Control systems 2

System Transient Performance

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Linear Time invariant system in Laplace and time domains:

Poles and Zeros of LTI Systems:
Given the transfer function of a proper system i(primitive rational function):

$$T(s) = \frac{\beta_m s^m + \beta_{m-1} s^{m-1} + \cdots + \beta_1 s^1 + \beta_0}{\alpha_n s^n + \alpha_{n-1} s^{n-1} + \cdots + \alpha_1 s^1 + \alpha_0} \quad \text{with } m < n$$

System Zeros:
A system zero is defined as the value s_z at which $|T(s_z)| = 0$.
A system zero can be a zero at finite or infinite.
A proper system has $n - m$ zeros at infinite, that is those that satisfy the relation $\lim_{s \to \infty} |T(s)| = 0$.

System Poles:
A system pole is defined as the value s_p at which $\lim_{s \to s_p} |T(s)| = \infty$.
A system pole can be a pole at finite or infinite.
An improper system (improper: $m \geq n$) has $m - n$ poles at infinite, that is those that satisfy the relation $\lim_{s \to \infty} |T(s)| = \infty$

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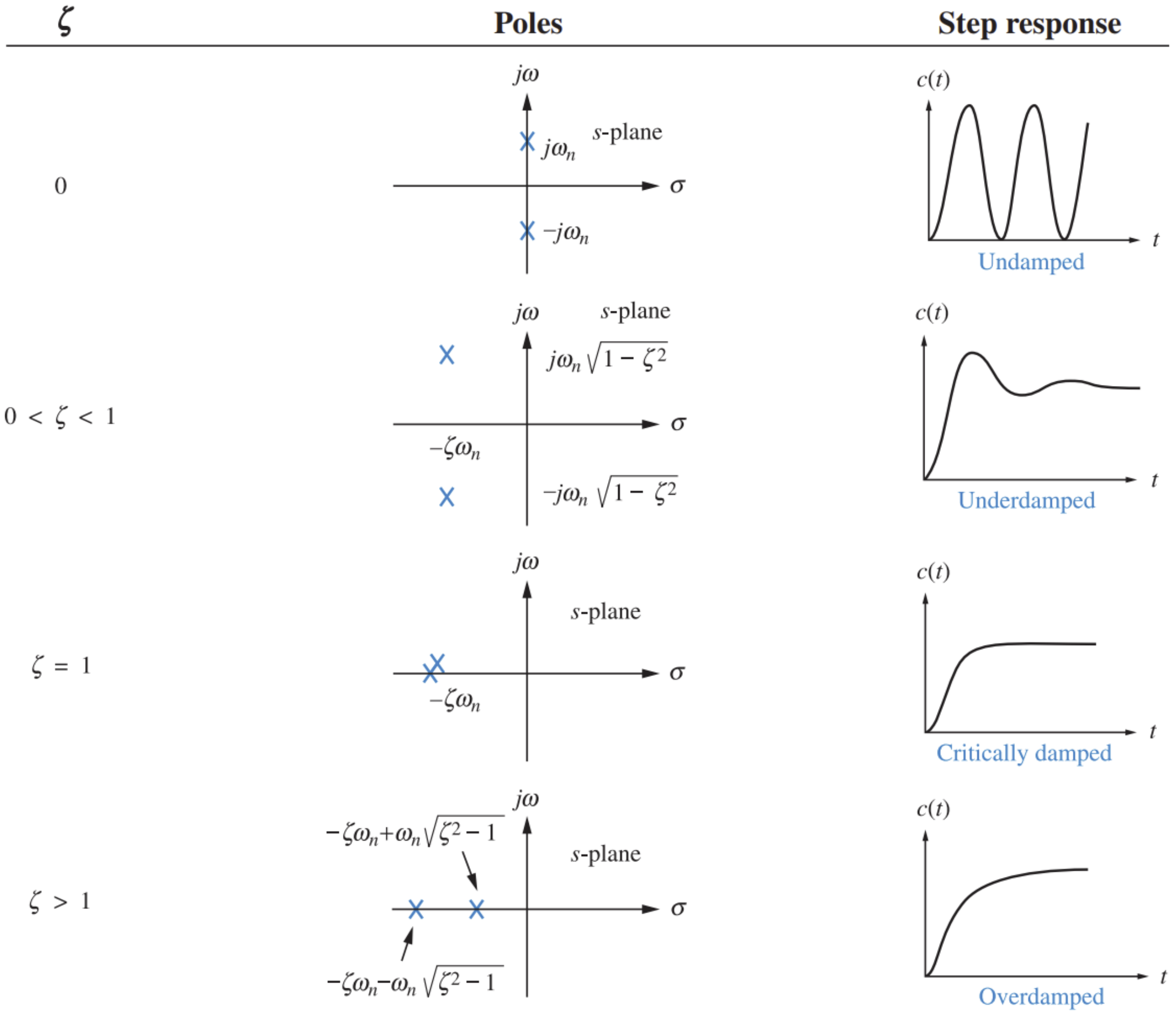
Effect of poles on system response:

The poles number and locations (system transfer function roots) determine the shape and the time performance of the transient response:

- Left-side poles generate a response that vanishes for $t \to \infty$, whereas right-side poles transient diverges
- Real-axe poles do not produce oscillation in the time response.
- Imaginary axe poles produce an undamped oscillation response.
- Complex poles produce oscillation in the response.



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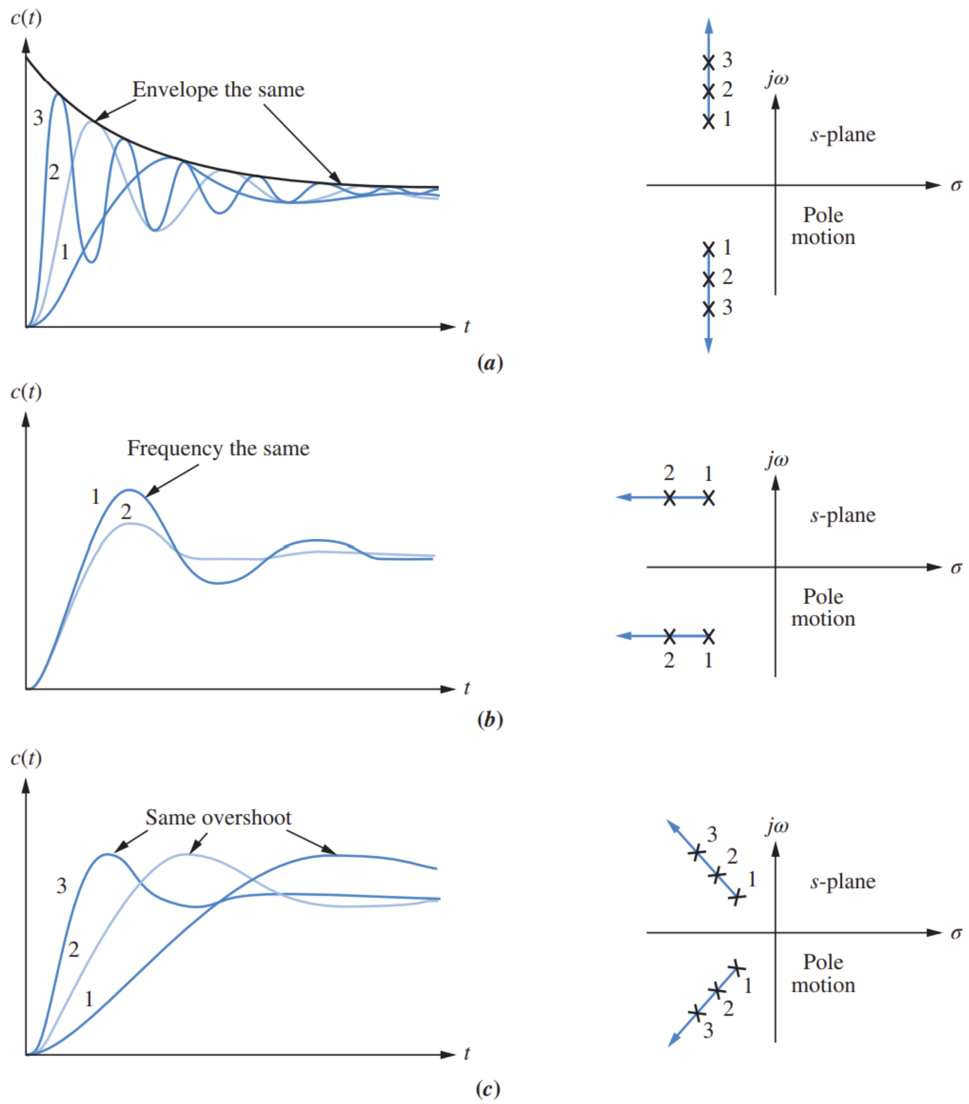
- Transient time performance depends on the relative distance between the imaginary axe and the pole location. That is the magnitude of the real part of the pole. Higher distance → Higher performance and faster transient. The time constant of a pole s_p is defined as $\tau_p = -\frac{1}{Re(s_p)}$
- Oscillation frequency depends on the relative distance between the real axe and the pole location. That is the magnitude of the imaginary part of the pole. Higher distance → Higher oscillation frequency and smaller period with higher density of oscillation cycles.
- Poles that are located on the same line have different time performance and oscillation frequency but equal damping ratio and relative overshoot value.



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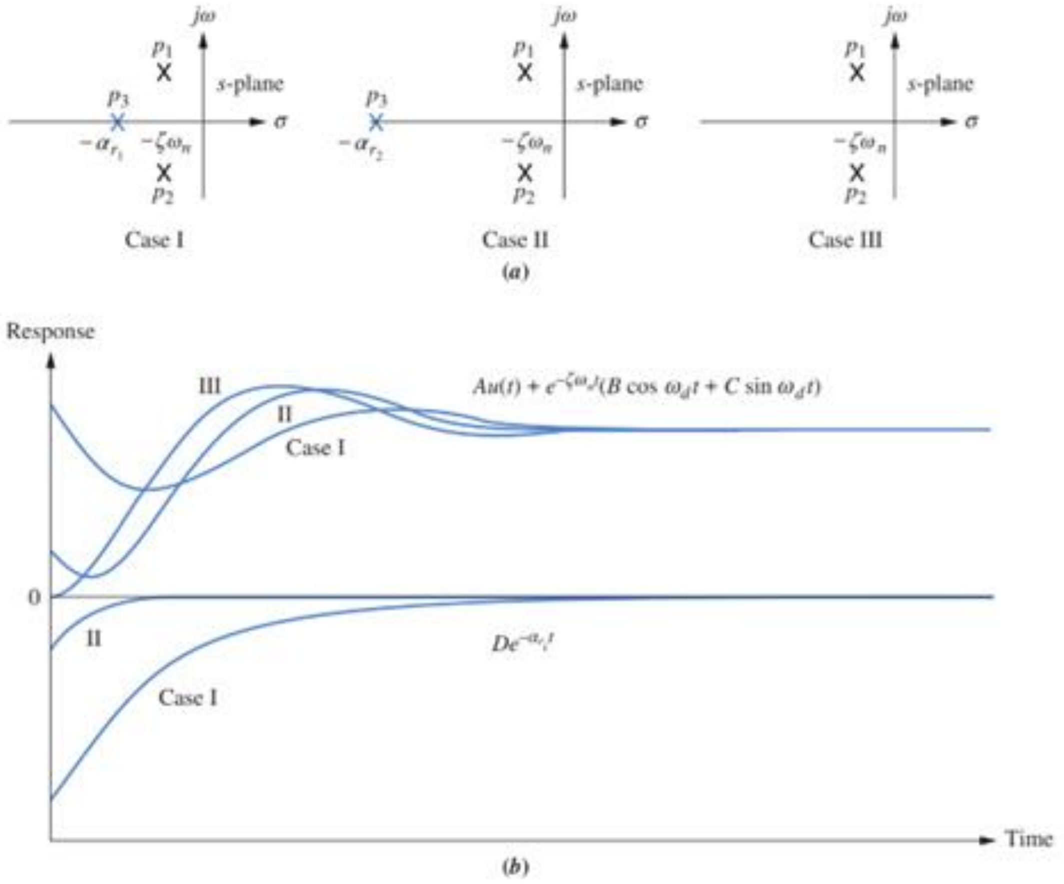
Dominant Poles:

- The set of dominant poles are those proximal to the imaginary axe and from the more distal poles with $\min(\tau_{dom}) > 5 \times \max(\tau_{non-dom})$. The dominant poles has slower transient response and thus they are the objective of the control problem.
- Considering the dominant poles reduces the order of the control system and the design of the controllers.



Component responses of a three-pole system: **a.** pole plot; **b.** component responses: Nondominant pole is near dominant second-order pair (Case I), far from the pair (Case II), and at infinity (Case III)

Pole-Zero Cancellation: zeros and poles can be set at the same position to cancel the effects of each other. Cancellation can be employed in controller design to cancel undesired effects or to reduce the order of the system(if it is a design degree of freedom)

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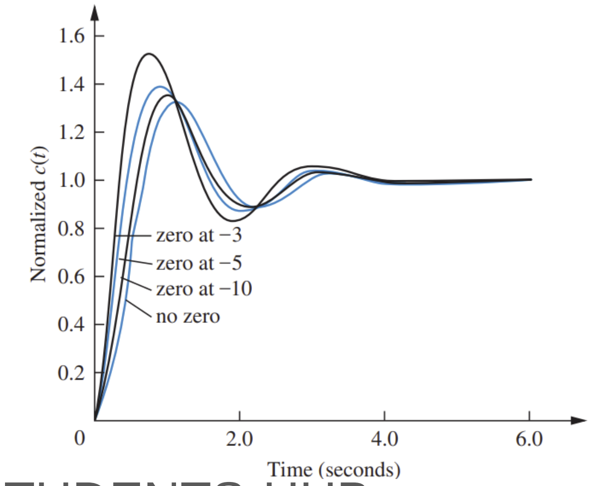
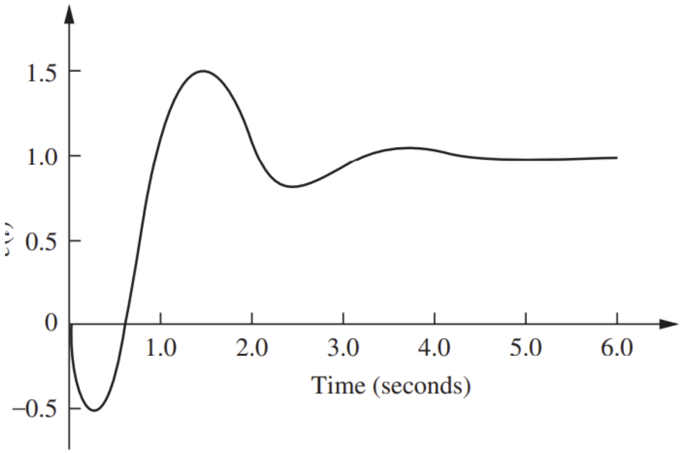
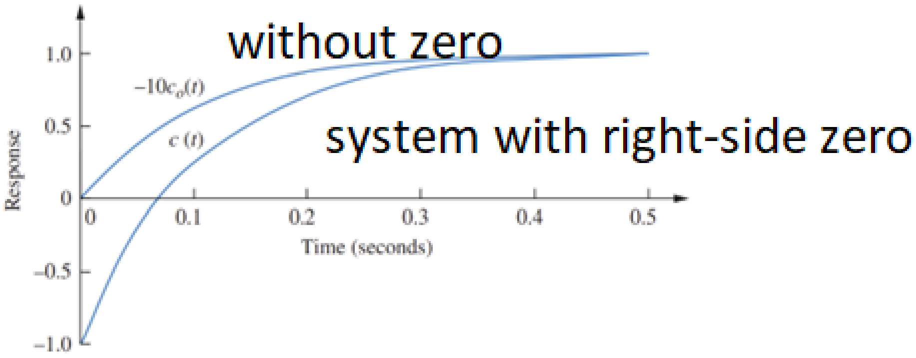
Effect of Zeros:

$$(s+a)C(s) = sC(s) + aC(s)$$

- The zeros affect the response amplitude.
- The effects of the zeros are more evident when they are more proximal to the dominant poles (zeros with smaller real part has a higher time constant and has a more evident effect on the system response).
- The zeros affect the response phase.
- A real zero (or the real part of a complex zero) introduces a derivative and proportional effect in the response without zero.
- For more distal zeros (from the imaginary axe) the proportional effect is higher than the derivative effect (fast zero effect). For the nearer zeros, the derivative effect is higher than the proportional one.
- Slower zeros cause higher signal overshoot because of the added positive value of the derivative.
- A left-side complex zero has a positive phase and thus an anticipation effect.
- A right-side complex zero has a negative phase and thus introduces a delay effect.
- A right-side zero with a smaller derivative effect than the proportional part may cause initial phase inversion.
- Asymptotically stable systems with only left-side zeros are said to be minimum-phase systems







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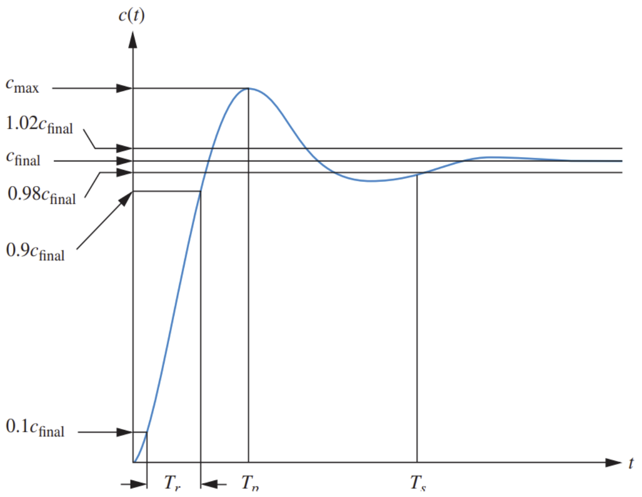
Performance parameters:

Performance parameters are used to set, evaluate, and compare the behavior of stable dynamic systems.

