1}{y^3+y}\,dy$

$$\begin{array}{r} y \\ y^3+y \,\big)\, \overline{y^4+y^2-1} \\ -y^4 \mp y^2 \\ \hline -1 \end{array}$$

$\int y\,dy - \int \frac{1}{y^3+y}\,dy$

$= \frac{y^2}{2} - \underbrace{\int \frac{1}{y(y^2+1)}\,dy}_{I_1}$

$= \frac{1}{2}y^2 - I_1$

For $I_1 = \int \frac{1}{y(y^2+1)}\,dy$ $\quad \boxed{\frac{1}{y(y^2+1)} = \frac{A}{y} + \frac{By+C}{y^2+1}}$

$1 = A(y^2+1) + (By+C)y$

$1 = Ay^2 + A + By^2 + Cy$

$A+B=0$, $\boxed{C=0}$, $\boxed{A=1}$, $\boxed{B=-1}$

$I_1 = \int \frac{1}{y}\,dy + \int \frac{-y}{y^2+1}\,dy = \ln|y| - \frac{1}{2}\ln|y^2+1| + C_1$

$\boxed{I = \frac{1}{2}y^2 - \ln|y| + \frac{1}{2}\ln(y^2+1) + C}$.

$\boxed{42}$ $$I = \int \frac{\sin\theta}{\cos^2\theta + \cos\theta - 2}\, d\theta$$

let $y = \cos\theta$
$dy = -\sin\theta\, d\theta$

$$I = \int \frac{-dy}{y^2 + y - 2}$$

$$= -\int \frac{1}{(y+2)(y-1)}\, dy$$

$$\boxed{\frac{1}{(y+2)(y-1)} = \frac{A}{y+2} + \frac{B}{y-1}}$$

$$1 = A(y-1) + B(y+2)$$

$\boxed{y=-2} \longrightarrow A = \frac{-1}{3}$

$\boxed{y=1} \longrightarrow B = \frac{1}{3}$

$$I = -\int \frac{-1/3}{y+2}\, dy - \int \frac{1/3}{y-1}\, dy$$

$$= \frac{1}{3}\int \frac{1}{y+2}\, dy - \frac{1}{3}\int \frac{1}{y-1}\, dy$$

$$= \frac{1}{3}\ln|y+2| - \frac{1}{3}\ln|y-1| + C$$

$$= \frac{1}{3}\ln\left|\frac{y+2}{y-1}\right| + C = \frac{1}{3}\ln\left|\frac{\cos\theta+2}{\cos\theta-1}\right| + C$$

$\boxed{47}$

$$I = \int \frac{\sqrt{x+1}}{x}\, dx$$

let $y = \sqrt{x+1} \longrightarrow x+1 = y^2$

$$\boxed{dx = 2y\, dy}$$

$$I = \int \frac{y}{y^2-1}\, 2y\, dy$$

$$= \int \frac{2y^2}{y^2-1}\, dy$$

$$\begin{array}{r|l} & 2 \\ \hline y^2-1 & 2y^2 \\ & -2y^2 \pm 2 \\ \hline & 2 \end{array}$$

$$= \int \left(2 + \frac{2}{y^2+1}\right) dy$$

$$= \int 2\, dy + 2\int \frac{1}{(y-1)(y+1)}\, dy$$

$$\frac{1}{(y-1)(y+1)} = \frac{A}{y-1} + \frac{B}{y+1}$$

$\boxed{y=1} \longrightarrow A = \frac{1}{2}$

$\boxed{y=-1} \longrightarrow B = \frac{-1}{2}$

$$= 2y + \frac{2}{2}\int \frac{1}{y-1} - \frac{2}{2}\int \frac{1}{y+1}$$

$$= 2y + \ln|y-1| - \ln|y+1| + C$$

$$= 2y + \ln\left|\frac{y-1}{y+1}\right| + C$$

$$\boxed{I = 2\sqrt{x+1} + \ln\left|\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right| + C}$$

$\boxed{49}$

$$I = \int \frac{1}{x(x^4+1)}\,dx$$

$$\left\{\begin{array}{l} y = x^4 + 1 \\ \dfrac{dy}{dx} = 4x^3 \end{array}\right.$$

$$= \int \frac{1}{y \cdot x}\,\frac{dy}{4x^3}$$

$$= \frac{1}{4}\int \frac{1}{y\,(y-1)}\,dy$$

$$\boxed{\frac{1}{y(y-1)} = \frac{A}{y} + \frac{B}{y-1}}$$

$$1 = A(y-1) + B(y)$$

$$\boxed{y=0} \longrightarrow \boxed{A=-1}$$

$$\boxed{y=1} \longrightarrow \boxed{B=1}$$

$$I = \frac{1}{4}\int \frac{-1}{y} + \frac{1}{4}\int \frac{1}{y-1}$$

$$I = \frac{-1}{4}\ln|y| + \frac{1}{4}\ln|y-1| + C$$

$$I = \frac{1}{4}\ln\left|\frac{y-1}{y}\right| + C$$

$$I = \frac{1}{4}\ln\left(\frac{x^4}{x^4+1}\right) + C$$

$\boxed{54}$ Solve the initial value problem (I.V.P)

$$(t+1)\frac{dx}{dt} = x^2+1 \quad , \quad t>-1 \quad , \quad x(0)=0$$

Sol:-

$$(t+1)\,dx = (x^2+1)\,dt$$

$$\int \frac{dx}{x^2+1} = \int \frac{dt}{t+1}$$

$$\tan^{-1}x + C_1 = \ln|t+1| + C_2$$

$$\boxed{\tan^{-1}x = \ln|t+1| + C}$$

$x(0)=0 \longrightarrow x=0 \ , \ t=0$

$$\tan^{-1}(0) = \ln 1 + C$$

$$0 = 0 + C \longrightarrow \boxed{C=0}$$

$$\longrightarrow \tan^{-1}x = \ln(t+1) \ , \ t>-1$$

$$\boxed{x = \tan(\ln(t+1))}$$