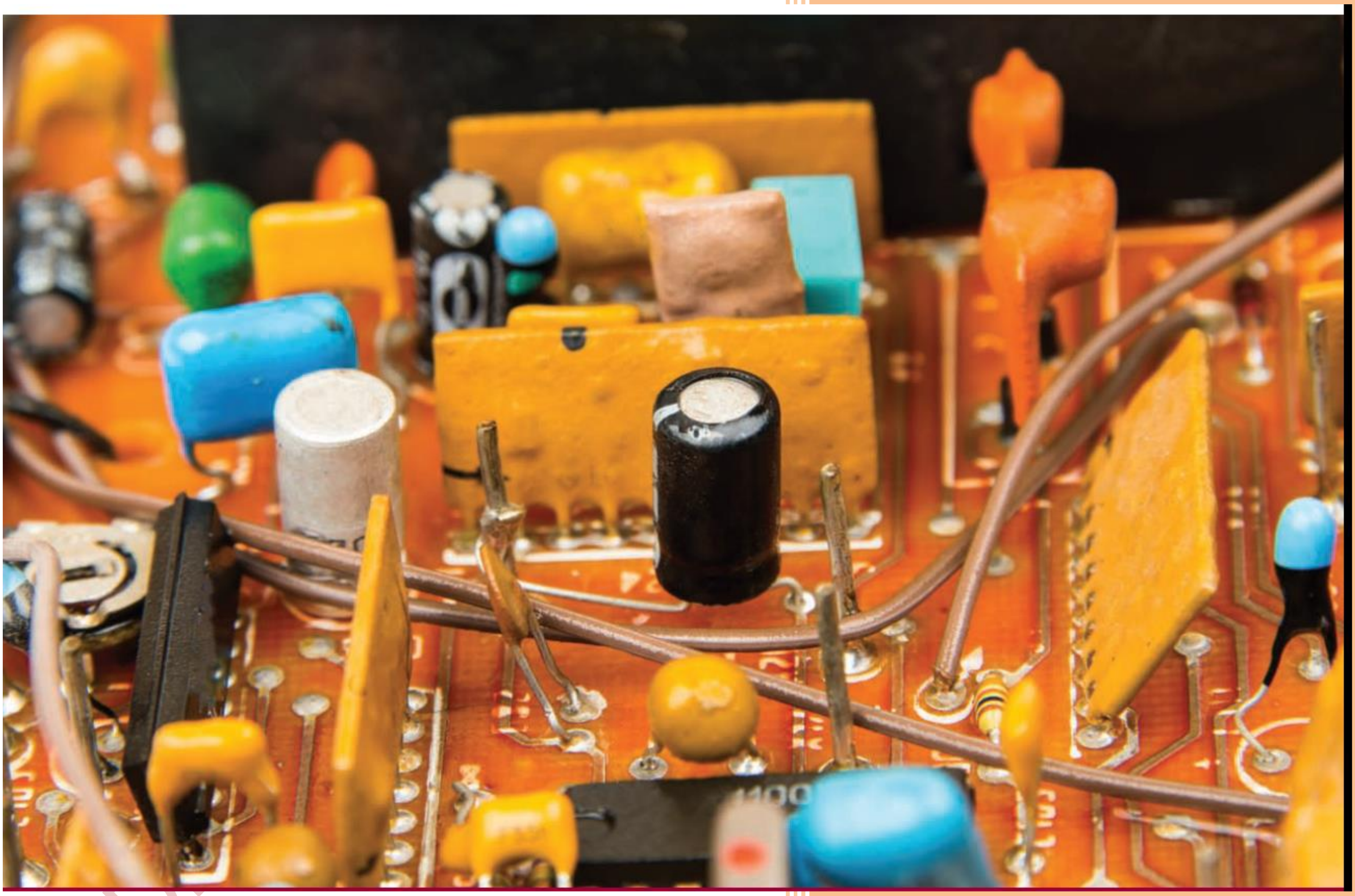
2024/25

Circuits Analysis - ENEE 2304



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Chapter 1

Introduction

An **electric circuit** is a mathematical model that approximates the behavior of an actual electrical system, such as computer, power, control or signal processing system. The model is employed to analyze and design such systems.

Voltage: is the energy per unit charge created by the separation, and is expressed as:

$$v = \frac{dw}{dq},$$

where, v is the voltage in **Volts**, w is the Work (energy) in joules, and q is the charge in coulombs.

Electric current: is the rate of charge flow, and is defined as:

$$i = \frac{dq}{dt},$$

where, *i* is the current in **Amperes**, *q* is the charge in coulombs, and t is the time in seconds.

Power: is the rate of doing work per time, and is defined as:

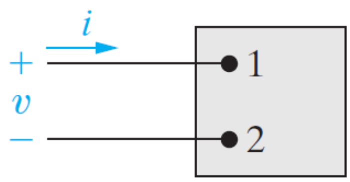
$$p = \frac{dw}{dt},$$

where, *p* is the power in **Watts**, *w* is the energy (work) in Joules, and *t* is the time in seconds. And also is defined as:

$$p = \frac{dw}{dt} = \left(\frac{dw}{dq}\right)\left(\frac{dq}{dt}\right),$$

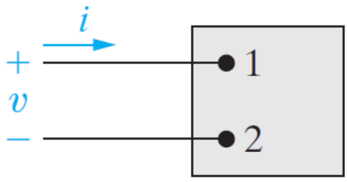
The power absorbed by a circuit element is:

$$p = vi$$

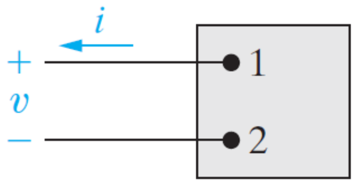


where, *v* is the voltage in **Volts** and *i* is the current in Amperes.