Time performance parameters:

- Rising time t_{ris}: the time necessary for the response to rise from 10% to 90% its final value.
- Delay-time t_d: the time necessary for the response to reach 50% of its final v
- Steady-state time (Settling time) t_{set}: at p% error: the time necessary for the response to reach and stay in $\pm 0.0p$ around its final value.
- Peak time t_{pn}: the time of the local maximum and minimum values of the response.
- Overshoot time t_{ov}: the time of the maximum deviation of the response from its final value.



Value Performance parameters:

- Overshoot (OV): the maximum deviation between the response and its final steady state value. $OV(t_{ov}) = y_{max}(t_{ov}) - y_{final}$. This parameter depends on the input value.

- Relative Overshoot (OV_r): the ratio of the overshoot and the response final value. That is $OV_r = \frac{y_{max}(t_{ov}) - y_{final}}{y_{final}}$ independent of the input value but requires the knowledge of the final value.

- Percentage Overshoot ($OV_r\%$): the ratio of the overshoot and the response final value. That is $OV_r = \frac{y_{max}(t_{ov}) - y_{final}}{y_{final}} \times 100\%$

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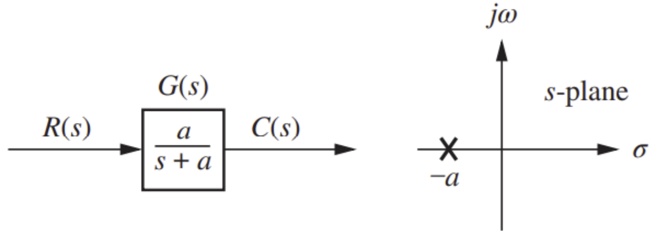
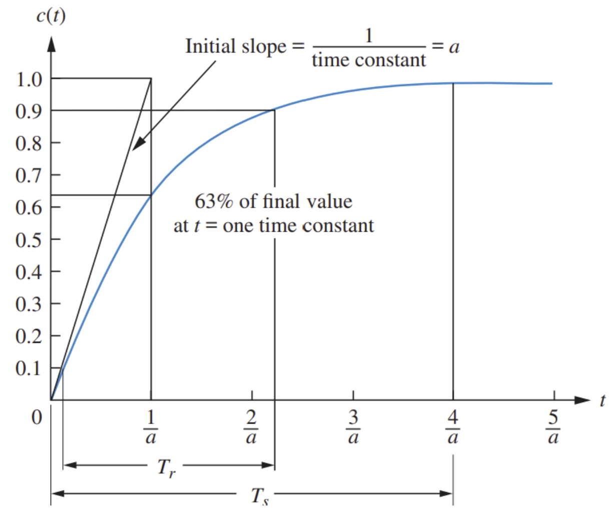
First Order System Step response:



(a) (b)

$$C(s) = R(s)G(s) = \frac{a}{s(s+a)}$$

$$c(t) = c_f(t) + c_n(t) = 1 - e^{-at}$$



Time constant and steady-state approximation:

- The time constant is defined as $\tau = \frac{1}{a}$
- Steady-state time: is the time at which the steady state response is assumed to be reached accepting and tolerating a defined maximum error value (because operations with the system can not be done for $t \to \infty$).
- The most used in Engineering is $t_{steady=}4\tau$ with approximately $error_{steady=}2\%$

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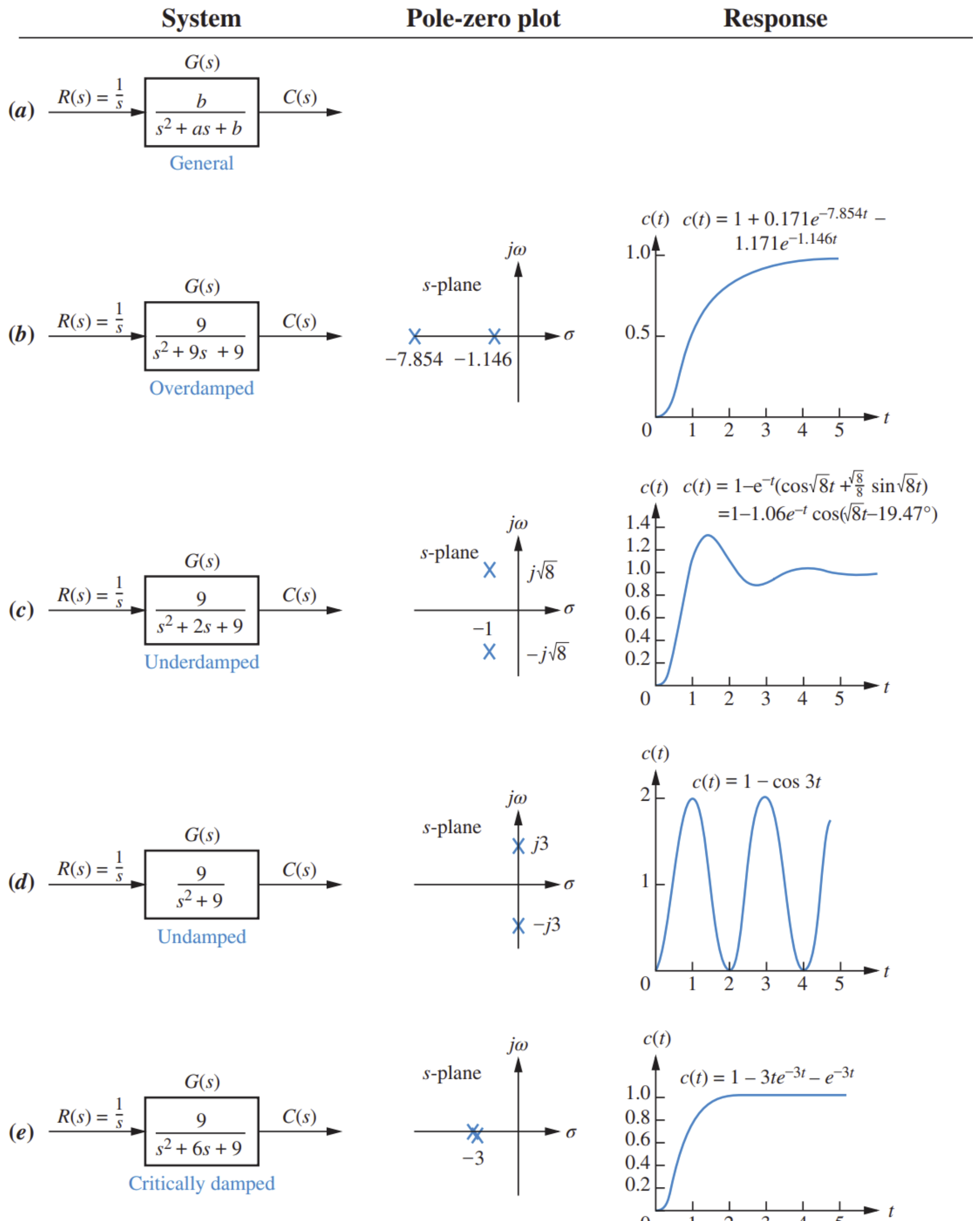
<u>Second Order System:</u>

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

The response of a stable second-order system has four shapes according to the classification of the poles of the system (roots of the characteristic algebraic equation.

- Overdamped oscillation response (no oscillation): real and different poles $\zeta > 1$ (positive discriminant)
- Critically damped response (change in convexity-start of oscillation: real and equal poles $\zeta = 1$ (discriminant=0)
- Undamped oscillation response (sustained oscillation): imaginary poles $\zeta = 0$ (negative discriminant with Re(pole)=0)
- Underdamped oscillation response:(damped oscillation): complex roots $0 < \zeta < 1$ (negative discriminant with $Re(pole) \neq 0$)



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Second-order system-performance parameters

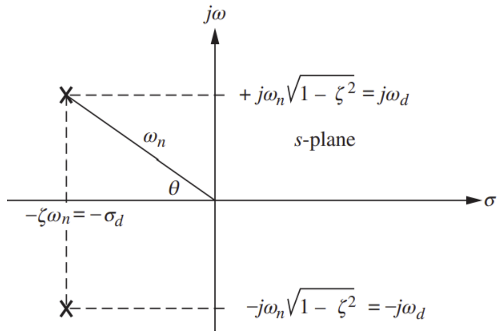
The underdamped response will be taken to determine the performance parameters because it has the maximum number of performance parameters.

Polar and cartesian representation:

a complex pair of system poles can be represented in:

cartesian form: $s_{1,2} = \alpha \pm j\omega_d$ $\quad$ α: *attenuation factor*, ω_d: *damped oscillation frequency*

Polar form: $s = \omega_n e^{j\theta}$ $\quad$ $\zeta = \cos\theta$: *attenuation ratio*, ω_n: *natural oscillation frequency*



Relations between polar and cartesian representations:

$$\alpha = \omega_n \cos\theta = \omega_n\zeta\,, \qquad \omega_d = \omega_n \sin\theta = \omega_n\sqrt{1-\zeta^2}$$

$$\omega_n = \sqrt{\alpha^2 + {\omega_d}^2}, \qquad \theta = -tan^{-1}\left(\frac{\omega_d}{\alpha}\right)$$

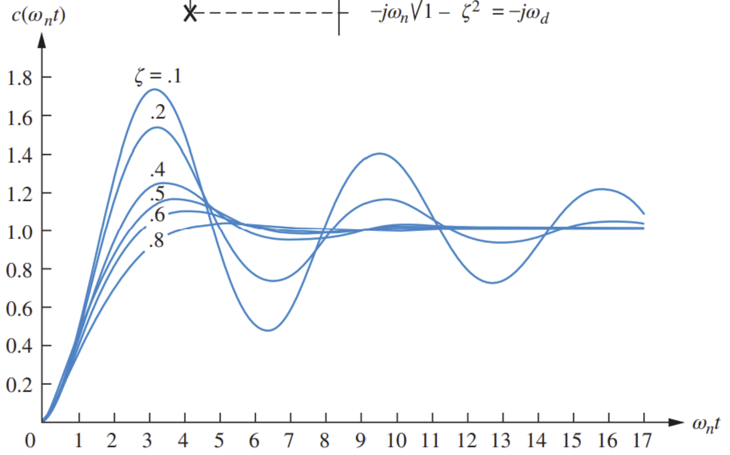
Under damped Step response:

$$C(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} = \frac{K_1}{s} + \frac{K_2 s + K_3}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

By computing the parameters using partial fractions and adjusting the function form to have the Laplace cosine and sine expression:

$$C(s) = \frac{1}{s} - \frac{(s+\zeta\omega_n) + \dfrac{\zeta}{\sqrt{1-\zeta^2}}\omega_n\sqrt{1-\zeta^2}}{(s+\zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)}$$

using the Laplace inverse we obtain:



$$c(t) = 1 - e^{-\zeta\omega_n t}\left(\cos\omega_n\sqrt{1-\zeta^2}\,t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_n\sqrt{1-\zeta^2}\,t\right)$$

$$= 1 - \frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\cos(\omega_n\sqrt{1-\zeta^2}\,t - \phi)$$

$$\phi = \tan^{-1}(\zeta/\sqrt{1-\zeta^2}).$$

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Performance parameters of the second-order system:

The most used performance parameters of the second-order system are the settling time, the overshoot time, and the overshoot value with all its variants.

Settling time at p% error:

To simplify computation it is assumed that the settling time is reaches at the first peak after the 0.0p, that is we consider a smaller error that satisfies the requirements. Thus, considering $c(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \cos(\omega_n \sqrt{1-\zeta^2}\, t - \phi)$

we have to solve the equation: $\frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t_{set}} = 0.0p$

Solving for t_{set} we obtain: $t_{set} = \frac{-\ln(0.op\sqrt{1-\zeta^2})}{\zeta\omega_n}$ at 2% this result is approximated as $t_{set\_2\%} = \frac{4}{|\zeta\omega_n|} = \frac{4}{|\alpha|} = 4\tau$

Peak and overshoot time:

The peak time is periodic and obtained by computing equating the derivative of the step response c(t) to zero.

Since the underdamped response of the second order system is strictly decreasing, the overshoot time is obtained at the first peak value. Considering: $C(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$ and the inverse of the derivative $\mathscr{L}[\dot{c}(t)] = sC(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

$$\mathscr{L}[\dot{c}(t)] = \frac{\omega_n^2}{(s+\zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)} = \frac{\frac{\omega_n}{\sqrt{1-\zeta^2}}\omega_n\sqrt{1-\zeta^2}}{(s+\zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)}$$

Applying Laplace inverse

$$\dot{c}(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin \omega_n \sqrt{1-\zeta^2}\, t$$

Setting the derivative equal to zero yields

$$\omega_n\sqrt{1-\zeta^2}\, t = n\pi \quad \leftrightarrow \quad t = \frac{n\pi}{\omega_n\sqrt{1-\zeta^2}}$$

Thus the overshoot time is: $T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\omega_d}$

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Overshoot evaluation: using $c(t) = 1 - e^{-\zeta\omega_n t}\left(\cos \omega_n \sqrt{1-\zeta^2}t + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \omega_n \sqrt{1-\zeta^2}t\right)$ and $\%OS = \frac{c_{max} - c_{final}}{c_{final}} \times 100$

compute $c_{max} = c(T_p) = 1 - e^{-(\zeta\pi/\sqrt{1-\zeta^2})}\left(\cos \pi + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \pi\right) = 1 + e^{-(\zeta\pi/\sqrt{1-\zeta^2})}$

Applying $c_{final} = 1$ in the percentage overshoot equation we obtain $\%OS = e^{-(\zeta\pi/\sqrt{1-\zeta^2})} \times 100$

Moreover, in the design problem, we can compute the damping ratio necessary to obtain a specific percentage overshoot by:

$$\zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}}$$

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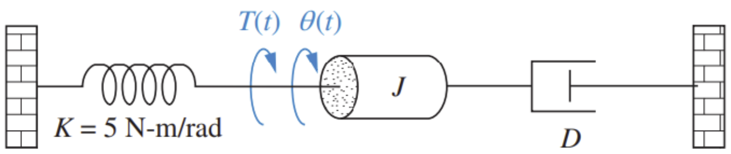
Example: Consider the following system and determine the moment of inertia and the damping coefficient to 20% overshoot and a 2% ERROR settling time of 2 seconds for a step torque input.



$$G(s) = \frac{1/J}{s^2 + \frac{D}{J}s + \frac{K}{J}} \rightarrow \quad \omega_n = \sqrt{\frac{K}{J}} \qquad 2\zeta\omega_n = \frac{D}{J}$$

$$T_s = 2 = \frac{4}{\zeta\omega_n} \rightarrow \zeta\omega_n = 2.$$

$$\zeta = \frac{4}{2\omega_n} = 2\sqrt{\frac{J}{K}} = 0.456 \rightarrow \frac{J}{K} = 0.052$$

$$\frac{J}{K} = 0.052$$

$$\frac{D}{J} = 4$$

$K = 5$ N-m/rad

$D = 1.04$ N-m-s/rad, and $J = 0.26$ kg-m^2

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Control Systems Representation

