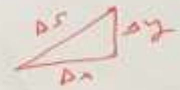
6.3 Lengths of Curves in the plane

$\Delta s_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$



$f'(c_k) = \frac{\Delta y_k}{\Delta x_k}$

$\Delta y_k = f'(c_k)\,\Delta x_k$

$L = \lim \sum \Delta s_k = \lim_{\Delta x_k \to 0} \sum \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$

$= \lim_{\Delta x_k \to 0} \sum \sqrt{(\Delta x_k)^2 + (f'(c_k)\,\Delta x_k)^2}$

$= \lim_{\Delta x_k \to 0} \sum \sqrt{1 + (f'(c_k))^2}\ \Delta x_k$

$= \int_a^b \sqrt{1 + (f'(x))^2}\, dx$

Ex $y = \frac{4\sqrt{2}}{3} x^{3/2} - 1 \qquad 0 \le x \le 1$

$y' = \frac{4\sqrt{2}}{3} \cdot \frac{3}{2} x^{1/2} = \frac{2\sqrt{2}}{1} x^{1/2}$

$(y')^2 = 8x$

$L = \int_0^1 \sqrt{1 + (8x)}\, dx = \frac{1}{8} \frac{(1+8x)^{3/2}}{3/2} \Big|_0^1$

$= \frac{2}{3} \cdot \frac{1}{8} (1+8x)^{3/2} \Big|_0^1$

$= \frac{1}{12}\left[(9)^{3/2} - 1\right] = \frac{13}{6}$

$\boxed{\text{Ex2}}$ Find the length of the graph of

$$f(x) = \frac{x^3}{12} + \frac{1}{x} \quad ; \quad 1 \le x \le 4$$

Soln $L = \int_1^4 \sqrt{1 + (f'(x))^2}\, dx$

$f'(x) = \frac{x^2}{4} - \frac{1}{x^2}$

$(f'(x))^2 = \left(\frac{x^2}{4} - \frac{1}{x^2}\right)^2 = \frac{x^4}{16} - 2\left(\frac{x^2}{4}\right)\left(\frac{1}{x^2}\right) + \frac{1}{x^4}$

$= \frac{x^4}{16} - \frac{1}{2} + \frac{1}{x^4}$

$1 + (f'(x))^2 = \frac{x^4}{16} + \frac{1}{2} + \frac{1}{x^4}$

$= \left(\frac{x^2}{4} + \frac{1}{x^2}\right)^2$

$\Rightarrow L = \int_1^4 \sqrt{1 + (f'(x))^2} = \int_1^4 \left(\frac{x^2}{4} + \frac{1}{x^2}\right) dx$

$\int x^{-2}$

$\frac{x^{-1}}{-1}$

$= \frac{x^3}{12} - \frac{1}{x} \Big|_1^4$

$= \left(\frac{64}{12} - \frac{1}{4}\right) - \left(\frac{1}{12} - 1\right)$

$= \frac{72}{12} = 6$

Some times $\frac{dy}{dx}$ is not continuous, so we can't use the formula, But we can consider $x = g(y)$ and use a similar formula

$$L = \int_c^d \sqrt{1+(g'(y))^2}\, dy = \int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$

Ex) L of $y = \left(\frac{x}{2}\right)^{2/3}$ from $x=0$, to $x=2$

$y' = \frac{2}{3}\left(\frac{x}{2}\right)^{-1/3} \cdot \frac{1}{2}$ (not continuous at $x=0$)

So we switch for x,

$y^{3/2} = \frac{x}{2}$

$x = 2y^{3/2}$

$x=0 \Rightarrow y=0$
$x=2 \Rightarrow y=1$

$\frac{dx}{dy} = \cancel{2}\left(\frac{3}{\cancel{2}}\right) y^{1/2}$

$\left(\frac{dx}{dy}\right)^2 = 9y$

$$L = \int_0^1 \sqrt{1+(x'(y))^2}\, dy = \int_0^1 \sqrt{1+9y}\, dy$$

$$= \frac{1}{9} \frac{(1+9y)^{3/2}}{\frac{3}{2}} \Big|_0^1 = \frac{2}{27}(10)^{3/2}$$

The differential Formula for Arc Length

If $f(x), f'(x)$ are continuous on $[a,b]$, then we can define the arc length function

$$S(x) = \int_a^x \sqrt{1+(f'(t))^2}\, dt$$

which gives the length from a to any point x

notice that $\frac{ds}{dx} = \sqrt{1+(f'(x))^2} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$

$$\Rightarrow \quad ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \sqrt{(dx)^2+(dy)^2}$$

which is called the differential formula of arc length

notice that

$$L = \int_a^b ds$$

Ex) Find the arc length function for the curve in example (2) taking the starting point $(1, \frac{13}{12})$

sol From Example (2)

$$(1+f'(x))^2 = \left(\frac{x^2}{4}+\frac{1}{x^2}\right)^2$$

$$S(x) = \int_1^x \sqrt{1+(f'(t))^2}\, dt = \int_1^x \left(\frac{t^2}{4}+\frac{1}{t^2}\right) dt = \frac{t^3}{12} - \frac{1}{t}\Big|_1^x$$

$$= \frac{x^3}{12} - \frac{1}{x} + \frac{11}{12},$$ To find the length from 1 to 4 we find $S(4) = 6$

From Book Exercises

#4) $x = \frac{y^{3/2}}{3} - y^{1/2}$ from $y=1$ to $y=9$

$$x' = \frac{1}{3}\cdot\frac{3}{2}y^{1/2} - \frac{1}{2}y^{-\frac{1}{2}} = \frac{1}{2}y^{1/2} - \frac{1}{2}y^{-\frac{1}{2}}$$

$$(x')^2 = \frac{1}{4}y - 2\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) + \frac{1}{4}y^{-1} = \frac{1}{4}y - \frac{1}{2} + \frac{1}{4}y^{-1}$$

$$1 + (x')^2 = \frac{1}{4}y + \frac{1}{2} + \frac{1}{4}y^{-1}$$
$$= \left(\frac{1}{2}y^{1/2} + \frac{1}{2}y^{-1/2}\right)^2$$

$$\Rightarrow L = \int_1^9 \sqrt{1+(x')^2}\, dy = \int_1^9 \frac{1}{2}y^{\frac{1}{2}} + \frac{1}{2}y^{-1/2}$$

$$= \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{3}y^{3/2} + \frac{1}{2}\cdot\frac{2}{1}y^{1/2}\Big|_1^9$$

$$= \left(\frac{1}{3}9^{3/2} + 9^{1/2}\right) - \left(\frac{1}{3} + 1\right) = (9+3) - \frac{4}{3} = \frac{32}{3}$$

#1a) Find a curve through $(1,1)$ whose Length is the integral

$$L = \int_1^4 \sqrt{1+\frac{1}{4x}}\, dx \Rightarrow (y')^2 = \frac{1}{4x}$$
$$\Rightarrow y' = \frac{1}{2\sqrt{x}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{x}} = \frac{1}{2}x^{-\frac{1}{2}} \Rightarrow y = \frac{1}{2}\int x^{-1/2}\, dx = \frac{1}{\cancel{2}}\frac{x^{1/2}}{1/\cancel{2}} = \sqrt{x} + C$$

$y(1) = 1 \Rightarrow C = 0 \Rightarrow y(x) = \sqrt{x}$