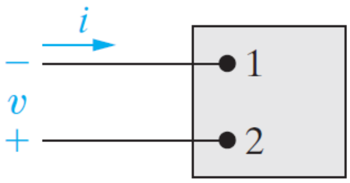
Note the polarity of power absorbed (or consumed) by a network for various polarities of voltage and directions of currents; *the power consumed is positive either if the current enters the positive terminal of the voltage, or the current leaves the negative terminal of the voltage!*



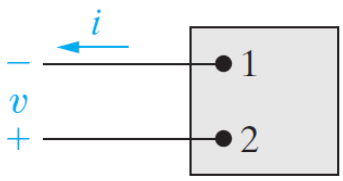
(a) $p = vi$



(b) $p = -vi$



(c) $p = -vi$



(d) $p = vi$

Chapter 2

Circuit Elements

2.1 Circuit Elements

An Ideal Circuit Element: is a two-terminal device, and can be classified as Active or Passive based simply on whether they supply energy to the circuit or absorb energy from the circuit.

Active elements: Batteries and Generators

Passive elements: Resistors, capacitors and inductors

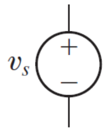
Active Circuit Elements (Sources):

They are two types; independent and dependent sources.

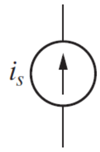
A) Independent Sources

An **electrical source** is a device that is capable of converting nonelectric energy to electric energy and vice versa.

1- **An Ideal Voltage Source** is a circuit element that maintains a specific voltage across its terminals regardless of the current through it.



2- **An ideal Current Source** is a circuit element that maintains a prescribed current through its terminals regardless of the voltage across its terminals.

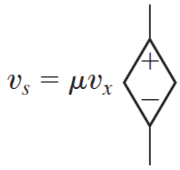


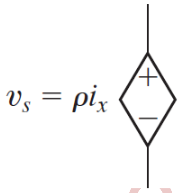
Note that, the arrow direction indicates the chosen reference direction of positive current charges.

B) Dependent Sources (Controlled Sources)

They generate a voltage or current whose value depends on the value of a voltage or current elsewhere in the circuit. They are represented by a diamond symbol. An example of dependent sources is the transistors and many electronic devices.

1) Dependent Voltage Sources

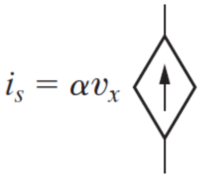


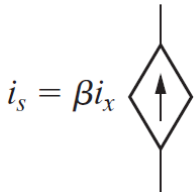


a) Voltage-controlled Voltage Source b) Current-controlled Voltage Source

Note that, μ is a multiplying constant that is dimensionless.

2) Dependent Current Sources





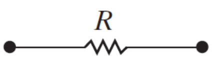
a) Voltage-controlled Current Source b) Current-controlled Current Source

Note that, β is a multiplying constant that is dimensionless.

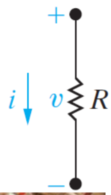
Passive Circuit Elements

A) Resistor (R):

- It impedes the current flow. It represents the part of the circuit in which energy entering the element, by the flow of current through it, is transformed into heat. Light can be emitted if the resistive element becomes hot enough to glow.



- It is measured in Ohms (Ω).



- The relationship between the current and voltage across the resistor is a linear relation, and is given by Ohm's Law as:

$$v = iR,$$

where, v is the voltage in Volts (V), R is the resistance in Ohms (Ω) and i is the current in Amperes (A).

Conductance (G): is the reciprocal of the resistance and is measured in Siemens (S), (℧) or Mho

$$G = \frac{1}{R} \text{ S}$$

Note that,

when $R = 0$ then $G = \infty$, and it is called a short circuit; $v = 0$

when $R = \infty$ then $G = 0$, and it is called an open circuit; $i = \frac{v}{\infty} = 0$, regardless of the voltage at the terminals!

- The energy absorbed in the resistor is dissipated in the form of heat or light.

- **The Instantaneous power** or the rate of energy absorption in the resistor is:

$$p = vi$$

or

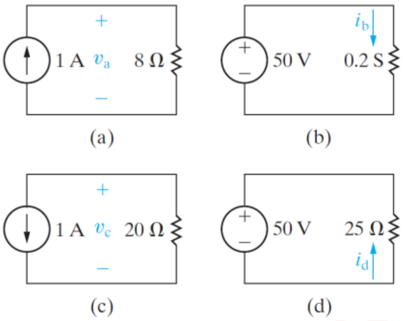
$$p = vi = (i\,R)i$$

$$\rightarrow \quad p = i^2 R \qquad p = \frac{i^2}{G}$$

Also,

$$p = \frac{v^2}{R} \quad \rightarrow \quad p = v^2 G$$

Example # 1:



a) Calculate the values of v and i.

b) Determine the power dissipated in each resistor.

Solution

a) The voltage v_a in Fig. 2.8(a) is a drop in the direction of the current in the resistor. Therefore,

$$v_a = (1)(8) = 8 \text{ V}.$$

The current i_b in the resistor with a conductance of 0.2 S in Fig. 2.8(b) is in the direction of the voltage drop across the resistor. Thus

$$i_b = (50)(0.2) = 10 \text{ A}.$$

The voltage v_c in Fig. 2.8(c) is a rise in the direction of the current in the resistor. Hence

$$v_c = -(1)(20) = -20 \text{ V}.$$

The current i_d in the 25 Ω resistor in Fig. 2.8(d) is in the direction of the voltage rise across the resistor. Therefore

$$i_d = \frac{-50}{25} = -2 \text{ A}.$$

b) The power dissipated in each of the four resistors is

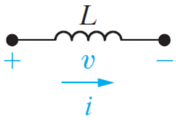
$$p_{8\Omega} = \frac{(8)^2}{8} = (1)^2(8) = 8 \text{ W},$$

$$p_{0.2S} = (50)^2(0.2) = 500 \text{ W},$$

$$p_{20\Omega} = \frac{(-20)^2}{20} = (1)^2(20) = 20 \text{ W},$$

$$p_{25\Omega} = \frac{(50)^2}{25} = (-2)^2(25) = 100 \text{ W}.$$

B) Inductor (L):



- It represents a two-terminal circuit element in which energy is stored in the magnetic field; examples of inductors are the coils of wires used to make an electro-magnet, or the windings of wire in an electric motor. The inductance is measured in Henry (H).
- The voltage across the inductor is proportional to the rate of change of the current

$$v = L\frac{di}{dt},$$

through it as:

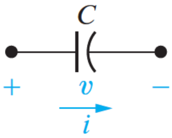
Therefore, the inductor's current is:

$$v\,dt = L\,di \quad \rightarrow \quad L\int_{i(t_0)}^{i(t)} dx = \int_{t_0}^{t} v\,d\tau$$

$$i(t) = \frac{1}{L}\int_{t_0}^{t} v\,d\tau + i(t_0),$$

where $i(t_0)$ is the initial inductor's current at time t_0.