Block and Signal-flow Diagrams

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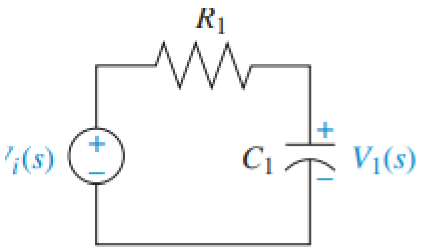
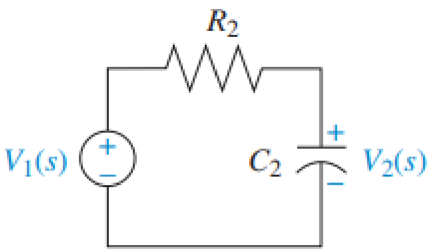
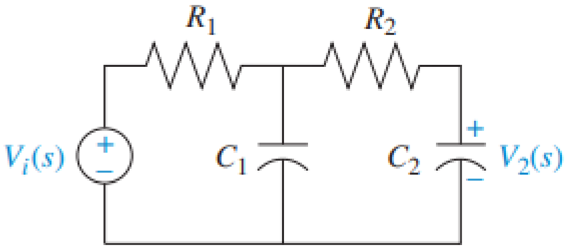
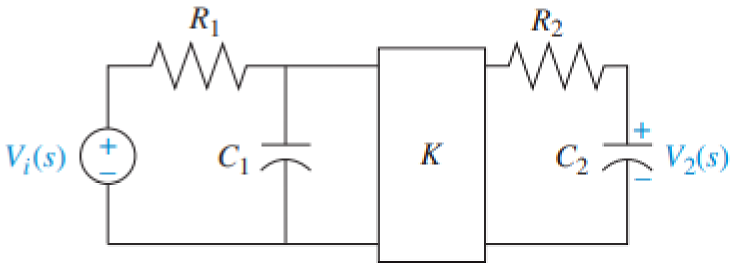
<u>Block Diagrams and Signal-flow Diagrams:</u>

- A control system is composed of several subsystems that interact and exchange signals and employ signal combination through sum nodes and distribution through derivation points.
- A simple representation that describes the subsystems interaction and signal flow become necessary to analyze and study.
- Block diagrams and Signal-flow diagrams are among the most used universal languages in control systems.
- An analogy relation exists between Block diagrams and signal flow diagrams. That is between the input and output vocabularies of the these languages and their grammar.
- Whenever, these representation are used, it is inherently assumed that the chain rule is satisfied, that is the connection of two subsystems does not affect the validity of their mathematical models. That is each system maintains its transfer relation.
- Specific subsystems interconnection (cascade, parallel, and feedback) and other rules related to signals combination (sum nodes) and extraction (derivation points) are used to reduce the system representation to an equivalent one that includes only the necessary components for the control systems objectives.









$$G_1(s) = \frac{V_1(s)}{V_i(s)}$$

$$G_2(s) = \frac{V_2(s)}{V_1(s)}$$

$$G_T(s) = \frac{V_2(s)}{V_i(s)} \neq G_2(s)G_1(s)$$

$$G_T(s) = \frac{V_2(s)}{V_i(s)} = KG_2(s)G_1(s)$$

$$G_1(s) = \frac{V_1(s)}{V_i(s)} = \frac{\frac{1}{R_1C_1}}{s + \frac{1}{R_1C_1}}$$

$$G_2(s) = \frac{V_2(s)}{V_1(s)} = \frac{\frac{1}{R_2C_2}}{s + \frac{1}{R_2C_2}}$$

$$G(s) = \frac{V_2(s)}{V_i(s)} = \frac{\frac{1}{R_1C_1R_2C_2}}{s^2 + \left(\frac{1}{R_1C_1} + \frac{1}{R_2C_2} + \frac{1}{R_2C_1}\right)s + \frac{1}{R_1C_1R_2C_2}}$$

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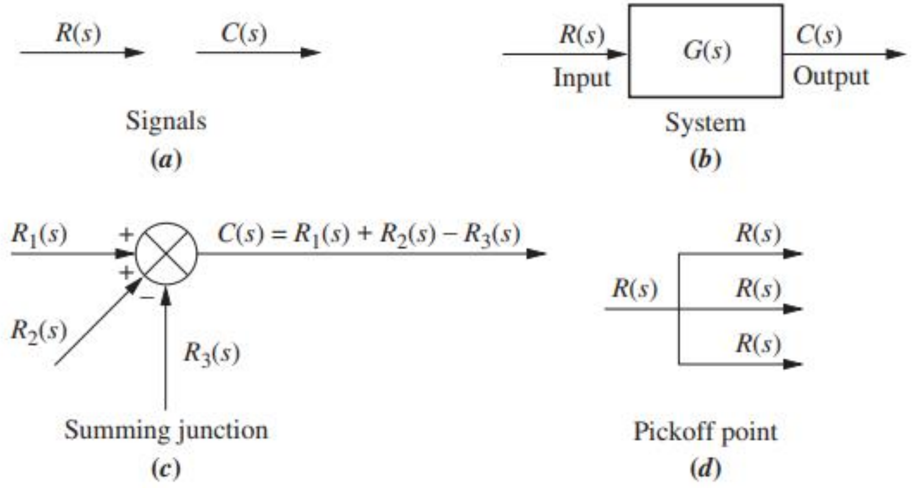
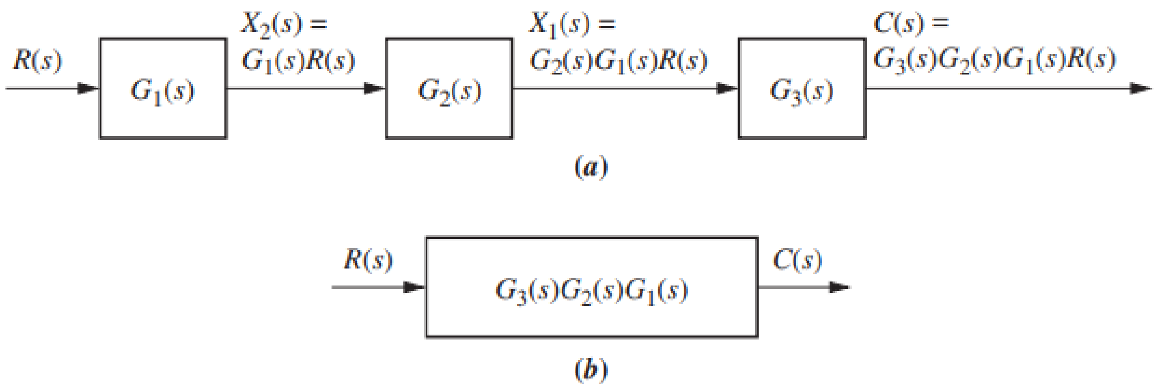
Block Diagrams:

The basic input vocabulary components of the block diagrams are shown in figure, with the signal represented by an arrow, system transfer relation (gain) represented by a block and signal combination and extraction represented by sum and derivation nodes.

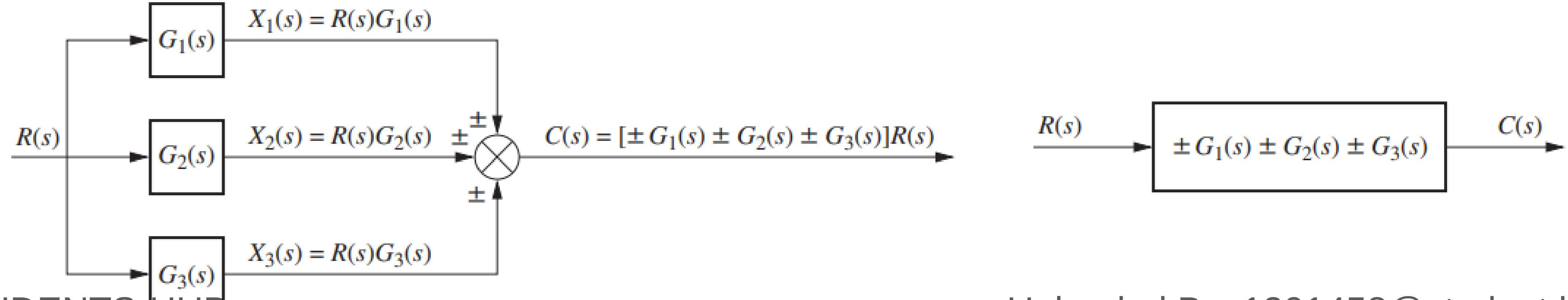


Block Algebra:

- Cascade connection



- Parallel connection



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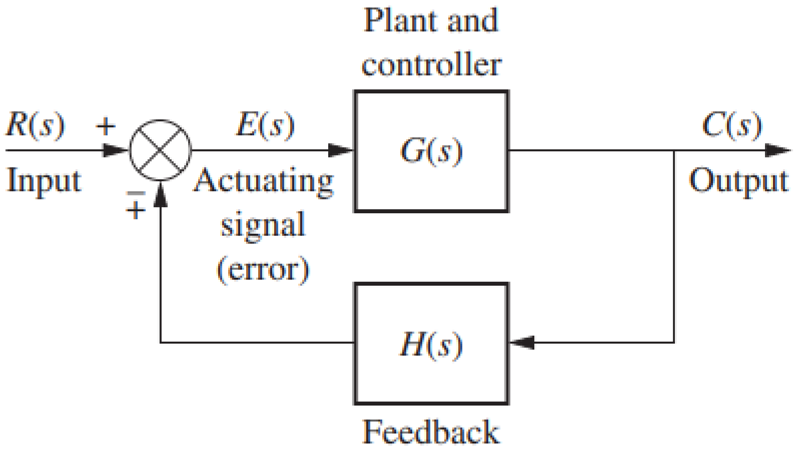
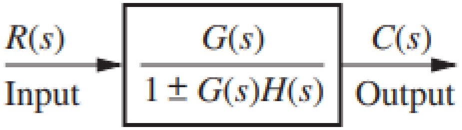
- Feedback connection



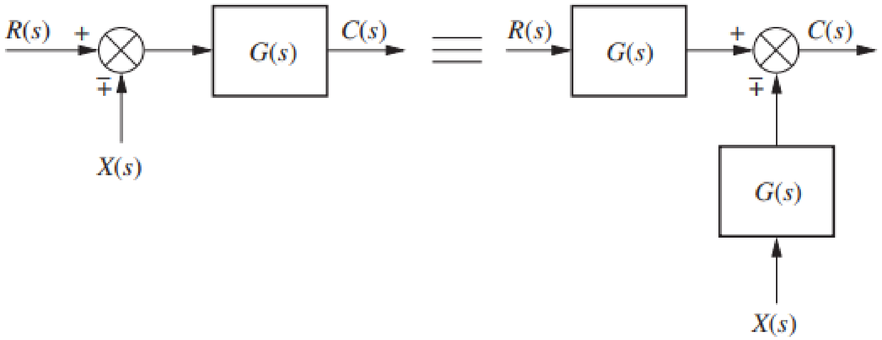
$$C(s) = G(s)E(s)$$
$$E(s) = R(s) - H(s)C(s) = R(s) - H(s)G(s)E(s) \rightarrow$$
$$E(s)\big(1 + H(s)G(s)\big) = R(s) \rightarrow E(s) = \frac{1}{1 + H(s)G(s)}R(s) \rightarrow$$
$$C(s) = \frac{G(s)}{1 + H(s)G(s)}R(s)$$



<u>Moving Blocks to Create Familiar Forms:</u>

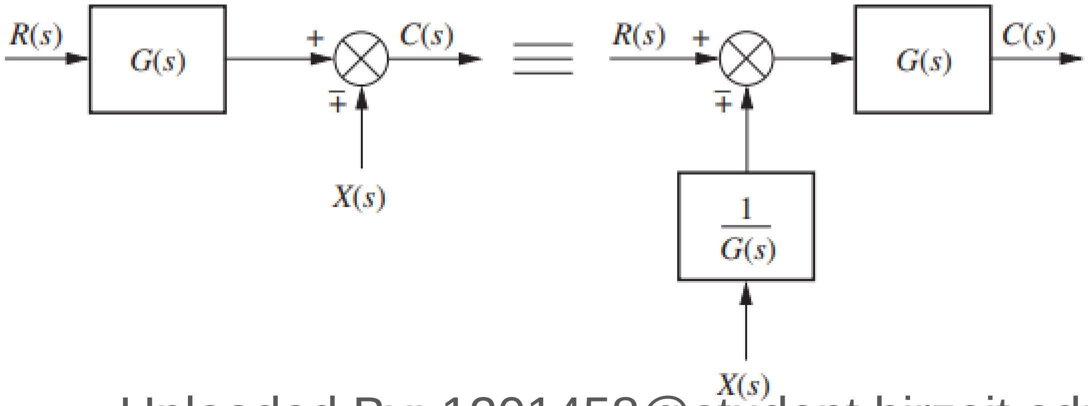
- Transfer of a sum node from the input to the output of a block:

$$C(s) = G(s)[R(s) + X(s)] = G(s)R(s) + G(s)X(s)$$



- Transfer of a sum node from the output to the input of a block:

$$C(s) = G(s)R(s) + X(s) = G(s)[R(s) + \frac{1}{G(s)}X(s)]$$



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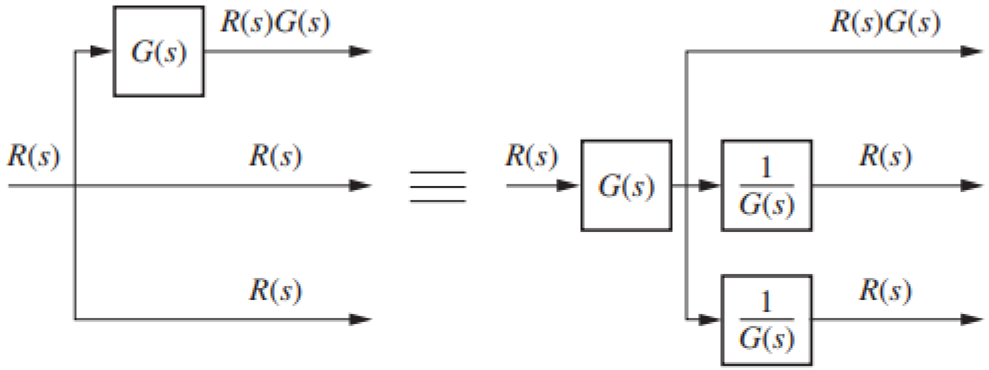
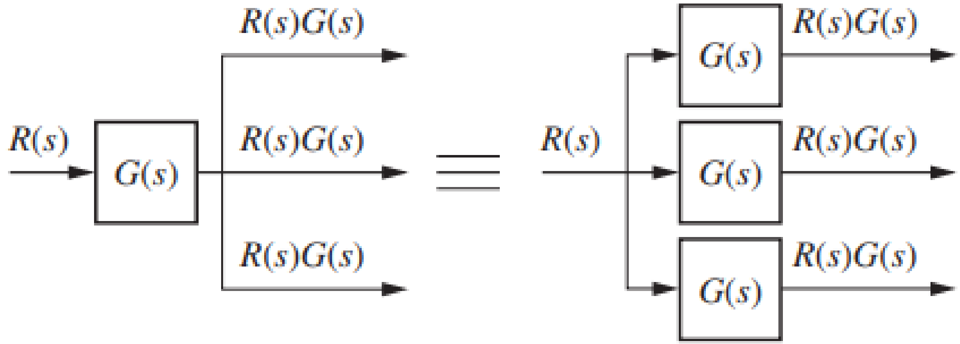
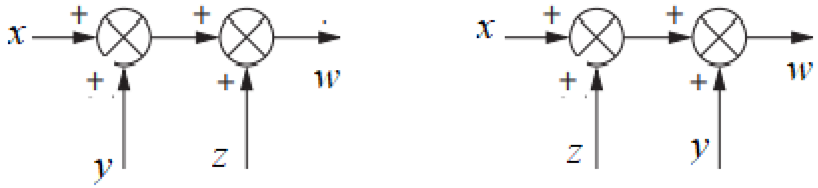
- Transfer of a derivation point from the input to the output of a block:



- Transfer of a derivation point from the output to the input of a block:



- Signal-node switching:



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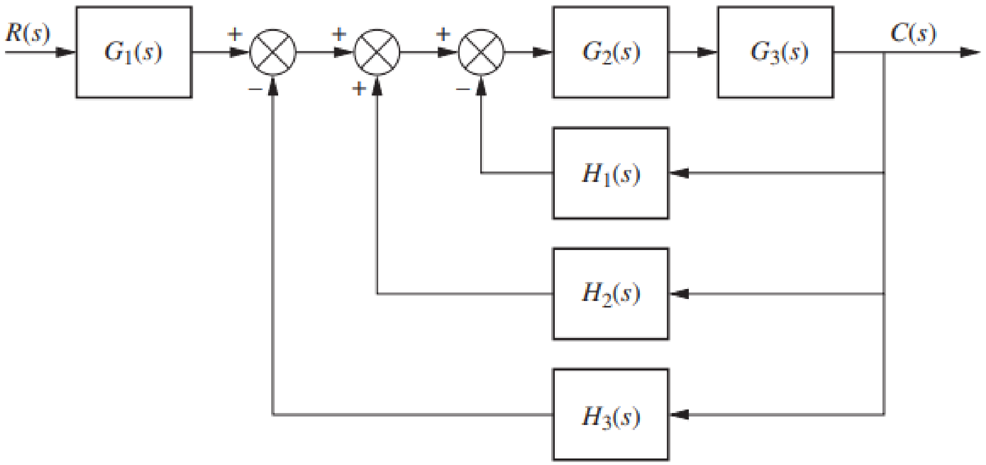
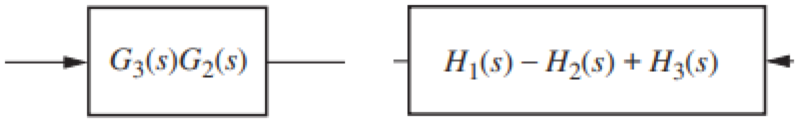
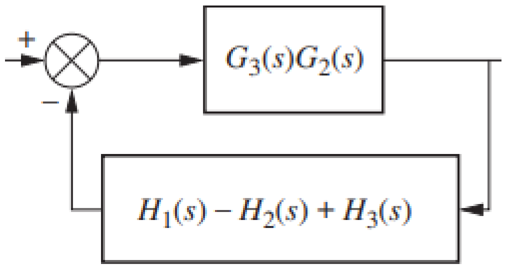
Example1: Reduce the block diagram shown in Figure to a single transfer function.

