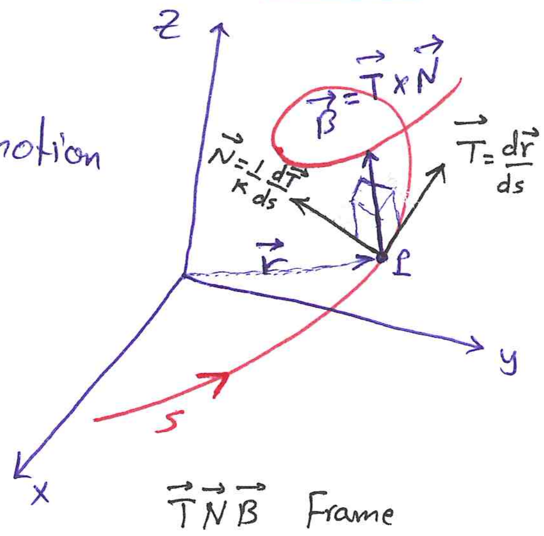
13.5 Tangential and Normal Components of Acceleration

- If a particle is traveling along a space curve s, then we can describe the motion of the particle interms of

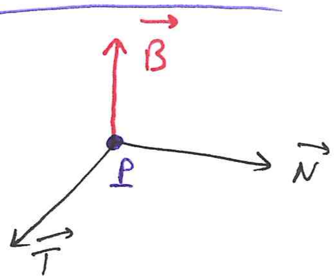
[1] the unit tangent vector $\vec{T}$ (forward direction)

[2] the unit normal vector $\vec{N}$ (the tendency of the motion)

[3] the unit binormal vector $\vec{B} = \vec{T} \times \vec{N}$ ($\perp$ to the plane created by $\vec{T}$ and $\vec{N}$)



- $\vec{T}, \vec{N}, \vec{B}$ define a right-handed frame used to calculate the paths of particles moving through space. This fram is also called $\vec{T}\vec{N}\vec{B}$ fram.



Def: If the acceleration vector is written as

$$\vec{a} = a_T \vec{T} + a_N \vec{N}, \text{ then}$$

the tangential scalar component is $a_T = \frac{d^2 s}{dt^2} = \frac{d}{dt}|\vec{v}|$ and

the normal scalar component is $a_N = \kappa \left(\frac{ds}{dt}\right)^2 = \kappa |\vec{v}|^2$.

- Note that $\vec{v} = \frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds}\frac{ds}{dt} = \vec{T}\frac{ds}{dt}$

$$\cancel{\vec{v}} = \vec{T}\frac{ds}{dt} = \frac{\cancel{\vec{v}}}{|\vec{v}|}\frac{ds}{dt} \Leftrightarrow \frac{ds}{dt} = |\vec{v}| \Leftrightarrow \frac{d^2 s}{dt^2} = \frac{d}{dt}|\vec{v}|$$

or $s(t) = \int_{t_0}^{t} |\vec{v}|\,dt \Rightarrow \frac{ds}{dt} = |\vec{v}|$

• Note also that

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\vec{T}\,\frac{ds}{dt}\right)$$

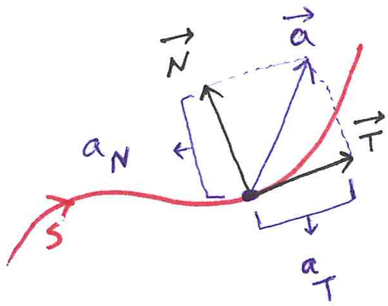
$$= \frac{d^2s}{dt^2}\,\vec{T} + \frac{ds}{dt}\,\frac{d\vec{T}}{dt}$$

$$= \frac{d^2s}{dt^2}\,\vec{T} + \frac{ds}{dt}\left(\frac{d\vec{T}}{ds}\,\frac{ds}{dt}\right)$$

$$= \frac{d^2s}{dt^2}\,\vec{T} + \frac{ds}{dt}\left(\kappa\vec{N}\,\frac{ds}{dt}\right) \qquad \frac{d\vec{T}}{ds} = \kappa\vec{N}$$

$$= \frac{d^2s}{dt^2}\vec{T} + \kappa\left(\frac{ds}{dt}\right)^2\vec{N} \quad \cdots\cdots *$$

$$= a_T\,\vec{T} + a_N\,\vec{N}$$



Notes:

1. the acceleration $\vec{a}$ always lies in the plane of $\vec{T}$ and $\vec{N}$
2. $\vec{a} \perp \vec{B}$
3. $*$ tells us how much of the acceleration takes place tangent to the motion (a_T) and how much takes place normal to the motion (a_N).
4. a_T measures the rate of change of the length of $\vec{v}$ (the change in the speed)
5. a_N measures the rate of change of the direction of $\vec{v}$.