C) Capacitor (C):



- ➢ It represents a two-terminal circuit element in which energy is stored in the electric field. It is fabricated by placing two parallel conducting plates separated by a layer of an insulating (dielectric) material.
- ➢ It is measured Farad (F).
- ➢ The "displacement current" current of a capacitor is proportional to the rate of change of the voltage across its terminals as:

$$i = C\frac{dv}{dt},$$

- ➢ Thus, the capacitor's voltage is:

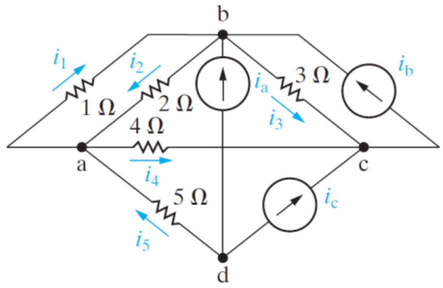
$$i\,dt = C\,dv \quad \text{or} \quad \int_{v(t_0)}^{v(t)} dx = \frac{1}{C}\int_{t_0}^{t} i\,d\tau.$$

$$v(t) = \frac{1}{C}\int_{t_0}^{t} i\,d\tau + v(t_0)$$

where $v(t_0)$ is the initial capacitor's voltage at time t_0.

2.2 Kirchhoff's Laws

They are used to analyse complex circuits.



A node: is a point of connection of two or more circuit elements; 4 nodes in the circuit: nodes: a, b, c, and d.

A Loop: is any closed path through the circuit in which no node is crossed more than once. For example, "aba", "bcb", "abca", "acda", "bdab", "bcdb", "abcda" are 7 loops.

A Mesh: is any loop that does not contain within it another loop.
Therefore, the first 6 loops are meshes, but the last (abcda) loop is not a mesh.

A Branch: is a portion of the circuit containing a single element and the nodes are at each end of the element; 8 branches are in the circuit.

2.2.1 Kirchhoff's Current Law (KCL)

The algebraic sum of all the currents entering any node in a circuit equals zero.

If the current leaving a node is assumed positive, then the current entering the node is assumed negative, and vice versa.

Assuming that the current leaving a node is positive, and applying KCL to the nodes in the figure above yields:

$$\text{node a} \qquad i_1 + i_4 - i_2 - i_5 = 0$$

$$\text{node b} \qquad i_2 + i_3 - i_1 - i_b - i_a = 0$$

$$\text{node c} \qquad i_b - i_3 - i_4 - i_c = 0$$

$$\text{node d} \qquad i_5 + i_a + i_c = 0$$

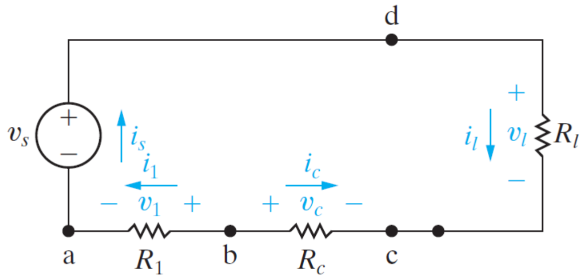
2.2.2 Kirchhoff's Voltage Law (KVL)

The algebraic sum of all the voltages around any closed path in a circuit equals zero.

It is important to keep track of the voltage polarity; either the decrease in voltage is assumed positive and then the increase is assumed negative, or vice versa.

Assuming a reduction in the voltage is positive, for the figure below, yields:

$$v_l - v_c + v_1 - v_s = 0$$



Example # 2:

Sum the voltages around each designated path in the circuit shown in Fig. 2.17.

Solution

In writing the equations, we use a positive sign for a voltage drop. The four equations are

path a $\quad -v_1 + v_2 + v_4 - v_b - v_3 = 0,$

path b $\quad -v_a + v_3 + v_5 = 0,$

path c $\quad v_b - v_4 - v_c - v_6 - v_5 = 0,$

path d $\quad -v_a - v_1 + v_2 - v_c + v_7 - v_d = 0.$

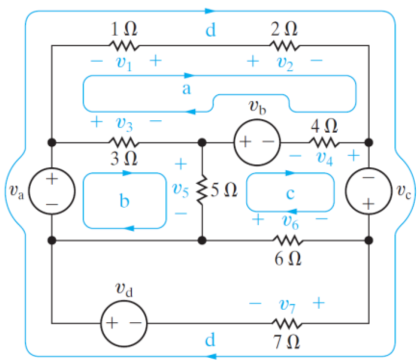


Figure 2.17 ▲ The circuit for Example 2.7.

Example # 3:

a) Use Kirchhoff's laws and Ohm's law to find i_o in the circuit shown in Fig. 2.18.

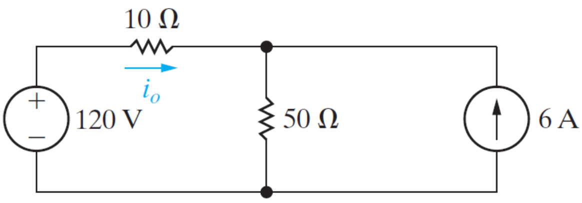


Figure 2.18 ▲ The circuit for Example 2.8.

b) Test the solution for i_o by verifying that the total power generated equals the total power dissipated.

Solution

a) We begin by redrawing the circuit and assigning an unknown current to the 50 Ω resistor and unknown voltages across the 10 Ω and 50 Ω resistors. Figure 2.19 shows the circuit. The nodes are labeled a, b, and c to aid the discussion.