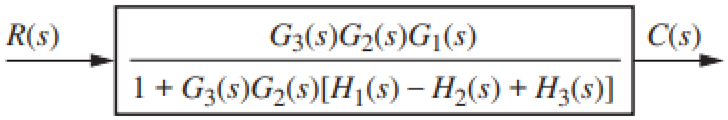


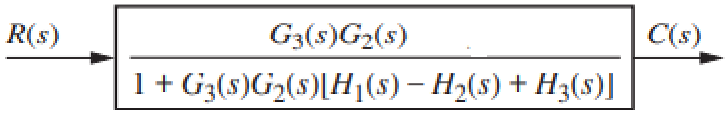
Reduction Steps:

1. $Cascade(G_3, G_2) = T_1$
2. $Parallel\ (H_1, H_2, H_3) = T_2$
3. $feedback(T_1, T_2) = T_3$
4. $Cascade(T_3, G_1) = T$









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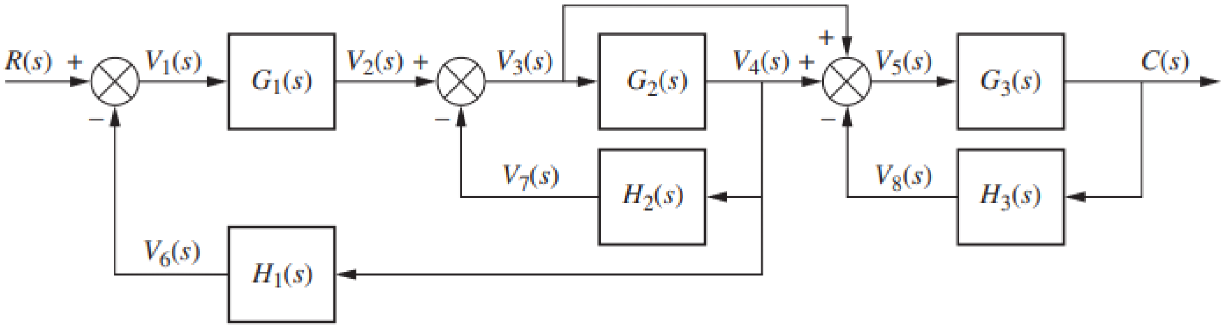
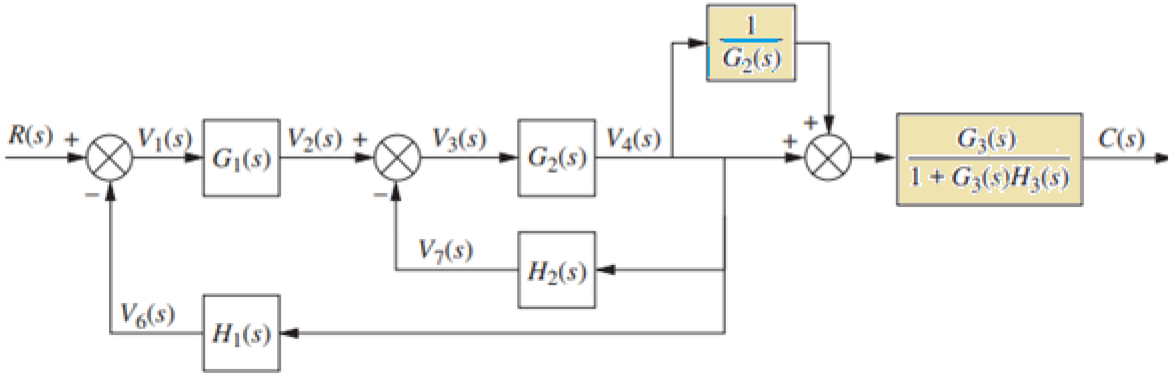
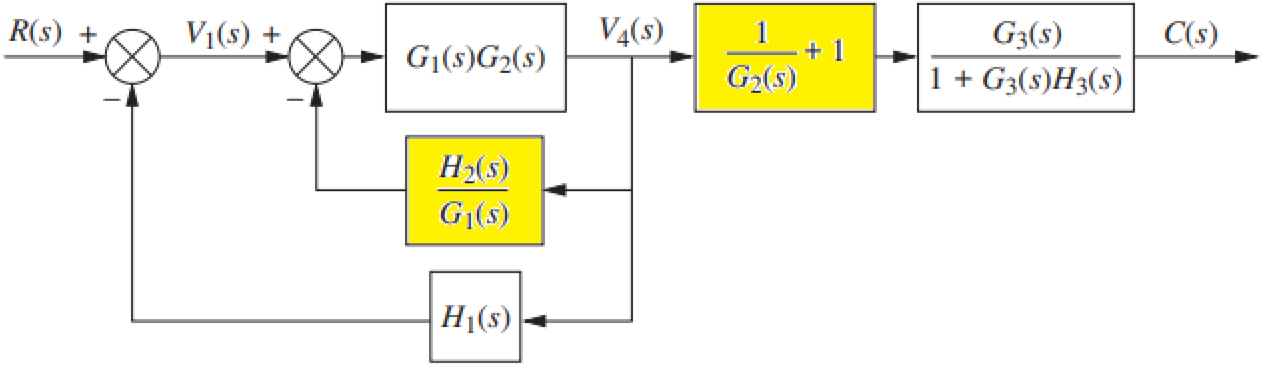
Example2: Reduce the block diagram shown in Figure to a single transfer function.

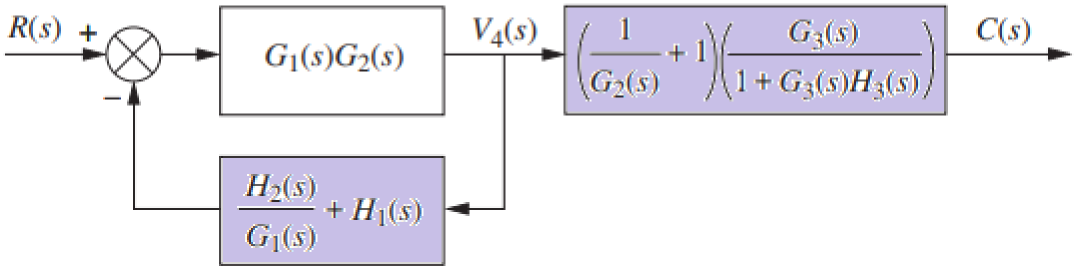


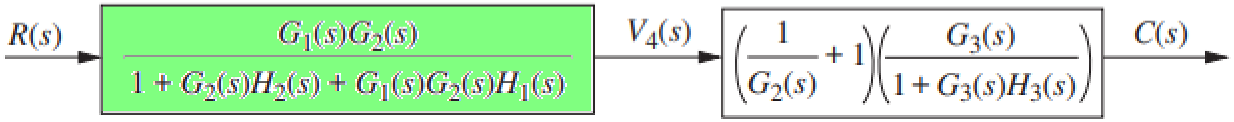


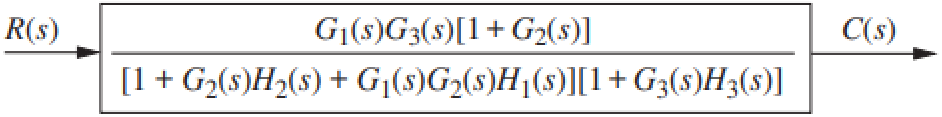
Reduction Steps:

- $Transfer\ the\ derivation\ point\ from\ the$
- $input\ of\ G_2\ to\ its\ output\ and\ apply$ $feedback(G_3, H_3)$
- $Transfer\ the\ sum\ node\ from\ the\ output\ of$ $G_1\ to\ its\ output + parallel(\frac{1}{G_2(s)}, 1)$
- $cascade\ and\ parallel$
- $feedback\ and\ then\ cascade$









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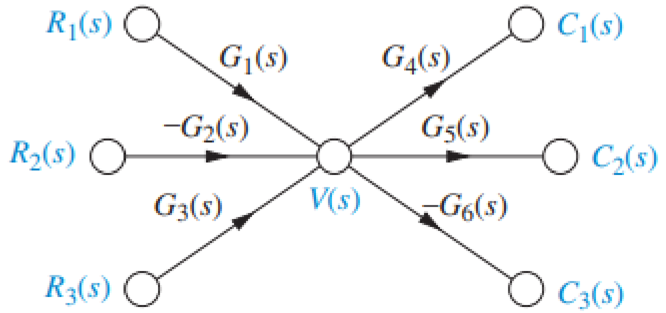
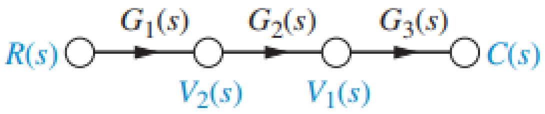
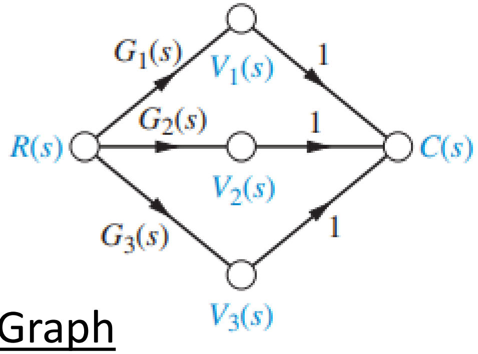
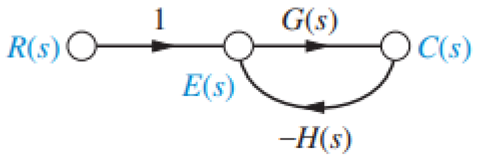
Signal-Flow diagrams:

The basic input vocabulary of the signal flow diagrams are shown in figure, with the signal represented by a node, system transfer relation (gain) represented by an arrow, and signal combination and extraction represented by rows converging in a node and rows diverging from it.

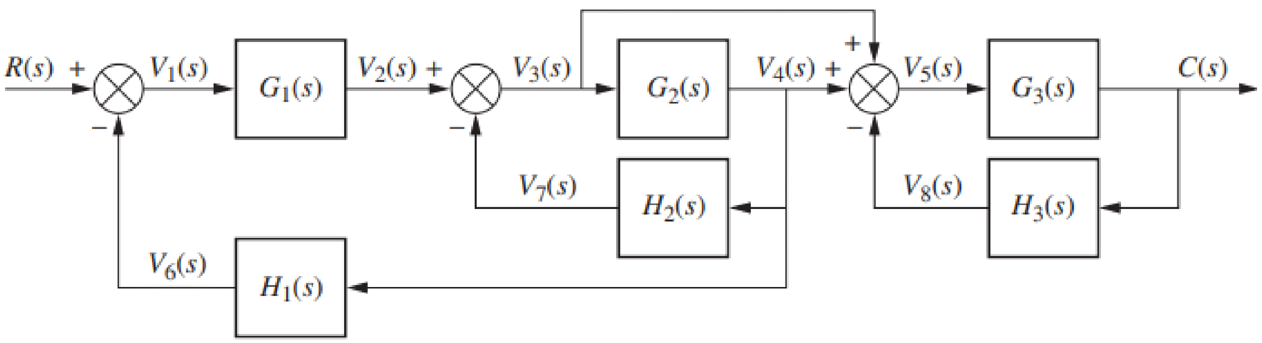


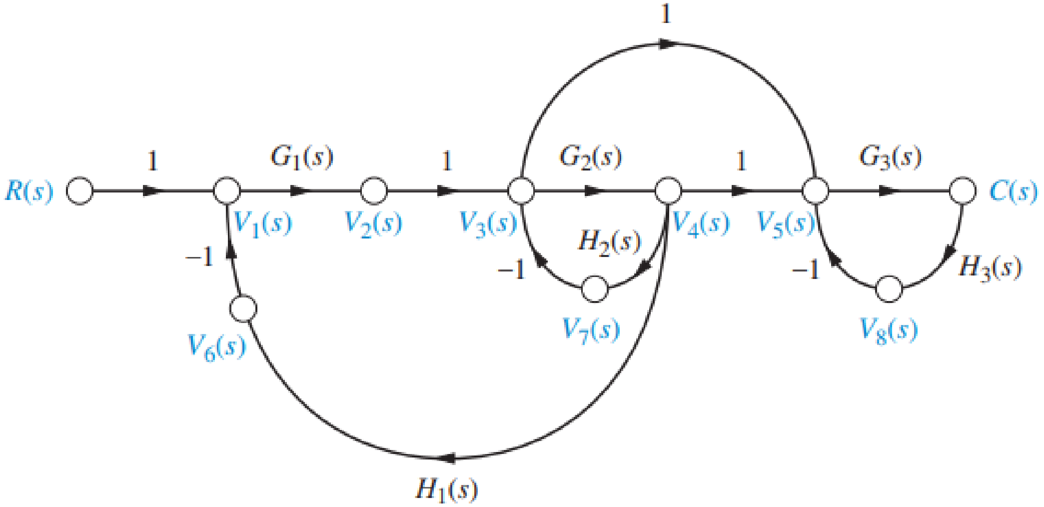






Converting a Block Diagram to a Signal-Flow Graph





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Mason's Rule:

We define the following:

- Pathi,j : the sequence of branching that connects node i and node j without going through any node more than one time.
- Path-gain: the product of the gains of all the branches of the path.
- Loop: a closed path.
- Loop gain: its relative path gain.
- Nontouching loops: Loops that do not have any nodes and branches in common. Nontouching loops are inspected as two, three, four, or more at a time.
- Nontouching loops gains: the product of nontouching loops taken as two, three, four, or more at a time.
- Loops and nontouching loops with a $path_{ij}$: the loops and nontoucing loops that do not have any nodes or branches in common.
- Loops and nontouching loops with a $path_{ij}$ gains: the product of the gains of the Loops and nontouching loops with the $path_{ij}$.

Masons Formula:

$$G(s) = \frac{C(s)}{R(s)} = \frac{\sum_k T_k \Delta_k}{\Delta}$$

where

k = number of forward paths

T_k = the kth forward-path gain

Δ = 1 − Σ loop gains + Σ nontouching-loop gains taken two at a time − Σ nontouching-loop gains taken three at a time + Σ nontouching-loop gains taken four at a time − . . .

Δ_k = Δ − Σ loop gain terms in Δ that touch the kth forward path. In other words, Δ_k is formed by eliminating from Δ those loop gains that touch the kth forward path.

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Example 1: determine the transfer function of the following system using Masons rule.

Solution:

graph elements gains will be written directly.

