3.4: uniform continuity.

Def 1: let E be a nonempty subset of $\mathbb{R}$ and $f: E \to \mathbb{R}$. Then f is said to be uniformly continuous on E iff $\forall \varepsilon > 0$, $\exists$ a $\delta > 0$ s.t

$$|x-a| < \delta \text{ and } x, a \in E \Rightarrow |f(x)-f(a)| < \varepsilon.$$

Notice that δ here depends on ε and f, But not on a and x.

expl: prove that $f(x) = x^2$ is uniformly continuous on $(0,1)$?

proof: given $\varepsilon > 0$ and set $\delta = \frac{\varepsilon}{2}$

If $x, a \in (0,1)$ and $|x-a| < \delta$, then

$$
\begin{aligned}
|f(x)-f(a)| &= |x^2-a^2| = |x+a|\,\underbrace{|x-a|}_{<\delta} \\
&\leq (|x|+|a|)\,|x-a| \\
&\leq (1+1)\,|x-a| \\
&= 2\,|x-a| \\
&< 2\delta \\
&< 2\left(\tfrac{\varepsilon}{2}\right) \\
&< \varepsilon \quad \boxed{\text{o}}
\end{aligned}
$$

$\rightarrow$

RMK :

1. The difference between the def'n of continuity and uniform continuity is that for a continuous function, δ may depend on a, where as for a uniformaly continuous function, δ must be choosen independently of a.

uniform منتظم

2. Every uniformly continuous function on E is also continuous on E. But the converse is not true. exp: $f(x) = x^2$ is cont. on $[0, \infty)$ But it is not uniformly cont.

→ Nonuniform continuity criteria :

let $E \subseteq \mathbb{R}$ and let $f : E \to \mathbb{R}$. Then the following statments are equivalent.

i. f is not uniformly cont. on E.

ii. $\exists$ an $\varepsilon_0 > 0$ s.t $\forall \delta > 0$ there are points $x, y \in E$ s.t $|x-y| < \delta$ and $|f(x) - f(y)| \geq \varepsilon_0$.

إذا بدي أثبت إنه مش uniform

iii. $\exists$ an $\varepsilon_0 > 0$ and two sequences $x_n, y_n \in E$ s.t $x_n - y_n \to 0$ as $n \to \infty$ and $|f(x_n) - f(y_n)| \geq \varepsilon_0$, $\forall n \in \mathbb{N}$

exp: Show that $f(x) = x^2$ is Not uniformly continuous on $[0,\infty)$.

Proof :

let $x_n = n + \frac{1}{n}$ and $y_n = n$

Then $|x_n - y_n| = \left|n + \frac{1}{n} - n\right| = \frac{1}{n} \longrightarrow 0$ as $n \to \infty$

But $|f(x_n) - f(y_n)| = \left|\left(n+\frac{1}{n}\right)^2 - n^2\right|$

$= \left|n^2 + 2 + \frac{1}{n^2} - n^2\right|$

$= 2 + \frac{1}{n^2}$

$\geq \underbrace{2}_{\varepsilon_0}, \quad \forall n$

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exp: Show that $f(x) = \frac{1}{x^2}$ is uniformly cont. on $[1,\infty)$ But it is NOT uniformly cont. on $(0,\infty)$.

proof.

$\rightarrow$ $f(x) = \frac{1}{x^2}$ is unifor. cont. on $[1,\infty)$.

let $\varepsilon > 0$, set $\delta = \frac{\varepsilon}{2}$

means
$x \geq 1$
$a \geq 1$

If $x, a \in [1,\infty)$ and $|x - a| < \delta$, Then

$|f(x) - f(a)| = \left|\frac{1}{x^2} - \frac{1}{a^2}\right| = \left(\frac{x+a}{x^2 a^2}\right)|x-a|$. توحيد مقامات وبعدها نفرق بين حدين

لما نكبر بالمقام
بتصير اقل

$= \left(\frac{1}{xa^2} + \frac{1}{x^2 a}\right)|x-a|$ نوزع البسط عالمقام

$\leq \left(\frac{1}{1} + \frac{1}{1}\right)|x-a|$

$= 2\,|x-a|$

$< 2\delta$

$< 2\left(\frac{\varepsilon}{2}\right)$

$< \varepsilon$

Continue :

$\rightarrow$ f is not unif. cont. on $(0,\infty)$

Set $x_n = \frac{1}{n}$, $y_n = \frac{1}{n+1}$

$$|x_n - y_n| = \left|\frac{1}{n} - \frac{1}{n+1}\right| = \frac{1}{n(n+1)} \rightarrow 0 \text{ as } n\rightarrow\infty .$$

$$|f(x_n) - f(y_n)| = |n^2 - (n+1)^2| = |n^2 - n^2 - 2n - 1|$$
$$= 2n+1$$
$$\geq 1 \quad \forall n .$$
$$\geq \varepsilon$$

exp: $f(x) = \sin\left(\frac{1}{x}\right)$ is Not unif. cont. on $(0,\infty)$.

proof:

let $x_n = \frac{1}{2n\pi}$ $\quad y_n = \frac{1}{2n\pi + \frac{\pi}{2}}$

$$|x_n - y_n| = \left|\frac{1}{2n\pi} - \frac{1}{2n\pi + \frac{\pi}{2}}\right| \rightarrow 0 \text{ as } n\rightarrow\infty$$