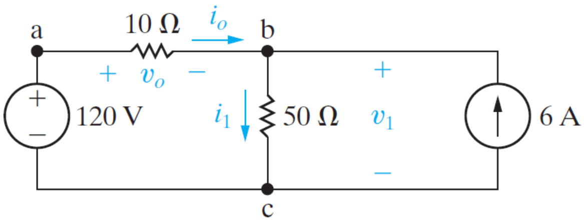


Figure 2.19 ▲ The circuit shown in Fig. 2.18, with the unknowns i_1, v_o, and v_1 defined.

Applying KCl at node "b" yields:

$$i_1 - i_o - 6 = 0 \qquad (1)$$

Applying KVL for the bath "cabc" yields:

$$-120 + 10i_o + 50i_1 = 0 \qquad (2)$$

Solving eqs. (1) and (2) yields:

$$i_o = -3 \text{ A} \qquad \text{and} \qquad i_1 = 3 \text{ A}$$

b) The power dissipated in the 50 Ω resistor is

$$p_{50\Omega} = (3)^2(50) = 450 \text{ W.}$$

The power dissipated in the 10 Ω resistor is

$$p_{10\Omega} = (-3)^2(10) = 90 \text{ W.}$$

The power delivered to the 120 V source is

$$p_{120V} = -120 i_o = -120(-3) = 360 \ \text{W}.$$

The power delivered to the 6 A source is

$$p_{6A} = -v_1(6), \quad \text{but} \quad v_1 = 50 i_1 = 150 \ \text{V}$$

Therefore

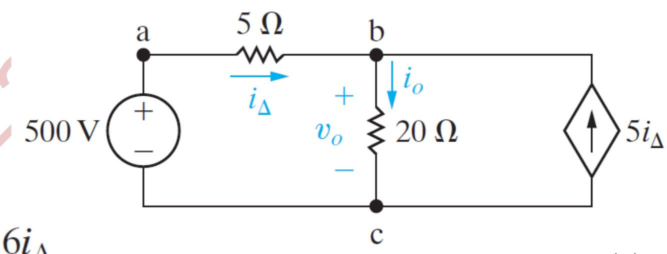
$$p_{6A} = -150(6) = -900 \ \text{W}.$$

The 6 A source is delivering 900 W, and the 120 V source is absorbing 360 W. The total power absorbed is 360 + 450 + 90 = 900 W. Therefore, the solution verifies that the power delivered equals the power absorbed.

Analysis of a Circuit Containing a Dependent Source:

Example # 4:

Find v_o in the circuit shown.



Solution:

Applying KCL at node "b" yields:

$$i_o = i_\Delta + 5i_\Delta = 6i_\Delta \qquad (1)$$

Applying KVL for the left hand loop "cabc" yields:

$$500 = 5i_\Delta + 20 i_o \qquad (2)$$

Solving eqs. (1) and (2) yields:

$$i_\Delta = 4 \ \text{A},$$

$$i_o = 24 \ \text{A}.$$

$$v_o = 20 i_o = 480 \ \text{V}$$

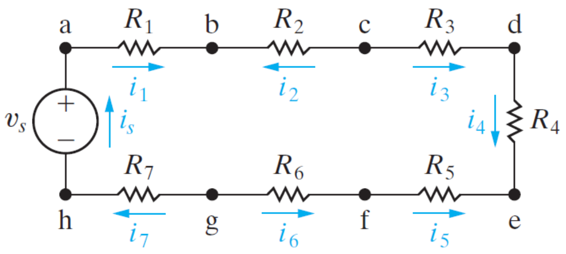
Chapter 3

Simple Resistive Circuits

3.1 Resistors in Series

They carry the same current (by applying KCL).

An example of a simple loop circuit is shown in the figure next.



Applying KCL at each node yields:

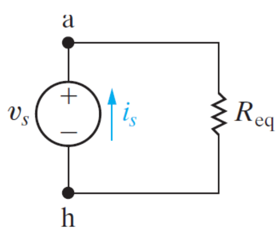
$$i_s = i_1 = -i_2 = i_3 = i_4 = -i_5 = -i_6 = i_7,$$

Applying KVL to the loop yields:

$$-v_s + i_sR_1 + i_sR_2 + i_sR_3 + i_sR_4 + i_sR_5 + i_sR_6 + i_sR_7 = 0,$$

or $$v_s = i_s(R_1 + R_2 + R_3 + R_4 + R_5 + R_6 + R_7).$$

The seven resistors can be replaced by a single resistor equivalent to all, as shown in the equivalent circuit of the figure next, such that;



$$R_{eq} = R_1 + R_2 + R_3 + R_4 + R_5 + R_6 + R_7$$

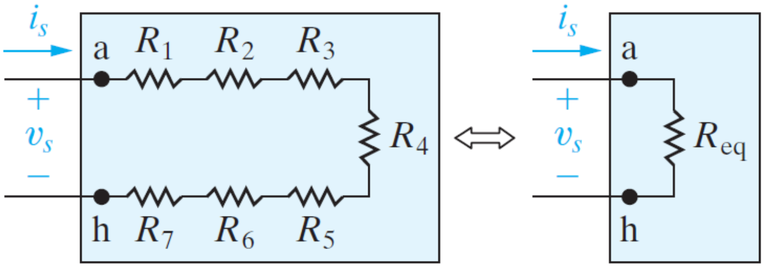
and

$$v_s = i_sR_{eq}$$

In general, if k resistors are connected in series, the equivalent single resistor has a resistance equal to the sum of the k resistances, or

$$R_{eq} = \sum_{i=1}^{k} R_i = R_1 + R_2 + \cdots + R_k$$

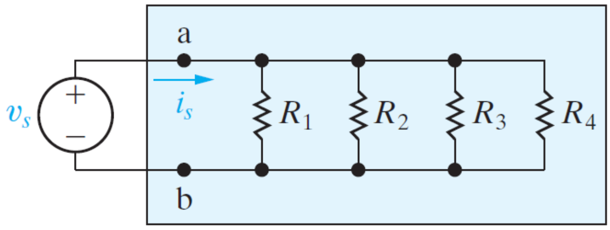
Note that, the resistance of the equivalent resistor is always larger than that of the largest resistor in the series connection.