Path gains:

$P_1 = G_1G_2G_3G_4G_5G_7$

$P_2 = G_1G_2G_3G_4G_6G_7$

Loop gains:

$L_1 = H_1G_2$

$L_2 = H_2G_4$

$L_3 = H_3G_4G_5$

$L_4 = H_3G_4G_6$

Nontouching Loops gains (2-2):

$L_{12} = H_1H_2G_2G_4$

$L_{13} = H_1H_3G_2G_4G_5$

$L_{14} = H_1H_3G_2G_4G_6$

Nontouching Loops gains (3-3):

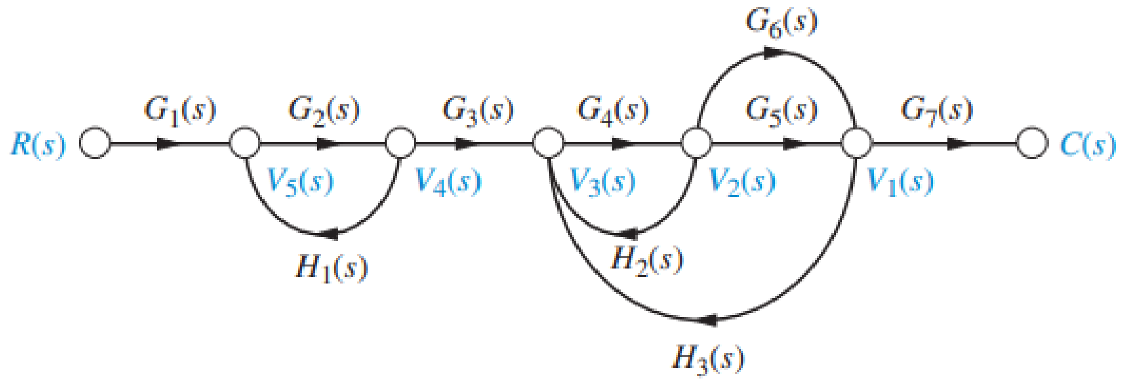
Do not exist

Nontouching Loops with P_1 gains:

Do not exist

Nontouching Loops with P_2 gains:

Do not exist



Computation:

$\Delta = 1 - (H_1G_2 + H_2G_4 + H_3G_4G_5 + H_3G_4G_6) + (H_1H_2G_2G_4 + H_1H_3G_2G_4G_5 + H_1H_3G_2G_4G_6)$

$\Delta_{P_1} = 1, \Delta_{P_2} = 1$

$$T(s) = \frac{C(s)}{R(s)} = \frac{G_1G_2G_3G_4G_7(G_5 + G_6)}{1 - (H_1G_2 + H_2G_4 + H_3G_4G_5) + (H_1H_2G_2G_4 + H_1H_3G_2G_4G_5)}$$

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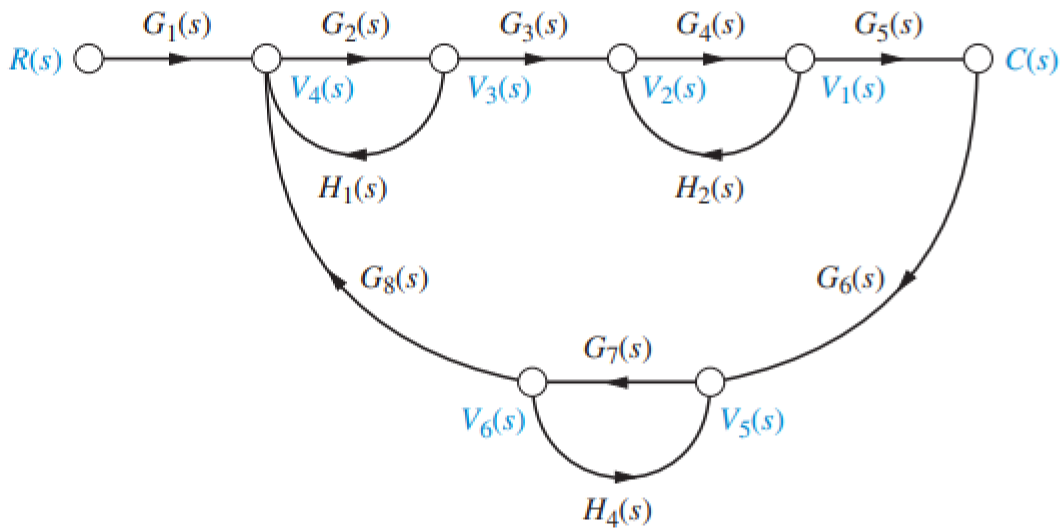
Example 2: determine the transfer function of the following system using Masons rule.



Solution:

graph elements gains will be written directly.

Path gains:

$P_1 = G_1G_2G_3G_4G_5$

Loop gains:

$L_1 = H_1G_2$

$L_2 = H_2G_4$

$L_3 = H_4G_7$

$L_4 = G_2G_3G_4G_5G_6G_7G_8$

Nontouching Loops gains (2-2):

$L_{12} = H_1H_2G_2G_4$

$L_{13} = H_1H_4G_2G_7$

$L_{23} = H_2H_4G_4G_7$

Nontouching Loops gains (3-3):

$L_{123} = H_1H_2H_4G_2G_4G_7$

Nontouching Loops with P_1 gains:

$L_{P_{1-3}} = H_4G_7$

Computation:

$\Delta = 1 - (H_1G_2 + H_2G_4 + H_4G_7 + G_2G_3G_4G_5G_6G_7G_8) + (H_1H_2G_2G_4 + H_1H_4G_2G_7 + H_2H_4G_4G_7) - (H_1H_2H_4G_2G_4G_7)$, $\quad \Delta_{P_1} = 1 - H_4G_7$

$$T(s) = \frac{C(s)}{R(s)} = \frac{G_1G_2G_3G_4G_5 \cdot (1 - H_4G_7)}{1 - (H_1G_2 + H_2G_4 + H_4G_7 + G_2G_3G_4G_5G_6G_7G_8) + (H_1H_2G_2G_4 + H_1H_4G_2G_7 + H_2H_4G_4G_7) - (H_1H_2H_4G_2G_4G_7)}$$

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State Space Representation

Dr. Jamal Siam

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State Space Representation:

- It is an internal system time-domain representation composed of a set of simultaneous first-order differential equations that describes the evolution of the internal state variables (memory elements variables or other related variables) and a second set of algebraic equations that set the relation between the input and the state.

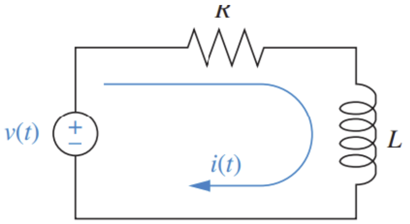
$$\dot{x} = f(x,u)$$
$$y = g(x,u)$$

- The number of independent state equations is equal to the order of the system.
- The natural selection of the independent state variables is the energy variable of the conservative elements.
- The state equation includes only state variables and input excitation.
- For a linear type invariant system of order n with m inputs and d outputs, the state equations representation is formulated as follow:

$$\begin{cases} \dot{x} = A_{n\times n}x_{n\times 1} + B_{n\times m}u_{m\times 1} \\ y_{d\times 1} = C_{d\times n}x + D_{d\times m}u \end{cases}$$

x : state vector, A: state Space matrix, B: state−input matrix, C: output−ste matrix, D:output−input Matrix

Example:



- The system is first order system, thus we need one state variable. The output is $v_R(t)$.
- Select the mesh current which is equal to the inductor current as state variable.
- The energy element equation is $v_L(t) = L\frac{di_L(t)}{dt}$ which is not a state equation because $v_L(t)$ is not a state variable an has to be eliminated.
- Applying KVL and the resistor characteristic equation we obtain $v_L(t) = v(t) - Ri(t)$
- Substituting in the energy equation, we obtain $\frac{di(t)}{dt} = \frac{1}{L}v(t) - \frac{R}{L}i(t)$ ....state equation

$$v_R(t) = Ri(t)$$ ......output equation

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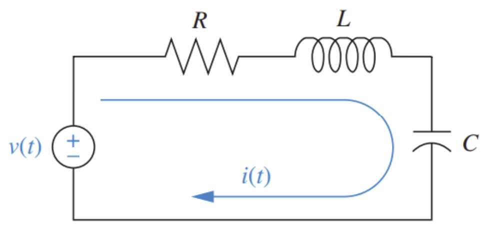
Example2

- The system of a second-order system, thus we need two independent state variables.
- The natural selection of state variables is $i_L(t)$ and $V_c(t)$.
- Assume the output variable is $V_c(t)$. The output equation becomes $y(t) = V_c(t)$
- Solution:
- The energy equations are $v_L(t) = L\frac{di_L(t)}{dt}$ and $i_c(t) = c\frac{dv_c(t)}{dt}$ which are both not state equations.
- From the node equations $i_c(t) = i_L(t)$ →the first state equation: $\frac{dv_c(t)}{dt} = \frac{1}{C}i_L(t)$
- From the KVL and the resistor characteristic equation: $v_L(t) = v(t) - Ri_L(t) - v_c(t)$
- Applying in the inductor characteristic equation and ordering we obtain:

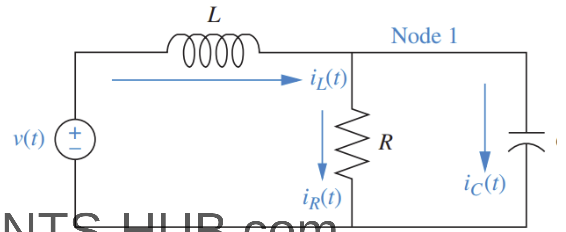


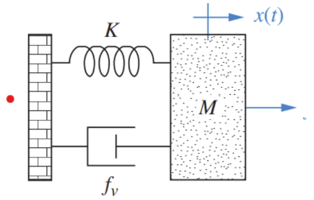
$$\frac{di_L(t)}{dt} = \frac{1}{L}v(t) - \frac{R}{L}i_L(t) - \frac{1}{L}v_c(t)$$

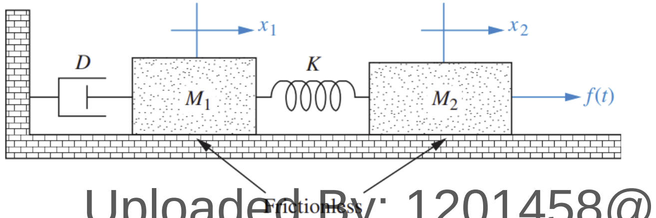
In matrix form
$$\begin{bmatrix} \dot{v}_C \\ \dot{i}_l \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix}\begin{bmatrix} V_c(t) \\ i_L(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} v_c(t)$$

$$y(t) = [1 \quad 0]\begin{bmatrix} V_c(t) \\ i_L(t) \end{bmatrix}$$

Exercise: Write the state equations of the following systems in algebraic matrix form. Outputs: $v_L(t), x, x_1 and x_2$, respectively.







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Transforming Internal representation to external representation (unique form)

- State space representation → Transfer Matrix/System of differential equations.
- For a SISO system: State space representation → Transfer function/ differential equations.

Given the state and output equations

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \quad (3.68a)$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \quad (3.68b)$$

take the Laplace transform assuming zero initial conditions:[8]

$$s\mathbf{X}(s) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s) \quad (3.69a)$$

$$\mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s) \quad (3.69b)$$

Solving for $\mathbf{X}(s)$ in Eq. (3.69a),

$$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{B}\mathbf{U}(s) \quad (3.70)$$

or

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) \quad (3.71)$$

where $\mathbf{I}$ is the identity matrix.

Substituting Eq. (3.71) into Eq. (3.69b) yields

$$\mathbf{Y}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) + \mathbf{D}\mathbf{U}(s) = \left[\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}\right]\mathbf{U}(s) \quad (3.72)$$

We call the matrix $\left[\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}\right]$ the transfer function matrix, since it relates the output vector, $\mathbf{Y}(s)$, to the input vector, $\mathbf{U}(s)$. However, if $\mathbf{U}(s) = U(s)$ and $\mathbf{Y}(s) = Y(s)$ are scalars, we can find the transfer function, Thus,

$$T(s) = \frac{Y(s)}{U(s)} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

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Example:

Determine the transfer function of the system defined by the following state space representation.

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \mathbf{x}$$

Solution:

$$(s\mathbf{I} - \mathbf{A}) = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 2 & s+3 \end{bmatrix} \rightarrow (s\mathbf{I} - \mathbf{A})^{-1} = \frac{\text{adj}(s\mathbf{I} - \mathbf{A})}{\det(s\mathbf{I} - \mathbf{A})} = \frac{\begin{bmatrix} (s^2+3s+2) & s+3 & 1 \\ -1 & s(s+3) & s \\ -s & -(2s+1) & s^2 \end{bmatrix}}{s^3 + 3s^2 + 2s + 1}$$

$$\mathbf{B} = \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} \quad \rightarrow \quad T(s) = \frac{Y(s)}{U(s)} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} \quad \rightarrow \quad T(s) = \frac{10(s^2 + 3s + 2)}{s^3 + 3s^2 + 2s + 1}$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

$$\mathbf{D} = 0$$

Exercise: determine the transfer function of the system represented by the following state space representation

$$\dot{\mathbf{x}} = \begin{bmatrix} -4 & -1.5 \\ 4 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u(t)$$

$$y = \begin{bmatrix} 1.5 & 0.625 \end{bmatrix} \mathbf{x}$$

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Converting external representation to internal representation:

differential equation/ transfer function $\rightarrow$ state space representation (not unique)

Phase-variable state space representation:

Consider the following differential equation and the following variable assignment:

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = b_0 u$$

$$\begin{aligned} x_1 &= y \\ x_2 &= \frac{dy}{dt} \\ x_3 &= \frac{d^2y}{dt^2} \\ &\vdots \\ x_n &= \frac{d^{n-1}y}{dt^{n-1}} \end{aligned} \quad \rightarrow \quad \begin{aligned} \dot{x}_1 &= \frac{dy}{dt} \\ \dot{x}_2 &= \frac{d^2y}{dt^2} \\ \dot{x}_3 &= \frac{d^3y}{dt^3} \\ &\vdots \\ \dot{x}_n &= \frac{d^n y}{dt^n} \end{aligned} \quad \rightarrow \quad \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= -a_0 x_1 - a_1 x_2 \cdots - a_{n-1} x_n + b_0 u \end{aligned}$$

In matrix form-The state matrix is called companion matrix because it includes the coefficient of the transfer equation :

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ \vdots & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & -a_4 & -a_5 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ b_0 \end{bmatrix} u \qquad y = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

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Example1: Consider the following transfer and

- determine the system differential equation and the phase variable representation.
- Plot the block diagram of the system

$$\frac{C(s)}{R(s)} = \frac{24}{(s^3 + 9s^2 + 26s + 24)}$$

Solution:

$$(s^3 + 9s^2 + 26s + 24)C(s) = 24R(s) \quad \rightarrow \quad \dddot{c} + 9\ddot{c} + 26\dot{c} + 24c = 24r$$

State variable assignment $\rightarrow$

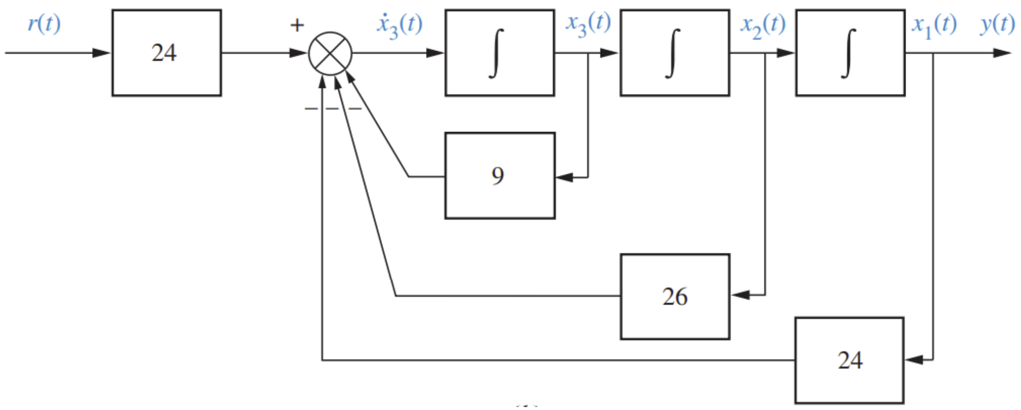
$$x_1 = c \qquad x_2 = \dot{c} \qquad x_3 = \ddot{c}$$

$$\rightarrow \quad \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= -24x_1 - 26x_2 - 9x_3 + 24r \\ y &= c = x_1 \end{aligned}$$

$$\rightarrow \quad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 24 \end{bmatrix} r$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Block Diagram:



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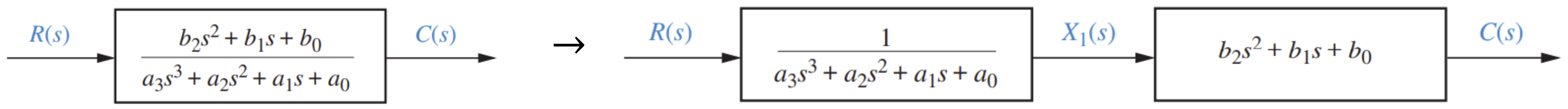
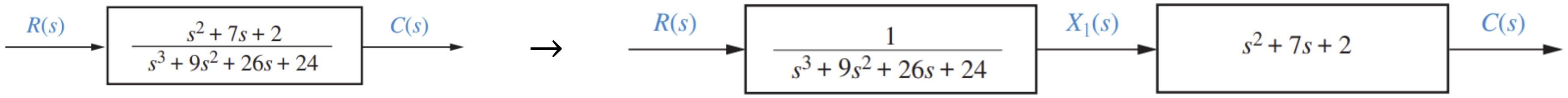
Example2: Transfer function with polynomial numerator

Determine the state space representation of the following system and plot the corresponding block diagram