But $|f(x_n) - f(y_n)| = |\underbrace{\sin(2n\pi)}_{0} - \underbrace{\sin(2n\pi + \frac{\pi}{2})}_{\rightarrow 1}| = 1 \geq \varepsilon_0$

lemma: suppose that $E \subseteq \mathbb{R}$ and that $f: E \to \mathbb{R}$ is uniformly continuouse.
If $x_n \in E$ is cauchy then $\{f(x_n)\}$ is cauchy.

proof:

let $\varepsilon > 0$, since f is unif. conti. on E, $\exists$ a $\delta > 0$ s.t

$|x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$, $\forall x, a \in E$.

since $\{x_n\}$ is cauchy, $\exists$ $N \in \mathbb{N}$ s.t

$m, n \geq N \Rightarrow |x_n - x_m| < \delta$

Then $|f(x_n) - f(y_n)| < \varepsilon$, $\forall m, n \geq N$

This means $\{f(x_n)\}$ is cauchy.

Note: هلأ لو بهاي lemma بدلت مكان unif. conti. بانه يكون conti بس بكون غلط

counter example:

$x_n = \frac{1}{n}$, $f(x) = \frac{1}{x}$

$f(x_n) = n$ ✓

Thm 1: suppose that I is a closed, bounded interval. If $f: I \to \mathbb{R}$ is continuous on I, then f is uniformly continuous on I.

exp: $f(x) = x^2$, $[0,1]$

باستخدام هاي النظرية بكون اثبتت انها unifo. conti. بدون ما استخدم التعريف.

proof: suppose to the contrary that f is cont. But not uniformly on I.

Then, $\exists$ an $\varepsilon_0 > 0$ and $x_n, y_n \in I$ s.t $|x_n - y_n| < \frac{1}{n}$

and $|f(x_n) - f(y_n)| \geq \varepsilon_0$, $\forall n \in \mathbb{N}$.

By the Bolzano-weiers. thm and the compasison Thm,

$\{x_n\}$ has a convergent subsequ. say $x_{n_k} \to x \in I$ as $k \to \infty$.

similarly, the seq. $\{y_n\}$ has a conv. subseq. say $x_{n_{k_j}} \to x$ as $j \to \infty$.

and f is conti. it follows that

$$|f(x) - f(y)| \geq \varepsilon_0 \quad \text{i.e } f(x) \neq f(y).$$

But $|x_n - y_n| < \frac{1}{n}$, $\forall n \in \mathbb{N}$, so By squeeze thm $x = y$.

Therefore $f(x) = f(y)$ a contradiction. $\blacksquare$

RMK: Thm 1 might Not hold if « closed » replaced By « open »

exp $f(x) = \frac{1}{x}$ is cont. on $(0,1)$ But not uniformly cont. on $(0,1)$.

($x_n = \frac{1}{n}$, $y_n = \frac{1}{n+2}$, $|x_n - y_n| \to 0$ as $n \to \infty$

But $|f(x_n) - f(y_n)| = |n-(n+2)| = 2 \geq 1 \ \varepsilon_0$

Thm 1 might Not true if « bdd » replaced By « unbdd ».

exp $f(x) = x^2$ is cont. on $[0,\infty)$ But is Not uniformly cont. on $[0,\infty)$

Thm 2 :

proof X

suppose that $a<b$ and that $f:(a,b) \to \mathbb{R}$. Then f is uniformly continuous on (a,b) iff f can be continuously extended to $[a,b]$, i.e iff there is a continuous function $g:[a,b] \to \mathbb{R}$ which satisfies $f(x) = g(x)$, $x \in (a,b)$.

exp: prove that $f(x) = \frac{x-1}{\ln x}$ is uniformly continuous

proof:

$$\lim_{x\to 1^-} f(x) = \lim_{x \to 1^-} \frac{x-1}{\ln x} \overset{\frac{0}{0} \to \text{Lopital}}{=} \lim_{x\to 1^-} \frac{1}{\frac{1}{x}} = 1$$

Define

$$g(x) = \begin{cases} f(x) & , \ 0<x<1 \\ \lim_{x\to 0^+} f(x) & , \ x=0 \\ \lim_{x\to 1^-} f(x) & , \ x=1 \end{cases}$$

$$= \begin{cases} \frac{x-1}{\ln x} & , \ x\in(0,1) \\ 0 & , \ x=0 \\ 1 & , \ x=1 \end{cases} \rightarrow$$

Notice that $g : [0,1] \to \mathbb{R}$ is a conti. function on $[0,1]$ and

$g(x) = f(x) \quad \forall x \in (0,1)$.

Hence, f is continuously extendedable on $[0,1]$. So By Thm 2

f is uniformly cont. on $(0,1)$.

RMK: let f be conti. on a bounded, open, nondegenerate interval (a,b)

Notice that f is continuously extendable to $[a,b]$ iff $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ exist. Indeed, where they exist we define g at $x = a$ and $x = b$ as $g(a) = \lim_{x \to a^+} f(x)$, $g(b) = \lim_{x \to b^-} f(x)$.

و المثال الي قبل RMK

Exercises 0 → 6