In other words, several voltage sources in series can be replaced by one source, whose value is the algebraic sum of individual sources. Furthermore, the equivalent resistance of any number of resistors in series is the sum of individual resistances.

3.2 Resistors in Parallel

When two (or more) elements are connected to the same single node pair, they are said to be in parallel; they have the same voltage.



i.e., *Parallel-connected circuit elements have the same voltage across their terminals.*

Applying KCL at the upper node, node "a", yields:

$$i_s = i_1 + i_2 + i_3 + i_4$$

where, the currents i_1, i_2, i_3, and i_4 are the currents in the resistors R_1 through R_4, respectively.

But,

$$i_1R_1 = i_2R_2 = i_3R_3 = i_4R_4 = v_s$$

Therefore, each current can be expressed in terms of the source voltage and respective resistor as:

$$i_1 = \frac{v_s}{R_1}, \quad i_2 = \frac{v_s}{R_2}, \quad i_3 = \frac{v_s}{R_3}, \quad \text{and} \quad i_4 = \frac{v_s}{R_4}.$$

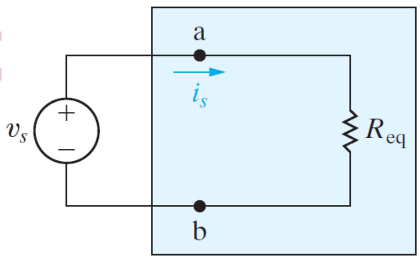
Substituting these current in the KCL equation ($i_s = i_1 + i_2 + i_3 + i_4$), yields:

$$i_s = v_s\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4}\right)$$

Rearranging yields:

$$\frac{i_s}{v_s} = \frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4}.$$

Thus, the four resistors in the previous circuit can be replaced by an equivalent resistor, as shown in the figure next, such that, this equivalent resistor draws the same current from the source and has the source voltage across it.



For "k" resistors connected in parallel, the equivalent resistor is:

$$\frac{1}{R_{eq}} = \sum_{i=1}^{k} \frac{1}{R_i} = \frac{1}{R_1} + \frac{1}{R_2} + \cdots + \frac{1}{R_k}$$

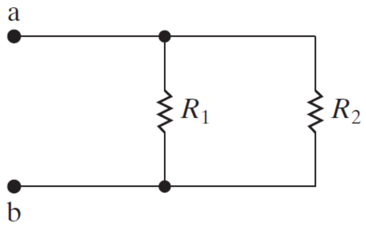
In terms of conductance:

$$G_{eq} = \sum_{i=1}^{k} G_i = G_1 + G_2 + \cdots + G_k$$

Note that, since the equivalent conductance is higher than the individual conductance of any resistor, then the resistance of the equivalent resistor is always **smaller** than the resistance of the smallest resistor in the parallel connection.

Two Parallel Resistors

The equivalent resistor is:



$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} = \frac{R_2 + R_1}{R_1 R_2}$$

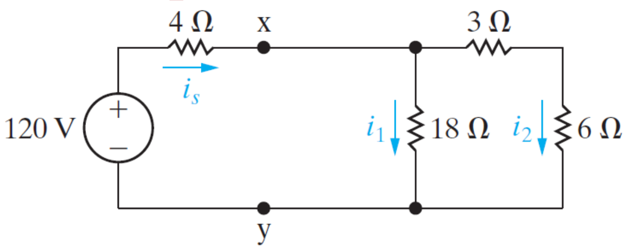
or

$$R_{eq} = \frac{R_1 R_2}{R_1 + R_2}$$

For two resistors in parallel, the equivalent resistance equals the product of the resistances divided by the sum of the resistances.

Example # 1:

Find i_s, i_1, and i_2, in the circuit shown in the Figure next, by simplifying the circuit using series-parallel reductions.

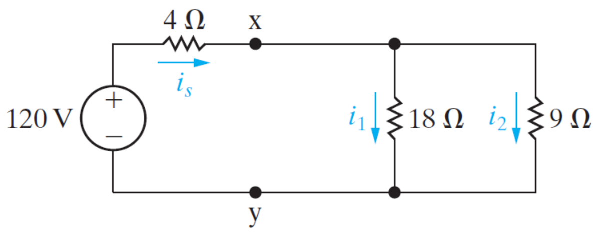


Solution:

The resistor 3Ω is in series with 6Ω resistor;

$$R_{eq1} = 3 + 6 = 9\Omega$$

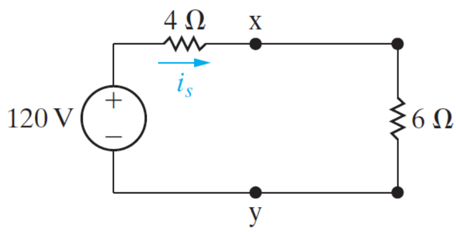
This equivalent resistor, 9Ω, is in parallel with 18Ω resistor;



$$R_{eq2} = \frac{18(9)}{18+9} = 6\Omega$$

The equivalent resistor, 6Ω, is in series with 4Ω resistor;

$$R_{eq3} = 4 + 6 = 10\Omega$$



The source current, i_s, is:

$$i_s = \frac{v_s}{R_{eq3}} = \frac{120}{10} = 12A$$

The voltage between x-y is the same as the voltage across the parallel branches, and is:

$$v_{xy} = R_{eq2} i_s = 6(12) = 72V$$

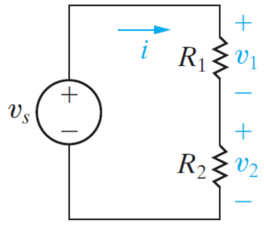
Therefore, $i_1 = \frac{v_{xy}}{R_1} = \frac{72}{18} = 4A$

$$i_2 = \frac{v_{xy}}{R_{eq1}} = \frac{72}{9} = 8A$$

Note that, $i_s = i_1 + i_2$ (KCL)!