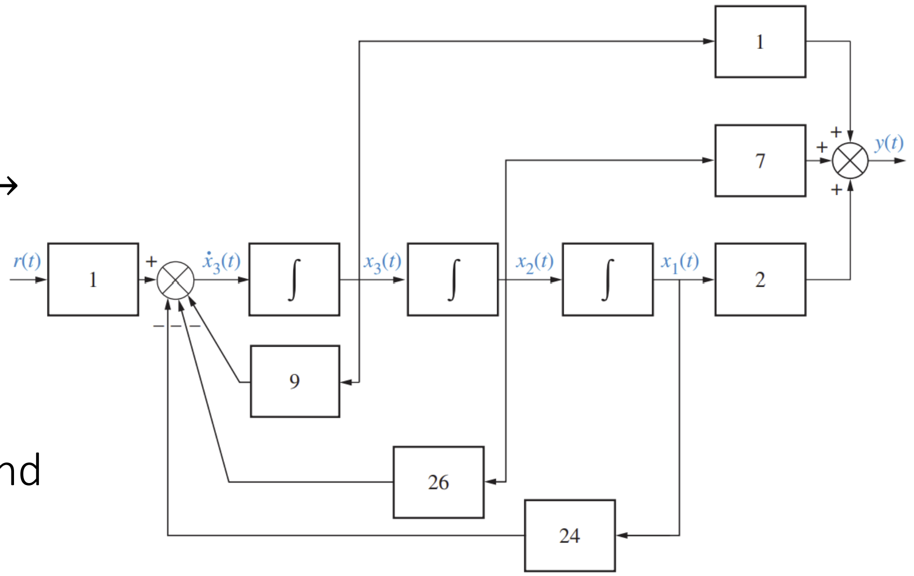
$$y(t)=b_0x_1+b_1x_2+b_2x_3$$



$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}0&1&0\\0&0&1\\-24&-26&-9\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}+\begin{bmatrix}0\\0\\1\end{bmatrix}r$$

$$C(s)=(b_2s^2+b_1s+b_0)X_1(s)=(s^2+7s+2)X_1(s)\ \rightarrow\ c=\ddot{x}_1+7\dot{x}_1+2x_1$$ using $$x_1=x_1,\quad \dot{x}_1=x_2,\quad \ddot{x}_1=x_3\ \rightarrow$$

$$y=\begin{bmatrix}b_0 & b_1 & b_2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}2 & 7 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$



Exercise: Determine the state space representation of the following system and plot the representation block diagram

$$G(s)=\frac{2s+1}{s^2+7s+9}.$$

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Alternative Representations in State Space:

Controller Canonical Form:(a variant of the phase variable representation with companion matrix)

$$G(s) = \frac{C(s)}{R(s)} = \frac{s^2 + 7s + 2}{s^3 + 9s^2 + 26s + 24}$$

Phase variable:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$y = \begin{bmatrix} 2 & 7 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The controller canonical representation is obtained by changing the numbers of the variables and reordering the equations

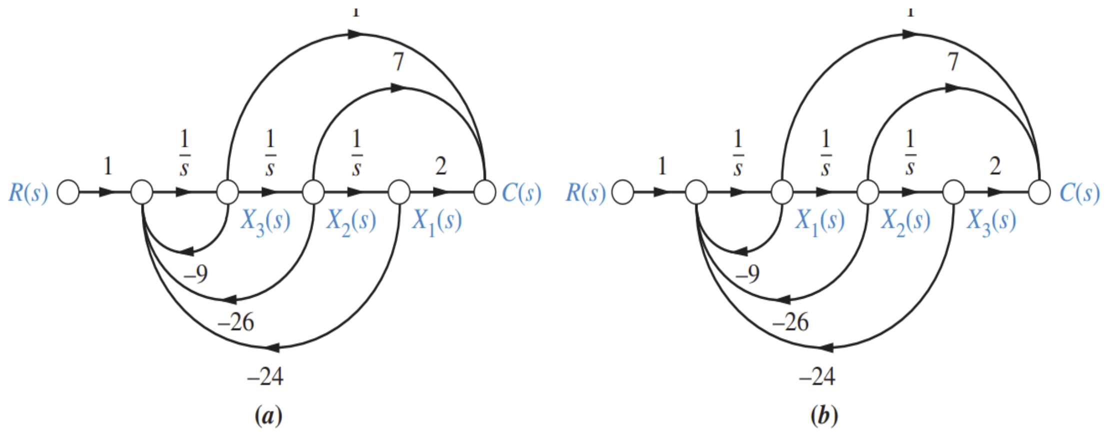
$$\begin{bmatrix} \dot{x}_3 \\ \dot{x}_2 \\ \dot{x}_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$

$$y = \begin{bmatrix} 2 & 7 & 1 \end{bmatrix} \begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix}$$

$\rightarrow$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -9 & -26 & -24 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} r$$

$$y = \begin{bmatrix} 1 & 7 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$



a. phase-variable form; **b.** controller canonical form

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Observer Canonical Form:

The transfer function/differential equation are written in integral form which is then written as a sequence of integration and variables are assigned accordingly.

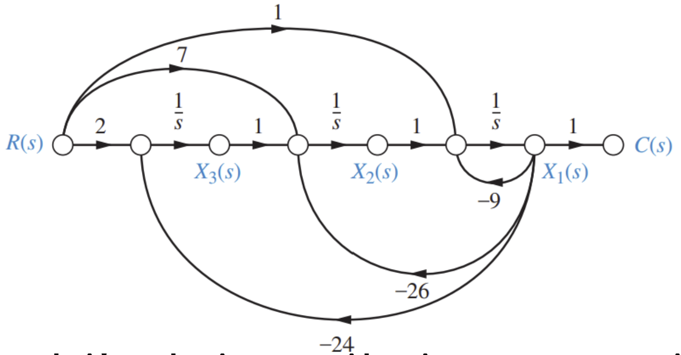
Example:

$$G(s)=\frac{C(s)}{R(s)}=\frac{s^2+7s+2}{s^3+9s^2+26s+24} \rightarrow \frac{C(s)}{R(s)}=\frac{\frac{1}{s}+\frac{7}{s^2}+\frac{2}{s^3}}{1+\frac{9}{s}+\frac{26}{s^2}+\frac{24}{s^3}} \rightarrow \left[\frac{1}{s}+\frac{7}{s^2}+\frac{2}{s^3}\right]R(s)=\left[1+\frac{9}{s}+\frac{26}{s^2}+\frac{24}{s^3}\right]C(s) \rightarrow$$

$$C(s)=\frac{1}{s}[R(s)-9C(s)]+\frac{1}{s^2}[7R(s)-26C(s)]+\frac{1}{s^3}[2R(s)-24C(s)] \rightarrow C(s)=\frac{1}{s}\left[[R(s)-9C(s)]+\frac{1}{s}\left([7R(s)-26C(s)]+\frac{1}{s}[2R(s)-24C(s)]\right)\right]$$

$$\begin{aligned}\dot{x}_1 &= -9x_1+x_2 \qquad +r\\ \dot{x}_2 &= -26x_1 \qquad +x_3+7r\\ \dot{x}_3 &= -24x_1 \qquad +2r\\ y &= c(t)=x_1\end{aligned} \rightarrow \dot{\mathbf{x}}=\begin{bmatrix}-9 & 1 & 0\\ -26 & 0 & 1\\ -24 & 0 & 0\end{bmatrix}\mathbf{x}+\begin{bmatrix}1\\7\\2\end{bmatrix}r$$

$$y=[1 \quad 0 \quad 0]\,\mathbf{x}$$



Controller-Observer Duality:

The controller representation of the same system is given by:

Observe the duality relation between the two representations:

$$\mathbf{A_D}=\mathbf{A}^T,\ \mathbf{B_D}=\mathbf{C}^T,\ \mathbf{C_D}=\mathbf{B}^T.$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}-9 & -26 & -24\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}+\begin{bmatrix}1\\0\\0\end{bmatrix}r$$

$$y=\begin{bmatrix}1 & 7 & 2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

Exercise: Determine the phase-variable, controller, and observer representation of the following system represented by state space and plot the signal flow diagrams

Hint: convert the state space representation to the transfer function

$$\dot{\mathbf{x}}=\begin{bmatrix}-105 & -506\\ 1 & 0\end{bmatrix}\mathbf{x}+\begin{bmatrix}1\\0\end{bmatrix}r \qquad y=[100 \quad 500]\,\mathbf{x}$$

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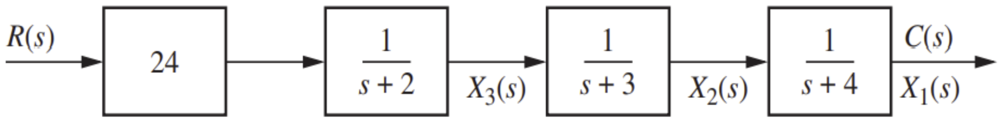
Cascade representation(for transfer functions with simple roots(Triangular Matrix Form):

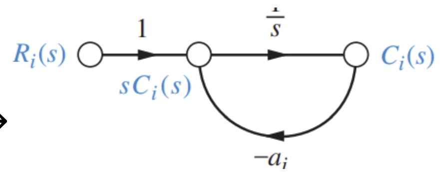
The Transfer function is written as the product of its basic-first-order terms and cascaded with the numerator term.

$$\frac{C(s)}{R(s)} = \frac{24}{(s+2)(s+3)(s+4)} \quad \rightarrow$$

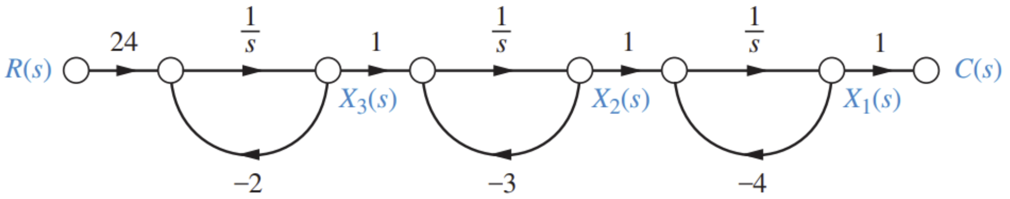


<u>Representation of the general first-order term</u>

$$\frac{C_i(s)}{R_i(s)} = \frac{1}{(s+a_i)} \rightarrow (s+a_i)C_i(s) = R_i(s) \rightarrow \frac{dc_i(t)}{dt} = -a_i c_i(t) + r_i(t) \rightarrow$$



The system can be represented using this representation as:



Writing the equations of each block we obtain:

$$\begin{aligned} \dot{x}_1 &= -4x_1 + x_2 \\ \dot{x}_2 &= \qquad -3x_2 + x_3 \\ \dot{x}_3 &= \qquad\qquad -2x_3 + 24r \end{aligned} \quad \rightarrow \quad \dot{\mathbf{x}} = \begin{bmatrix} -4 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -2 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 24 \end{bmatrix} r$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\mathbf{x}$$

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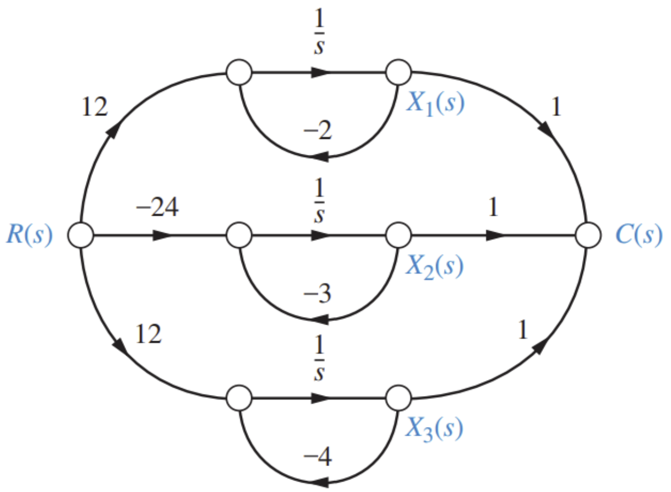
Parallel representation: systems with simple roots written in the form of the partial fraction(Diagonal Matrix)

$$\frac{C(s)}{R(s)} = \frac{24}{(s+2)(s+3)(s+4)} = \frac{12}{(s+2)} - \frac{24}{(s+3)} + \frac{12}{(s+4)} \rightarrow C(s) = R(s)\frac{12}{(s+2)} - R(s)\frac{24}{(s+3)} + R(s)\frac{12}{(s+4)}$$



Using the general first-order representation we can obtain the parallel plot in the figure. Writing the equation of each block we obtain:

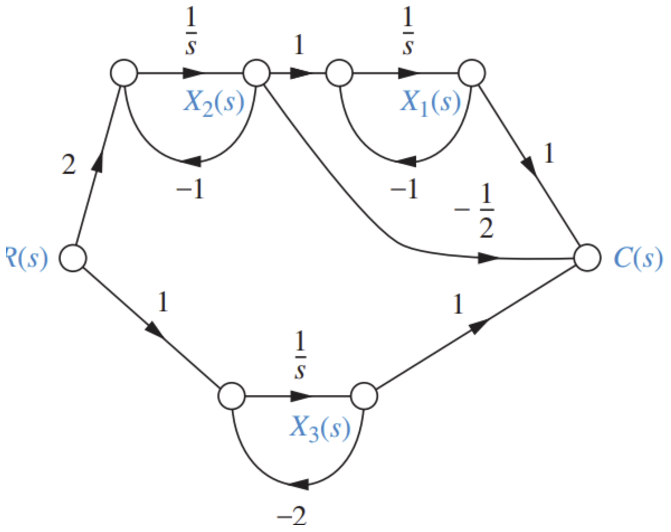
$$\begin{aligned} \dot{x}_1 &= -2x_1 + 12r \\ \dot{x}_2 &= -3x_2 - 24r \\ \dot{x}_3 &= -4x_3 + 12r \\ y = c(t) &= x_1 + x_2 + x_3 \end{aligned} \rightarrow \dot{\mathbf{x}} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -4 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 12 \\ -24 \\ 12 \end{bmatrix} r$$

$$y = [1 \quad 1 \quad 1]\mathbf{x}$$

Mixed Parallel-Cascade representation: partial fractions with repeated roots(Jordan Matrix):

$$\frac{C(s)}{R(s)} = \frac{(s+3)}{(s+1)^2(s+2)} \rightarrow \frac{C(s)}{R(s)} = \frac{2}{(s+1)^2} - \frac{1}{(s+1)} + \frac{1}{(s+2)}$$

Plotting using the first-order cell and reading the equation we obtain:



$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2 \\ \dot{x}_2 &= -x_2 + 2r \\ \dot{x}_3 &= -2x_3 + r \\ y = c(t) &= x_1 - \frac{1}{2}x_2 + x_3 \end{aligned} \rightarrow \dot{\mathbf{x}} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} r$$

$$y = \begin{bmatrix} 1 & -\frac{1}{2} & 1 \end{bmatrix}\mathbf{x}$$

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Stability

Dr. Jamal Siam

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System Stability-Review:

Definition: An LTI system is said to be asymptotically stable if its transient response goes to zero and a steady state response is reached for t goes to infinity.

Theorem1: an LTI system with impulse response $h(t)$ is asymptotically stable ↔ $\lim_{t\to\infty} h(t) = 0$.

Theorem2: adynamic LTI system is asymptotically stable↔ all the roots of its characteristic equation/ the poles of its transfer function have a negative real part (located in the left semi plan of the complex plan)

Theorem3: an LTI system is unstable if it has at least a positive real-part root or a repeated root with zero real part.

Definition:(BIBO stability) an LTI system is said to be BIBO (Bounded Input/Bounded Output)↔ $\forall\, input\ x(t) with\ |x(t)| \le N,\ \ \exists M < \infty$ so that the respons $|y(t)| \le M, \forall t$ (weak stability)

Theorom4: a system is BIBO stable ↔ $\int_{-\infty}^{\infty} |h(t)| dt < \infty$ that is if its impulse response is absolutely integrable.

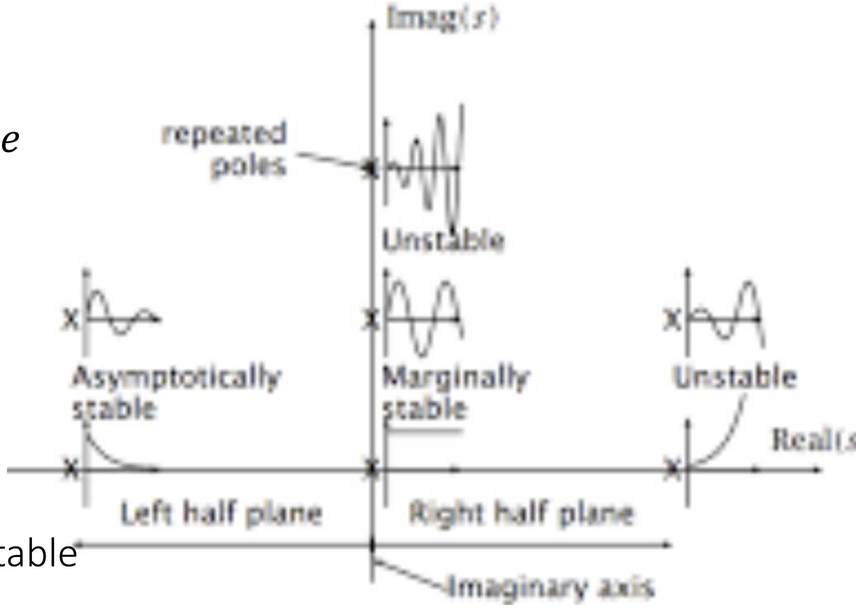
Exercise: prove this theorem.