$*$ We can calculate a_N without finding κ by:

$$|\vec{a}|^2 = a_T^2 + a_N^2 \iff \boxed{a_N = \sqrt{|\vec{a}|^2 - a_T^2}}$$

Exp Let $\vec{r}(t) = (1+3t)\vec{i} + (t-2)\vec{j} - 3t\vec{K}$

write $\vec{a}$ in the form $\vec{a} = a_T \vec{T} + a_N \vec{N}$ without finding $\vec{T}$ and $\vec{N}$.

- $\vec{V} = 3\vec{i} + \vec{j} - 3\vec{K} \Rightarrow |\vec{V}| = \sqrt{9+1+9} = \sqrt{19}$
- $a_T = \frac{d}{dt}|\vec{V}| = 0$
- $\vec{a} = \vec{0} \Rightarrow a_N = \sqrt{|\vec{a}|^2 - a_T^2} = 0$
- $\vec{a} = (0)\vec{T} + (0)\vec{N} = \vec{0}$

Def Let $\vec{B} = \vec{T} \times \vec{N}$. The torsion function of a smooth curve is $\tau = -\frac{d\vec{B}}{ds} \cdot \vec{N}$

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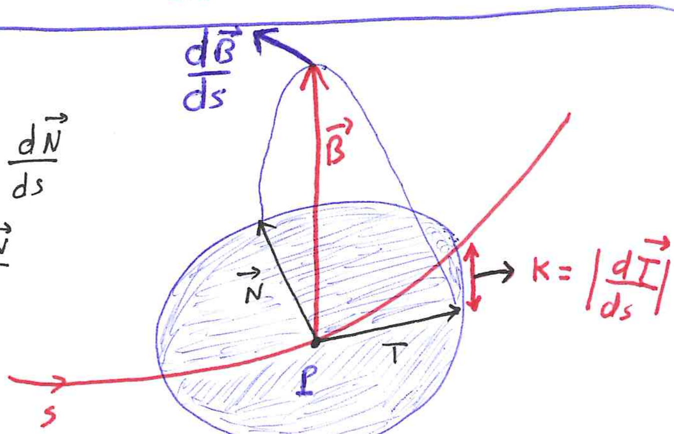
Torsion



$$\frac{d\vec{B}}{ds} = \frac{d}{ds}(\vec{T} \times \vec{N}) = \frac{d\vec{T}}{ds} \times \vec{N} + \vec{T} \times \frac{d\vec{N}}{ds}$$

$$= \vec{0} + \vec{T} \times \frac{d\vec{N}}{ds}$$

$$\frac{d\vec{B}}{ds} = \vec{T} \times \frac{d\vec{N}}{ds}$$

this is because $\vec{N}$ is the direction of $\frac{d\vec{T}}{ds}$ (that is $\vec{T}$ has constant length $\Rightarrow \frac{d\vec{T}}{ds}$ is $\perp \vec{T}$). That is, $\vec{N} = \frac{1}{K}\frac{d\vec{T}}{ds}$

Hence, $\frac{d\vec{B}}{ds} \perp \vec{T}$ "cross product"

but $\frac{d\vec{B}}{ds} \perp \vec{B}$ "since $\vec{B}$ has constant length"

$\Rightarrow \frac{d\vec{B}}{ds} \perp$ plane of $\vec{B}$ and $\vec{T}$

$\Rightarrow \frac{d\vec{B}}{ds} \parallel \vec{N}$. Hence $\frac{d\vec{B}}{ds} = -\tau \vec{N}$ "scalar multiple of $\vec{N}$"

the negative is traditional

$\Rightarrow \tau$ is called the torsion.

Note that now; $\frac{d\vec{B}}{ds} \cdot \vec{N} = -\tau \vec{N} \cdot \vec{N} = -\tau$

Hence $\tau = -\frac{d\vec{B}}{ds} \cdot \vec{N}$ ✓

To calculate the torsion, we can use

$$\tau = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|\vec{v} \times \vec{a}|^2} \quad \text{if} \quad \vec{v} \times \vec{a} \neq \vec{0}$$

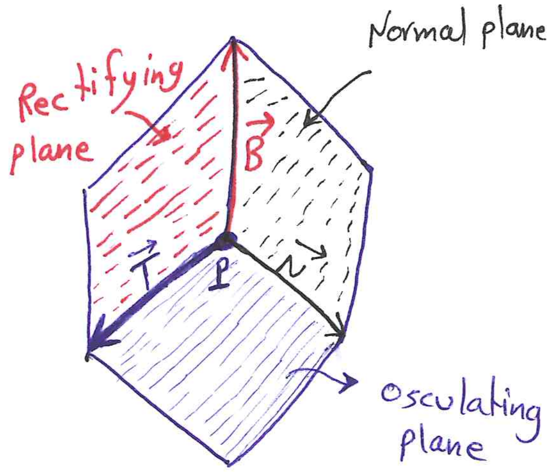
$\dot{x} = \frac{dx}{dt}$

$\ddot{x} = \frac{d^2x}{dt^2}$

where one derivative w.r.t t for each dot. $\dddot{x} = \frac{d^3x}{dt^3}$

Proof is omitted.