3.3 Voltage Divider Circuit

Applying KVL for the circuit shown in the figure yields:



$$v_s = iR_1 + iR_2.$$

Thus,
$$i = \frac{v_s}{R_1 + R_2}$$

The voltages v_1 and v_2 can be found by applying Ohm's law;

$$v_1 = iR_1 = v_s \frac{R_1}{R_1 + R_2}$$

and

$$v_2 = iR_2 = v_s \frac{R_2}{R_1 + R_2}$$

v_1 and v_2 are functions of v_s.

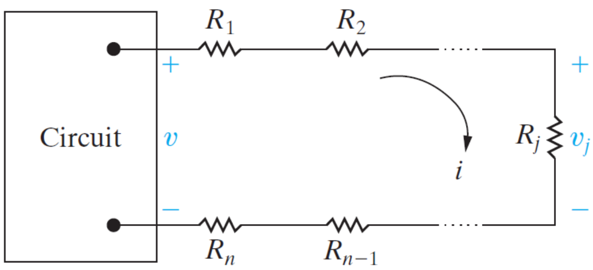
v_s *is divided between the resistors R_1 and R_2 in a direct proportion to their resistances.*

For "n" series resistors, the voltage division is as follows:

The current i is:

$$i = \frac{v}{R_1 + R_2 + \cdots + R_n} = \frac{v}{R_{eq}}$$

where R_{eq} is the sum of all resistors!

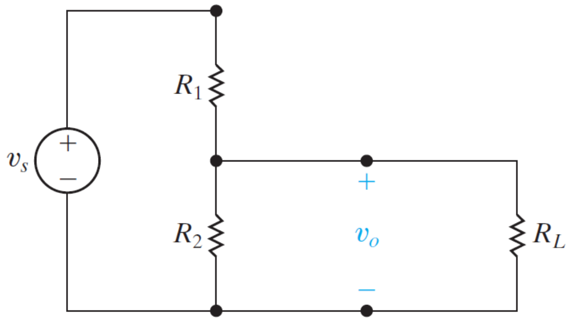


The voltage drop v_j across the resistor R_j is:

$$v_j = iR_j = \frac{R_j}{R_{eq}}v$$

Loading Effect on the Potential Divider

Consider the circuit shown in the figure next.



The output voltage, v_o, across the load, is:

$$v_o = \frac{R_{eq}}{R_1 + R_{eq}}v_s$$

where

$$R_{eq} = \frac{R_2 R_L}{R_2 + R_L}$$

Thus,

$$v_o = \frac{R_2}{R_1[1 + (R_2/R_L)] + R_2}v_s$$

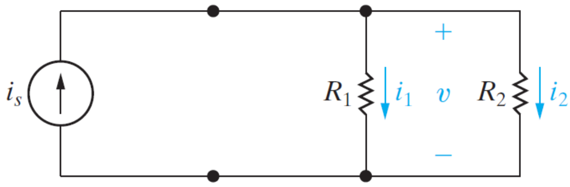
Note that, as $R_L \to \infty$, then $v_o = \frac{R_2}{R_1+R_2}v_s$!

As long as $R_L \gg R_2$, then the voltage ratio $\frac{v_o}{v_s}$ is undisturbed by the addition of the load on the divider.

In other words, R_L should be much larger than R_2 to avoid loading effect!

3.4 Current Divider Circuit

Consider the circuit shown in the figure next



The voltage across the parallel resistors is:

$$v = i_1 R_1 = i_2 R_2 = \frac{R_1 R_2}{R_1 + R_2} i_s$$

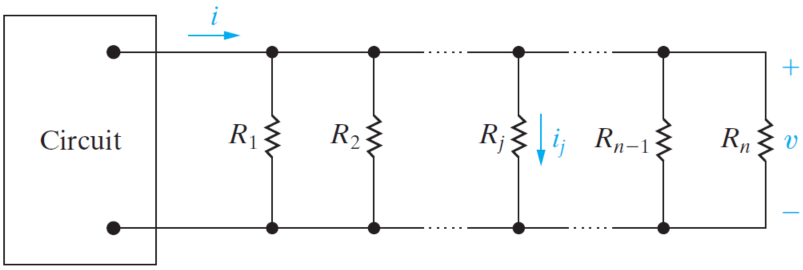
Rearranging for the branch currents yields:

$$i_1 = \frac{R_2}{R_1 + R_2} i_s \quad \text{and} \quad i_2 = \frac{R_1}{R_1 + R_2} i_s$$

These relationships are valid only for two parallel resistors.

The current in one resistor is directly proportional to the value of the other parallel resistor.

For "n" parallel resistors, the current division is as follows:



The voltage *v* is:

$$v = i(R_1 \| R_2 \| \ldots \| R_n) = iR_{eq}$$

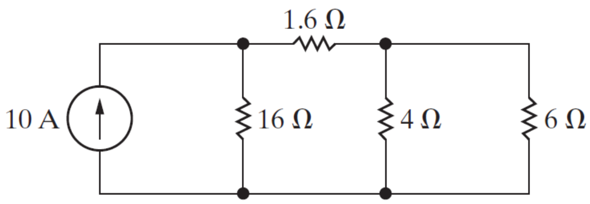
where R_{eq} is the equivalent resistor of all resistors!

The current i_j through the resistor R_j is:

$$i_j = \frac{v}{R_j} = \frac{R_{eq}}{R_j} i$$

Example # 2:

Find the power dissipated in the 6Ω resistor in the circuit shown in the figure below, by simplifying the circuit using series-parallel reductions.



Solution:

The resistors 4Ω //6Ω

Thus, $R_{eq1} = \frac{4(6)}{4+6} = 2.4\Omega$

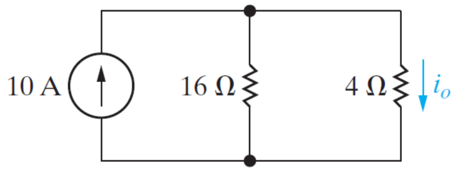
and 2.4Ω is in series with 1.6Ω,

Therefore,

$$R_{eq2} = 2..4 + 1.6 = 4\Omega$$

The equivalent circuit will be as shown next.