Theorom5: a system is BIBO stable if it has no roots with positive real parts and all the roots with zero real part are not repeated roots.

Example1: discuss the stability of the following dynamic systems:

- $\frac{d^2y}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = x(t)$, the roots of are $\sigma_1 = -1, \sigma_2 = -2 \rightarrow asymptotically\ stable$
- $H(s) = \frac{(s+4)}{(s^2+4)(s+1)} \rightarrow BIBO\ stable$
- $H(s) = \frac{(s+4)}{(s^2+4)^2(s+1)} \rightarrow Unstable$
- $H(s) = \frac{(s+4)}{(s+2)(s-1)} \rightarrow Unstable$

Asymptotically stable →BIBO Stable



Example2: Prove that the system with the following $h(t)$ achieves the BIBO stability theorem

$h(t) = 10e^{-3t}u(t) \rightarrow \int_{-\infty}^{\infty} |10e^{-3t}u(t)| dt \rightarrow \int_0^{\infty} 10e^{-3t} dt = \frac{10e^{-3t}}{-3}\Big|_0^{\infty} = \frac{10}{3} < \infty$ →BIBO stable

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Stability- a different perspective

Observation:

- It is noted that the system stability depends of the locations of the roots of the characteristic equation of the LTI system in the complex plane.
- The system roots locations depend on the coefficients of the system characteristic equation (system parameters). For example, the roots of the second order characteristic equation $as^2 + bs + c = 0$ are $s_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2}$

Accordingly, It is possible to set stability algorithms based on the equation coefficients (roots location and not values).

Coefficients-Based Stability Theorems and Methods:

Theorem I:

It is necessary for an LTI system to be stable that the coefficients of the characteristic polynomial has all the same signs.

Example1: Discuss the stability of the LTI system with the following characteristic equation.

$$D(s) = 5s^4 + 3s^3 - 2s^2 + s + 1$$

The system is unstable because the coefficients have different signs.

Example2: Discuss the stability of the LTI system with the following characteristic equation.

$$D(s) = 5s^4 + 3s^3 + 2s^2 + s + 1$$

The system stability can not be determined based on Theorem I.

Observation: Routh and Hurwitz developed a stability criteria to set a necessary and sufficient conditions for LTI system Stability. The Routh-Hurwitz(RH) criterion is applied to proper polynomials which do not have poles in the origin. Poles in the origin implies that the system can not be asymptotically stable(can be BIBO stable or Unstable based on the number of origin poles or the application of the RH criterion in the extracted proper polynomials (after extracting the origin poles).

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Routh-Hurwitz Stability Criterion:

Method Formulation:

- Construction of the Routh-Hurwitz (RH) table.
- Test of the sign changes in elements of the first column of the table.
- Any change in sign of the elements of the first column implies system instability.
- The number of first-column-elements sign changes equals the number of right side poles.

RH-table regular construction:

- The table has $n+1$ rows, where n is the polynomial order.
- The characteristic polynomial is divided in two polynomials, the first the odd terms power polynomial and the second is the even power polynomial.
- The coefficients of the higher power polynomial (odd or even polynomials) are distributed on the first row of the table and the lower power polynomial coefficients on the second row.
- The difference in terms power between of two consecutive columns always equals two (thus the coefficients of the power terms that do not appear in the equation should be set to zero.
- The coefficient r^{th} element of the k^{th}+2 row is determined by the negative value of the determinant of a second order matrix (composed of the first column of these k^{th} and k^{th}+1 rows and the two elements of the r+1 column of the same rows) divided by the first element of the k^{th}+1 row. That is, $x_{k+2,r} = -\frac{\begin{vmatrix} x_{k,1} & x_{k,r+1} \\ x_{k+1,1} & x_{k+1,r+1} \end{vmatrix}}{x_{k+1,1}}$

Observation:

- The multiplication of the elements of a row with a positive constant (to simplify the computation) does not compromise the table construction or change the RH stability-test outcomes.
- The table construction can not be continued in the following two cases, and thus alternative ways can be adopted for the RH table construction and test.

1. A zero in the first column with at least one non-zero element in the other columns.
2. A row of zero-elements.

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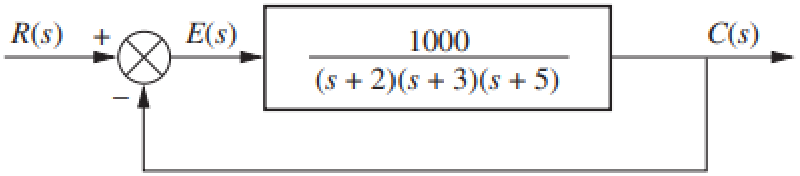
Stability Test:

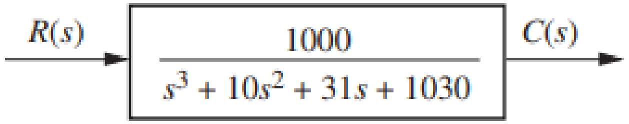
- Check the elements of the first column for possible sign changes.
- The existence of sign changes implies the existence of right-side poles which means that the LTI system is unstable.
- The number of right-side poles equals the number of sign changes.

Observation:

The method can also be useful to determine the number of right-side poles of an unstable LTI system according to Theorem I.

Example1: Discuss the stability of the following LTI system using The RH criterion.





RH-table:

s^3	1	31	0
s^2	~~10~~ 1	~~1030~~ 103	0
s^1	$-\dfrac{\begin{vmatrix}1 & 31\\1 & 103\end{vmatrix}}{1} = -72$	$-\dfrac{\begin{vmatrix}1 & 0\\0 & 0\end{vmatrix}}{1} = 0$	$-\dfrac{\begin{vmatrix}1 & 0\\1 & 0\end{vmatrix}}{1} = 0$
s^0	$-\dfrac{\begin{vmatrix}1 & 103\\-72 & 0\end{vmatrix}}{-72} = 103$	$-\dfrac{\begin{vmatrix}1 & 0\\-72 & 0\end{vmatrix}}{-72} = 0$	$-\dfrac{\begin{vmatrix}1 & 0\\-72 & 0\end{vmatrix}}{-72} = 0$

characteristic equation:

$$D(s) = s^3 + 10s^2 + 31s + 1030$$

RH-test: The system is unstable with two right side poles because of the existence of two changes in the sign of the first column elements

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Example2: Discuss the stability of the LTI system with the following transfer function using The RH criterion.

$$T(s) = \frac{200}{s^4 + 6s^3 + 11s^2 + 6s + 200}$$

$$D(s) = s^4 + 6s^3 + 11s^2 + 6s + 200$$

RH-table:

s^4		1		11	200
s^3	~~6~~	1	~~6~~	1	
s^2	~~10~~	1	~~200~~	20	
s^1		−19			
s^0		20			

RH-test: The system is unstable with two right side poles because of the existence of two changes in the sign of the first column elements

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Routh-Hurwitz Stability Criterion-Special cases:

A zero in the first column:

In this case the construction of the table falls because of the division by zero error. Two ways are used to overcome this problem

- Substitution of a small number ε and computation of $\text{sign}\left(\lim_{\varepsilon\to 0} func(\varepsilon)\right)$ in all the first columns elements where it appears.
- Construction of the reciprocal which have reversed coefficient order and consequently the inverse of the original polynomial poles with the same signs.

Example1 $\varepsilon - method$ Discuss the stability of the LTI system with the following transfer function using The RH criterion.

$$T(s) = \frac{10}{s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3} \longrightarrow D(s) = s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3$$

RH-table construction:

s^5	1	3	5
s^4	2	6	3
s^3	~~0~~ ϵ	$\frac{7}{2}$	0
s^2	$\frac{6\epsilon - 7}{\epsilon}$	3	0
s^1	$\frac{42\epsilon - 49 - 6\epsilon^2}{12\epsilon - 14}$	0	0
s^0	3	0	0

RH-table elements sign computation:

Label	First column	$\epsilon = +$	$\epsilon = -$
s^5	1	+	+
s^4	2	+	+
s^3	~~0~~ ϵ	+	−
s^2	$\frac{6\epsilon - 7}{\epsilon}$	−	+
s^1	$\frac{42\epsilon - 49 - 6\epsilon^2}{12\epsilon - 14}$	+	+
s^0	3	+	+

RH-test: The system is unstable with two right side poles because of the existence of two changes in the sign of the first column elements

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Example2 $Reciprocal\ polynome - method$:

Discuss the stability of the LTI system with the following transfer function using The RH criterion.

$$D(s) = s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3 \longrightarrow D_{rec}(s) = 3s^5 + 5s^4 + 6s^3 + 3s^2 + 2s + 1$$

RH-table

s^5	3	6	2
s^4	5	3	1
s^3	4.2	1.4	
s^2	1.33	1	
s^1	−1.75		
s^0	1		

RH-test: The system is unstable with two right side poles because of the existence of two changes in the sign of the first column elements

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A row of zeros:

- A row of zeros (with proper polynomials) can occur only at an odd-indexed rows and is caused by imaginary roots.
- The problem is solved by operating on the Auxiliary polynomial by:

1. Solving the auxiliary polynomial equation that is determining the corresponding system roots.
2. Differentiating the auxiliary polynomial, applying the coefficient at the zero elements row, and continuing the regular construction process.

Construction of the auxiliary polynomial:

The auxiliary polynomials has even power terms with order equals to the index of the row that precedes the row of zeros and coefficients equal to the preceding row elements (according to the difference of 2 columns rule)

Example1: Discuss the stability of the LTI system with the following transfer function using The RH criterion

$$T(s) = \frac{10}{s^5 + 7s^4 + 6s^3 + 42s^2 + 8s + 56} \longrightarrow D(s) = s^5 + 7s^4 + 6s^3 + 42s^2 + 8s + 56$$

Auxiliary polynomial and its derivative

$$P(s) = s^4 + 6s^2 + 8 \longrightarrow \frac{dP(s)}{ds} = 4s^3 + 12s + 0$$

s^5			1			6			8
s^4		~~7~~	1		~~42~~	6		~~56~~	8
s^3	~~0~~	~~4~~	1	~~0~~	~~12~~	3	~~0~~	~~0~~	0
s^2			3			8			0
s^1			$\frac{1}{3}$			0			0
s^0			8			0			0

RH-test: There are no changes in the signs of the first column elements. However, the system can not be asymptotically stable because the table has a row of zeros which implies the existence of four imaginary poles, which imply that the system is BIBO stable since the imaginary roots has multiplicity one because the polynomial is not a perfect polynomial (can not be written as $(s^2 + \alpha^2)^2$. The fifth pole is in the semi-left plane.

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<u>Example2:</u> Discuss the stability of the LTI system with the following transfer function using The RH criterion (number of poles)

$$T(s) = \frac{20}{s^8 + s^7 + 12s^6 + 22s^5 + 39s^4 + 59s^3 + 48s^2 + 38s + 20}$$

s^8	1	12	39	48	20
s^7	1	22	59	38	0
s^6	$-\cancel{10}\ -1$	$-\cancel{20}\ -2$	$\cancel{10}\ \ 1$	$\cancel{20}\ \ 2$	0
s^5	$\cancel{20}\ \ 1$	$\cancel{60}\ \ 3$	$\cancel{40}\ \ 2$	0	0
s^4	1	3	2	0	0
s^3	$\cancel{0}\ \ \cancel{4}\ \ 2$	$\cancel{0}\ \ \cancel{6}\ \ 3$	$\cancel{0}\ \ \cancel{0}\ \ 0$	0	0
s^2	$\cancel{\frac{3}{2}}\ \ 3$	$\cancel{2}\ \ 4$	0	0	0
s^1	$\frac{1}{3}$	0	0	0	0
s^0	4	0	0	0	0

Auxiliary polynomial and its derivative

$$P(s) = s^4 + 3s^2 + 2 \longrightarrow \frac{dP(s)}{ds} = 4s^3 + 6s + 0$$

RH-test: the system is unstable because of the existence of two changes in the signs of the first column elements which means two right side poles. Continuing the table construction helps in determining the poles distribution. The construction with auxiliary polynomial (row of zeros) shows the existence of four different imaginary poles. Thus the two remaining poles should be at the semi left plane.

<u>Observation:</u> The polynomial equation with only even power terms has always symmetric roots around the origin, which means that the system poles should have positive and negative real parts (changes in first column sign) or zero real part (imaginary poles) the roots are repeated imaginary roots only if the polynomial is a perfect polynomial and can be written as $(s^2+\alpha^2)^r$.

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LTI System Conditional Stability:

A system stability that depends on parameters with values that variates in a defined parameters space.

Observation: the conditional stability analysis should cover all the values of the parameters space and can include different special cases.

Example1: Discuss the conditional stability of the LTI system with the following transfer function using The RH criterion

$$T(s) = \frac{K}{s^3 + 18s^2 + 77s + K}$$

s^3	1	77
s^2	18	K
s^1	$\frac{1386 - K}{18}$	
s^0	K	

Discussion:

To have asymptotic stability the first column parametric terms should be positive (no signs change). Thus we solve for $parametric\ terms \geq 0$ and determine the changes in sign in each part of the sequence of signs distribution.

$first\ term\ 1386 - k \geq 0 \rightarrow k \leq 1386, \quad second\ term \quad k \geq 0$

At the value $k = 1386$ we have a special case with a row of zeros at odd indexed row.

At $k = 0$ in the last row (the only case with row of zeros in an odd indexed row) the special case indicates that the polynomial has roots in the origin, therefore, these poles should be extracted, discussed, and the stability analysis continues with the proper polynomial .

Stability analysis

Columns(3,2): +++

Column(1) :+++++++++++++++++++++++++(1386)---------------

Column(0) :-----------(0)+++++++++++++++++++++++++++++++

Thus: the system is unstable for $k < 0$ with one right side pole (RSP).

the system is asymptotically stable in the interval $]0, 1386[$

the system is unstable for $k > 1386$ with two right side poles

Special cases:

For $k = 1386$ we have a row of zeros with auxiliary equation $18s^2 + 1386 = 0 \rightarrow s_{1,2} = \pm j8.77$ so the system is BIBO stable

For $k = 0$ the extracted polynomial becomes: $D(s) = s(s^2 + 18s + 77)$ with roots $s_1 = 0, s_2 = -11, s_3 = -7$

The system is BIBO stable.

Observation: the polynomial equations were simple and solved to obtain the remaining system roots.

Exercise Use the RH-table with the auxiliary polynomial (for $k = 1386$) and the extracted polynomial (for $k = 0$) to reach the same conclusions.

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Example2: Discuss the conditional stability of the LTI system with the following transfer function using The RH criterion

$$T(s) = \frac{0.25K_1(s+0.435)}{s^4 + 3.456s^3 + 3.457s^2 + (0.719 + 0.25K_1)s + (0.0416 + 0.109K_1)}$$

s^4	1	3.457	$0.0416 + 0.109K_1$
s^3	3.456	$0.719 + 0.25K_1$	
s^2	$11.228 - 0.25K_1$	$0.144 + 0.377K_1$	
s^1	$\frac{-0.0625K_1^2 + 1.324K_1 + 7.575}{11.228 - 0.25K_1}$		
s^0	$0.144 + 0.377K_1$		

Row(1) sign (division of numerator and denominator signs)

Numerator sign: -------(-4.685)++++(25.87)------------

Denominator sign: ++++++++++++++++++++++(44.91)-----

Stability analysis:

Rows(4,3) sign: ++

Row(2) sign: +++(44.91)---------------------------------

Row(1) sign: ----------(-4.685)++++++++++++++++++++++++(25.87)--------------------(44.91)+++++++++++++++++++++++++

Row(0) sign: ----------------------------(-0.382)+++

The system is unstable with one RSP for $k_1 < -4.685$

The system is unstable with one RSP for $-4.685 < k_1 < -0.382$

The system is Asymptotically stable for $-0.382 < k_1 < 25.87$

The system is unstable with two RSP for $25.87 < k_1 < 44.91$

The system is unstable with two RSP for$k_1 > 44.91$

Exercise: Study the stability of the system at the critical points $k_1 = -0.382,\ -4.685, 25.87, and\ 44.91$

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Steady-State Error

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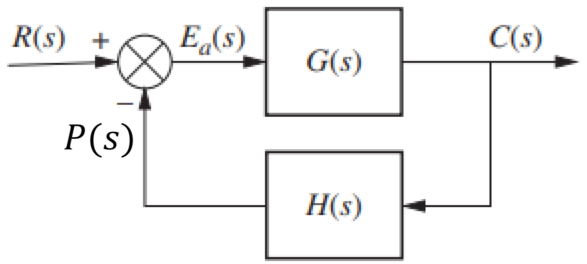
Steady-State Error:

Definition (system error):

Given a linear feedback control system with response $c(t)$ and reference $r(t)$, the system error function is defined as:

$$e(t) = r(t) - c(t).$$



Definition (actuation error):

Given a linear feedback control system with feedback signal $p(t)$ and reference $r(t)$, the system error is defined as: $e_a(t) = r(t) - p(t)$.