* The planes determined by $\vec{T}, \vec{N}, \vec{B}$:



Exp [1] Find $\vec{r}, \vec{T}, \vec{N}, \vec{B}$ at $t=0$ for
$\vec{r}(t) = (\cos t)\vec{i} + (\sin t)\vec{j} + t\vec{K}$

[2] Find the equations of the osculating plane, normal plane and rectifying plane.

[3] Find the torsion

[1] • $\vec{V} = (-\sin t)\vec{i} + (\cos t)\vec{j} + \vec{K}$

$|\vec{V}| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$

• $\vec{T} = \frac{\vec{V}}{|\vec{V}|} = \left(\frac{-1}{\sqrt{2}} \sin t\right)\vec{i} + \left(\frac{1}{\sqrt{2}} \cos t\right)\vec{j} + \left(\frac{1}{\sqrt{2}}\right)\vec{K}$

$\vec{T}(0) = \left(\frac{1}{\sqrt{2}}\right)\vec{j} + \left(\frac{1}{\sqrt{2}}\right)\vec{K}$

• $\frac{d\vec{T}}{dt} = \left(\frac{-1}{\sqrt{2}} \cos t\right)\vec{i} - \left(\frac{1}{\sqrt{2}} \sin t\right)\vec{j} \Rightarrow \left|\frac{d\vec{T}}{dt}\right| = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$

$\vec{N} = \frac{\frac{d\vec{T}}{dt}}{\left|\frac{d\vec{T}}{dt}\right|} = (-\cos t)\vec{i} - (\sin t)\vec{j}$

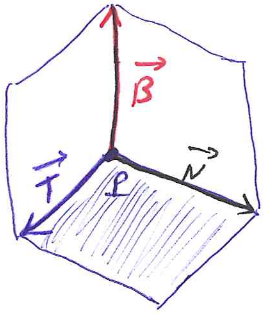
$\vec{N}(0) = -\vec{i}$.

• $\vec{B}(0) = \vec{T}(0) \times \vec{N}(0) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{K} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -1 & 0 & 0 \end{vmatrix} = \frac{-1}{\sqrt{2}}\vec{j} + \frac{1}{\sqrt{2}}\vec{K}$

• $\vec{r}(0) = \vec{i}$.

$\boxed{2}$ Since $\vec{r}(0) = \vec{i} \Rightarrow$ The point is $P(1,0,0)$

$\vec{B}(0) = \frac{-1}{\sqrt{2}}\vec{j} + \frac{1}{\sqrt{2}}\vec{k} \perp$ osculating plane



- The equation for the osculating plane is

$$\frac{-1}{\sqrt{2}} y + \frac{1}{\sqrt{2}} z = (1)(0) + (0)\left(\frac{-1}{\sqrt{2}}\right) + (0)\left(\frac{1}{\sqrt{2}}\right)$$

$\boxed{y - z = 0}$ is the osculating plane.

- $\vec{T}(0) = \left(\frac{1}{\sqrt{2}}\right)\vec{j} + \left(\frac{1}{\sqrt{2}}\right)\vec{k} \perp$ normal plane

$\Rightarrow$ The equation of the normal plane is

$$\frac{1}{\sqrt{2}} y + \frac{1}{\sqrt{2}} z = 0 \Leftrightarrow \boxed{y + z = 0}$$

- $\vec{N}(0) = -\vec{i} \perp$ rectifying plane

$\Rightarrow$ The equation of the rectifying plane is

$$-x = (1)(-1) \Leftrightarrow \boxed{x = 1}$$

$\boxed{3}$ • $\vec{a} = (-\cos t)\vec{i} - (\sin t)\vec{j}$

$$\vec{v} \times \vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin t & \cos t & 1 \\ -\cos t & -\sin t & 0 \end{vmatrix} = (\sin t)\vec{i} - (\cos t)\vec{j} + \vec{k}$$

$$|\vec{v} \times \vec{a}| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$$

- $\frac{d\vec{a}}{dt} = (\sin t)\vec{i} - (\cos t)\vec{j}$

- $$\tau = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|\vec{v} \times \vec{a}|^2} = \frac{\begin{vmatrix} -\sin t & \cos t & 1 \\ -\cos t & -\sin t & 0 \\ \sin t & -\cos t & 0 \end{vmatrix}}{2} = \frac{1}{2}$$