The current in 4Ω of the simplified circuit is found by current divider;



$$i_o = \frac{R_1}{R_1 + R_{eq2}} i_s$$

$$i_o = \frac{16}{16+4} 10 = 8A$$

Thus, the current in 6Ω of the original circuit is found by current divider of i_o between 4Ω & 6Ω;

$$i_6 = \frac{4}{4+6} i_o$$

Thus,

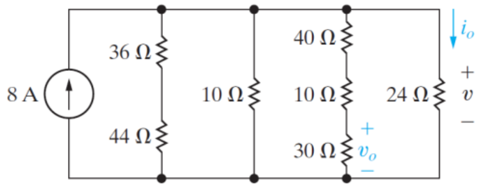
$$i_6 = \frac{4}{4+6} 8 = 3.2A$$

The power dissipated in the 6Ω resistor is:

$$p = Ri^2 = 6(3.2)^2 = \mathbf{61.44W}$$

Example # 3:

Use current division to find the current i_o and use voltage division to find the voltage v_o for the circuit



Solution:

The equivalent resistor of all parallel branches is:

$$R_{eq} = (36 + 44)\|10\|(40 + 10 + 30)\|24$$

$$= 80\|10\|80\|24 = \frac{1}{\frac{1}{80} + \frac{1}{10} + \frac{1}{80} + \frac{1}{24}} = 6\ \Omega$$

The output current i_o is:

$$i_o = \frac{v}{24}$$

where v is the voltage across the parallel branches such that:

$$v = R_{eq} i_s$$

$$v = 6(8) = 48V$$

Therefore,

$$i_o = \frac{6}{24}(8\ \text{A}) = 2\ \text{A}$$

and the output voltage, v_o, can be obtained by voltage (potential) divider as:

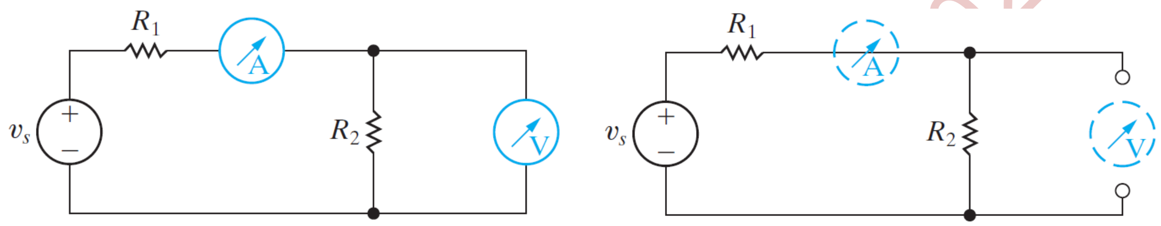
$$v_o = \frac{30}{80}(48\ \text{V}) = 18\ \text{V}$$

3.5 Measuring Voltage and Current

- **An ammeter** is an instrument designed to measure current; it is placed in series with the circuit element whose current is being measured. Ideally, it has zero equivalent resistance!

- **A voltmeter** is an instrument designed to measure voltage; it is placed in parallel with the element whose voltage is being measured. Ideally, it has infinite equivalent resistance!

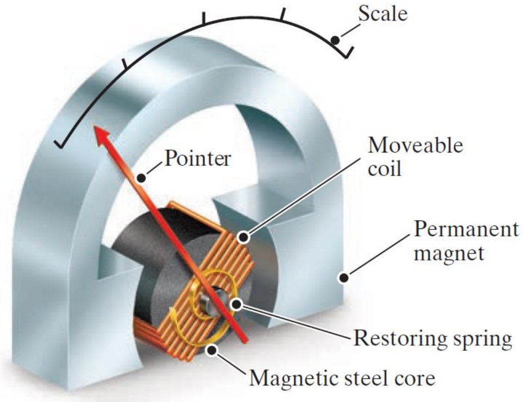
- Note that, an ideal ammeter or voltmeter has no effect on the circuit variable it is designed to measure.



- These meters are classified in two categories: Analog and digital

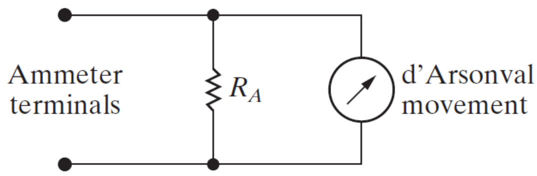
3.5.1 Analog Meters

- They are based on the d'Arsonval meter movement, which implements the readout mechanism.
  - A d'Arsonval meter movement consists of a movable coil placed in the field of a permanent magnet. When current flows in the coil, it creates a torque on the coil, causing it to rotate and move a pointer across a calibrated scale.



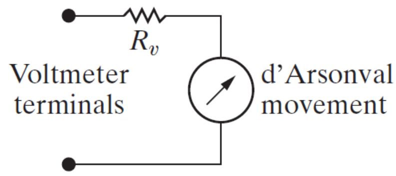
- By design, the deflection of the pointer is directly proportional to the current in the movable coil.
- The coil is characterized by both a voltage rating and a current rating. For example, one commercially available meter movement is rated at 50 mV and 1 mA. This means that when the coil is carrying 1 mA, the voltage drop across the coil is 50 mV and the pointer is deflected to its full-scale position.

- An analog DC ammeter has a parallel resistor, R_A, to limit the amount of current through the coil.



- Note that, a real ammeter has an equivalent resistance that is not zero, and it thus effectively adds resistance to the circuit in series with the element whose current the ammeter is reading.
- The effective resistance of an ammeter should be no more than $^1/_{10^{th}}$ of the value of the smallest resistance in the circuit.

- An analog DC voltmeter has a series resistor, R_v, to limit the amount of voltage across the coil.
- Note that, a real voltmeter has an equivalent resistance that is not infinite, so it effectively adds resistance to the circuit in parallel with the element whose voltage is being read.



3.5.2 Digital Meters

- They measure a continuous voltage or current at discrete points of time, called sampling time. It converts an analog signal to a digital signal.
- They have several advantages over analog meters:
    1. They introduce less effect on the resistance of the circuit.
    2. They are easier to connect.
    3. They are more precise due to the nature of the readout mechanism.