Definition (system steady-state error):

Given a linear feedback control system with response $c(t)$, reference $r(t)$, and a system steady-state error function $e(t)$, the system steady-steady error e_{steady} is defined under the existence condition of the limit as:

$$e_{steady} = \lim_{t\to\infty} e\,(t)$$

Steady-State Control Objective:

To adjust the steady-state error to a value that follows the steady state error specifications.

Theorem:

It is necessary for the existence of the steady-state error that the system be asymptotically stable.

Observation:

The computation of the steady state error with a conditional stability parameter space has a meaningful value only in the asymptotic stability parameter-space region.

Lemma:

Under the asymptotic stability condition, the steady-state error can be computed using the Laplace transform final value theorem. That is, $e_{steady} = \lim_{s\to 0} s\,E(s)$.

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Steady-State error (Basic Concept):

The value and type of the steady-state error depends on the order of the order of the dynamic of the reference r(t) to be tracked and the order of the dynamic of the system in the following sense:

- If the order of the dynamic of the reference is higher than that of the system, that is the reference changes are faster than the system dynamic, then $e_{steady} \to \infty$.
- If the order of the dynamic of the reference is has the same order of that of the system, that is the reference changes are of the same order of the system dynamic, then $e_{steady} = finite - \text{value}$.
- If the order of the dynamic of the reference is lower than that of the system, that is the reference changes are slower than the system dynamic, then $e_{steady} = 0$.

Reference Dynamic Order-Error-type:

The order of $r(t)$ as an infinite function of t.

Zero-order dynamic: $r(t) = unit\ step\ input = u(t) \to R(s) = \frac{1}{s}$.

Error-type: position error $e_{steady} = e_p$.

First-order dynamic: $r(t) = ramp\ input = r(t) \to R(s) = \frac{1}{s^2}$.

Error-type: velocity error $e_{steady} = e_v$.

Second -order dynamic: $r(t) = parabolic\ \ input = p(t) \to R(s) = \frac{1}{s^3}$.

Error-type: acceleration error $e_{steady} = e_a$.

System Dynamic-System-Type:

The system type equals the number of zero-roots of the dynamic characteristic error gain function or equivalently the number of the singularity points of the error-gain Laplace transfer function at the origin.

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Singularity points of $f(x)$:

A point x_0 is said to be a singularity point for $f(x)$ if $\lim_{x \to x_0} |f(x)| \to \infty$

Error gain function-Laplace domain:

The error gain transfer function is the equivalent direct path gain function that sets the feedback control system in its unity feedback form. $G_E(s) = \frac{G(s)}{1+G(s)[H(s)-1]}$

System of Type n:

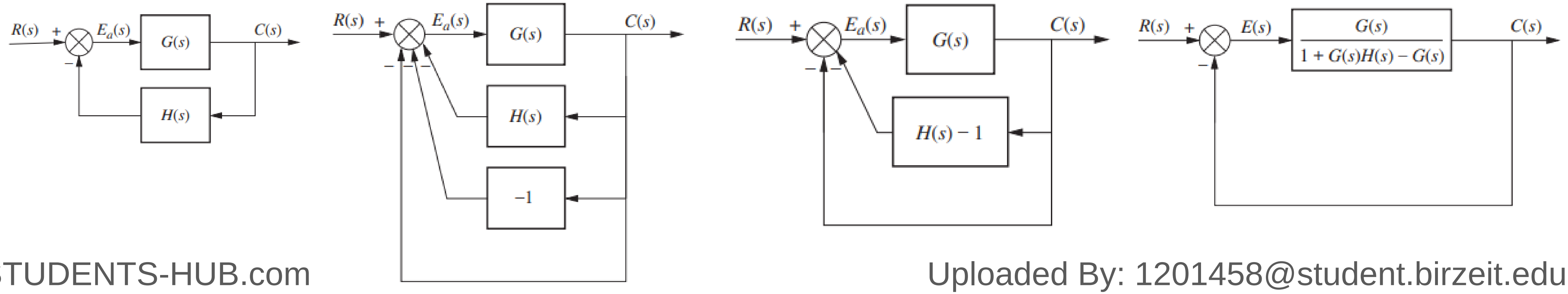
$G_E(s) = \frac{a(s)}{s^n b(s)}$ with $G_E(s)$ in its primitive form and $b(s) = 0$ has no solutions at $s = 0$.

Observation 1:

The error gain-function is used for error computation purposes and not used for stability analysis unless the complete closed loop transfer function is computed.

Observation 2:

In a unity feedback system $G_E(s) = G(s)$ as can be deduced by substituting $H(s) = 1$ in $G_E(s)$



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Steady-State Error Computation (Error Static Gain):

$$e(t) = r(t) - c(t) \rightarrow E(s) = R(s) - C(s) = R(s) - G_E(s)E(s) \rightarrow E(s) = \frac{R(s)}{1 + G_E(s)}$$

$$e_{steady} = \lim_{s \to 0} sE(s)$$

Position-Error e_p: $(R(s) = \frac{1}{s})$

$$e_p = \lim_{s \to 0} s \cdot \frac{\frac{1}{s}}{1 + G_E(s)} = \frac{1}{1 + \lim_{s \to 0} G_E(s)}$$

System type cases:

- Type0: $\lim_{s \to 0} G_E(s) = constant = k_p \rightarrow e_p = \frac{1}{1+k_p} = finite,\ \ k_p$ is called position static gain. (reference dynamic =type)
- Type1: $\lim_{s \to 0} G_E(s) \rightarrow \infty \rightarrow e_p = \frac{1}{1+\lim_{s \to 0} G_E(s)} = 0$, (reference dynamic < type)
- In general Type(n)≥1: $\lim_{s \to 0} G_E(s) \rightarrow \infty \rightarrow e_p = \frac{1}{1+\lim_{s \to 0} G_E(s)} = 0$, (reference dynamic < type→ system is faster than reference)

Velocity-Error e_v: $(R(s) = \frac{1}{s^2})$

$$e_v = \lim_{s \to 0} s \cdot \frac{\frac{1}{s^2}}{1 + G_E(s)} = \lim_{s \to 0} \frac{1}{s + sG_E(s)} = \frac{1}{\lim_{s \to 0} sG_E(s)}$$

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System type cases:

- Type0: $\lim_{s\to 0} sG_E(s) = 0 \to e_v \to \infty$, (reference dynamic > system type)
- Type1: $\lim_{s\to 0} sG_E(s) = finite = k_v \to e_v = \frac{1}{k_v}$, k_v *is called velocity static gain* (reference dynamic =system type)
- In general Type(n)>1: $\lim_{s\to 0} sG_E(s) \to \infty \to e_v = 0$, (reference dynamic < system type→ system is faster than reference)

Acceleration-Error e_a: $(R(s) = \frac{1}{s^3})$

$$e_a = \lim_{s\to 0} s \cdot \frac{\frac{1}{s^3}}{1 + G_E(s)} = \lim_{s\to 0} \frac{1}{s^2 + s^2 G_E(s)} = \frac{1}{\lim_{s\to 0} s^2 G_E(s)}$$

System type cases:

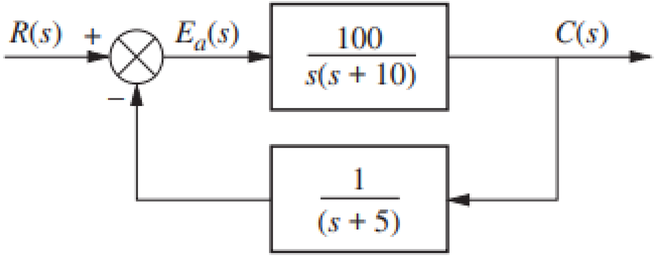
- Type0 and Type 1: $\lim_{s\to 0} s^2 G_E(s) = 0 \to e_a \to \infty$, (reference dynamic > system type)
- Type2: $\lim_{s\to 0} s^2 G_E(s) = finite = k_a \to e_a = \frac{1}{k_a}$, k_a *is called acceleration static gain* (reference dynamic =system type)
- In general Type(n)>2: $\lim_{s\to 0} s^2 G_E(s) \to \infty \to e_a = 0$, (reference dynamic < system type→ system is faster than reference)

Example: Compute the position, velocity and acceleration errors of the following feedback system.

Exercise: prove that the system is stable using the RH criterion.

The error gain function of this asymptotically stable system is given by:

$$G_E(s) = \frac{G(s)}{1 + G(s)[H(s) - 1]} = \frac{100(s+5)}{s^3 + 15s^2 - 50s - 400}$$



The error gain function has no roots at the origin → the system type is zero→ $e_v = e_a \to \infty$

$$k_p = \lim_{s\to 0} G_e(s) = \frac{500}{-400} = -\frac{5}{4} \to e_p = \frac{1}{1 + k_p} = \frac{1}{1 - \frac{5}{4}} = -4$$

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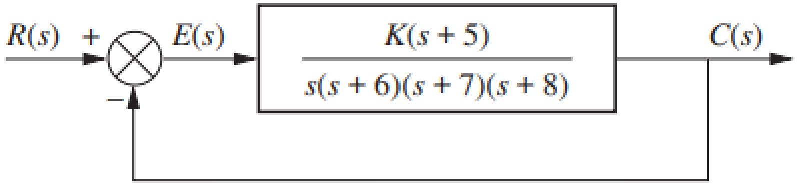
Example2: Consider the following static gain and determine the type of the system and the system static gain so that the steady state error = 10%.

Solution:

The system error gain function $G_E(s) = G(s)$the direct path gain function because of the unity feedback condition.
Accordingly the system is of type1→ $e_p = 0\ and\ e_a \to \infty$.

Thus, K should be computed using the velocity steady state error $e_v = \frac{1}{\lim_{s\to 0} sG_E(s)} = 0.1 = \frac{1}{\frac{5k}{6\cdot 7\cdot 8}} \to k = \frac{6\cdot 7\cdot 8}{0.5} = 672.$



Important Observation: The result is not valid until it is proved that the system is asymptotically stable for $k = 672.$

Exercise1:

Compute the system transfer function of the system, study the conditional stability of the system using RH, and prove that the system is asymptotically stable for $k = 672.$

Exercise1:

Compute the velocity steady state error as a function of k and determine the minimum and maximum velocity errors in the asymptotic stability interval of the parameter k.

Steady state error in presence of disturbance:

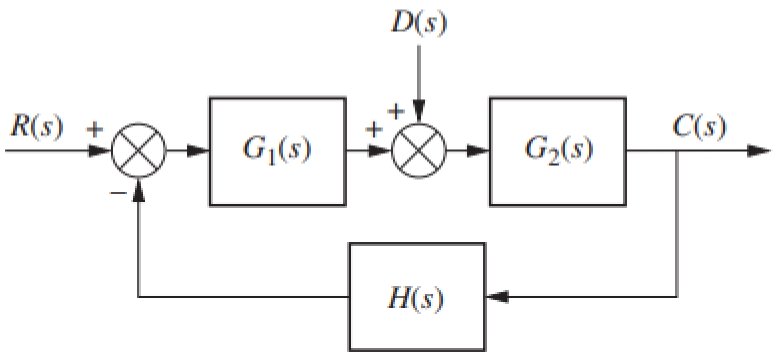
Superposition is applied to compute the steady state error in presence of deterministic disturbance.

$C(s) = C_R(s) + C_D(s)$

$E(s) = R(s) - C(s)$

$$C_R(s) = \frac{G_1(s)G_2(s)}{1 + G_1(s)G_2(s)H(s)} \cdot R(s)$$

$$C_D(s) = \frac{G_2(s)}{1 + G_1(s)G_2(s)H(s)} \cdot D(s)$$



$$E(s) = R(s) - \left[\frac{G_1(s)G_2(s)}{1 + G_1(s)G_2(s)H(s)} \cdot R(s) + \frac{G_2(s)}{1 + G_1(s)G_2(s)H(s)} \cdot D(s)\right]$$

$$R(s)\left[1 - \frac{G_1(s)G_2(s)}{1 + G_1(s)G_2(s)H(s)}\right] - \frac{G_2(s)}{1 + G_1(s)G_2(s)H(s)} \cdot D(s) =$$

$$= R(s)\left[\frac{1 + G_1(s)G_2(s)(H(s) - 1)}{1 + G_1(s)G_2(s)H(s)}\right] - \frac{G_2(s)}{1 + G_1(s)G_2(s)H(s)} \cdot D(s)$$

$$E_R(s) = R(s)\left[\frac{1 + G_1(s)G_2(s)(H(s) - 1)}{1 + G_1(s)G_2(s)H(s)}\right] \rightarrow e_{ref\_steady} = \lim_{s\to 0} sE_R(s)$$

$$E_D(s) = \frac{G_2(s)}{1 + G_1(s)G_2(s)H(s)} \cdot D(s) \rightarrow e_{dist\_steady} = \lim_{s\to 0} sE_D(s)$$

$e_{total\_steady} = e_{ref\_steady} - e_{dist\_steady}$

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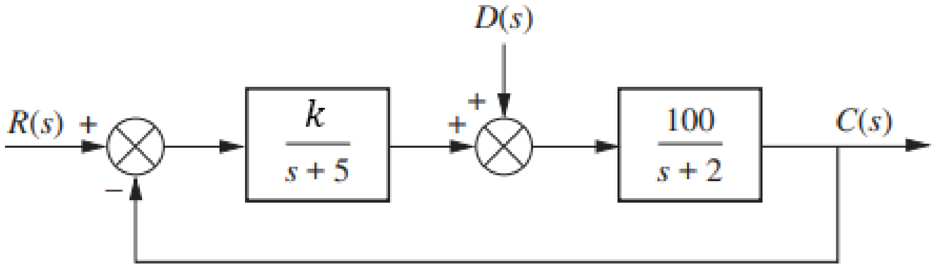
Observation 1:
The stability analysis must be done before error computation, and is done using the denominator of one of the transfer functions $\frac{C_R(s)}{R(s)}$ or $\frac{C_D(s)}{D(s)}$

Observation 2:
The type of the system is determined by the $R(s)$ reference input that makes e_{ref_steady} finite. That is if it is finite for $R(s) = \frac{1}{s}$ then the system is type0, for $R(s) = \frac{1}{s^2}$ then the system is type1, and for $R(s) = \frac{1}{s^3}$ the system is type2.

Observation 3:
The rules related to the comparison of the reference dynamic and system type continue to be valid also in this case. For example, if the error e_{ref_steady} is finite for $R(s) = \frac{1}{s} \rightarrow type0$ then $e_p = finite,\ e_v = e_a \rightarrow \infty$

Observation 3:
e_{dist_steady} remains the same and computed one time whatever is the reference, that is independent of computing the position, velocity, or acceleration errors.

Example: Compute the total steady state error for a unit step disturbance and $k \in R$.



$$E_R(s) = R(s)\left[\frac{1 + G_1(s)G_2(s)(H(s) - 1)}{1 + G_1(s)G_2(s)H(s)}\right] = R(s)\frac{1}{1 + \frac{100k}{(s+5)(s+2)}}$$

$$E_R(s) = \frac{s^2 + 7s + 10}{s^2 + 7s + 10 + 100k} \cdot R(s)$$

$$E_D(s) = \frac{\frac{100}{(s+2)}}{1 + \frac{100k}{(s+5)(s+2)}} \cdot D(s) = \frac{100(s+5)}{s^2 + 7s + 10 + 100k} \cdot D(s)$$

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Stability analysis:

The characteristic equation is $D_k(s) = s^2 + 7s + 10 + 100k$

2	1	$10 + 100k$
1	7	0
0	$10 + 100k$	

From the RH table the region of asymptotic stability is defined by $10 + 100k > 0 \rightarrow k > -0.1$

Error analysis:

position error: $e_p = \lim_{s\to 0} s \cdot \frac{1}{s} \cdot \frac{s^2+7s+10}{s^2+7s+10+100k} = \frac{10}{10+100k} = \frac{1}{1+10k}$.

Thus the system is of type zero for $k > -0.1 \rightarrow e_v = e_a \rightarrow \infty$.

$$e_D = \lim_{s\to 0} s \cdot \frac{1}{s} \cdot \frac{100(s+5)}{s^2 + 7s + 10 + 100k} = \frac{500}{10 + 100k} = \frac{50}{1 + 10k}$$

The total error $e_{tot\_stedy} = e_p - e_D = \frac{1}{1+10k} - \frac{50}{1+10k} = \frac{-49}{1+10k}$

Exercise: Determine the absolute values of minimum and maximum total steady state error with k.

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