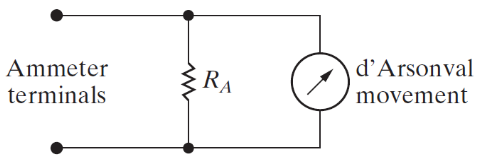
Example # 4:

a) A 50 mV, 1 mA d'Arsonval movement is to be used in an ammeter with a full-scale reading of 10 mA. Determine R_A.

b) Repeat (a) for a full-scale reading of 1 A.

c) How much resistance is added to the circuit when the 10 mA ammeter is inserted to measure current?

d) Repeat (c) for the 1 A ammeter.



Solution:

a) By Ohm's law, the coil resistance is $R_c = \frac{50m}{1m} = 50\Omega$

Since the meter is to measure 10mA, then 9mA must pass through the parallel resistor, R_A. The voltage across the meter at 10mA will be 50mV. Therefore, applying Ohm's law

$$(9mA)R_A = 50mV \rightarrow R_A = \frac{50}{9} = 5.555\Omega$$

b) Since the meter is to measure 10A, then 0.999A must pass through the parallel resistor, R_A. The voltage across the meter at 1A will be 50mV. Therefore, applying Ohm's law

$$(0.999A)R_A = 50mV \rightarrow R_A = \frac{50m}{0.999} = 50.55m\Omega$$

c) Let R_m represent the equivalent resistance of the ammeter. For the 10 mA ammeter,

$$R_m = \frac{50\text{ mV}}{10\text{ mA}} = 5\ \Omega,$$

or, alternatively,

$$R_m = \frac{(50)(50/9)}{50 + (50/9)} = 5\ \Omega.$$

d) For the 1 A ammeter

$$R_m = \frac{50\text{ mV}}{1\text{ A}} = 0.050\ \Omega,$$

or, alternatively,

$$R_m = \frac{(50)(50/999)}{50 + (50/999)} = 0.050\ \Omega.$$

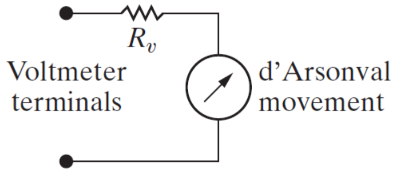
Example # 5:

a) A 50 mV, 1 mA d'Arsonval movement is to be used in a voltmeter in which the full-scale reading is 150 V. Determine R_v.

b) Repeat (a) for a full-scale reading of 5 V.

c) How much resistance does the 150 V meter insert into the circuit?

d) Repeat (c) for the 5 V meter.



Solution:

a) The current through the meter at 150V will be 1mA. Since the meter is to measure 150V, then 50mV appears across the coils resistance (50Ω) and 149.950V appears across the series resistor, R_v. Therefore, applying potential divider yields:

$$V_c = \frac{R_c}{R_c+R_v}V_{measured} \rightarrow 50m = \frac{50}{50+R_v}150 \rightarrow R_v = 149.95k\Omega$$

b) Since the meter is to measure 5V, then 50mV appears across the coils resistance (50Ω) and 4.950V appears across the series resistor, R_v. Therefore, applying potential divider yields:

$$V_c = \frac{R_c}{R_c+R_v}V_{measured} \rightarrow 50m = \frac{50}{50+R_v}150 \rightarrow R_v = 4.95k\Omega$$

c) The resistance of the meter R_m is:

$$R_m = \frac{150 \text{ V}}{10^{-3} \text{ A}} = 150{,}000\ \Omega,$$

or, alternatively,

$$R_m = 149{,}950 + 50 = 150{,}000\ \Omega.$$

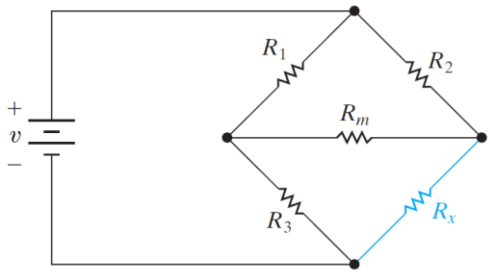
d) Then,

$$R_m = \frac{5 \text{ V}}{10^{-3} \text{ A}} = 5000\ \Omega,$$

or, alternatively,

$$R_m = 4950 + 50 = 5000\ \Omega.$$

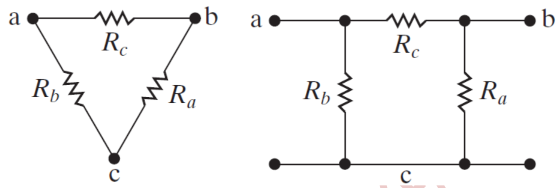
3.6 Delta-to-Wye (Pi-to-Tee) Equivalent Circuits



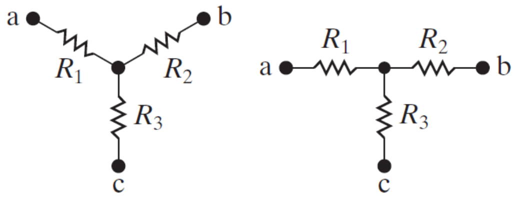
In the opposite circuit, there is no series or parallel connection.

However, the equivalent can be found by implementing $\Delta - Y$ transformation.

The Delta configuration can be viewed as π!



The Y (WYE)- or T- connection is shown in the Figure next.



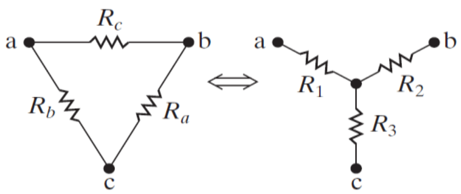
A Delta interconnection is equivalent to a WYE interconnection if the resistance between the corresponding terminal pairs is the same for each connection (*third terminal is open circuit*).

The resistance between each pair of terminals is:

$$R_{ab} = \frac{R_c(R_a + R_b)}{R_a + R_b + R_c} = R_1 + R_2,$$

$$R_{bc} = \frac{R_a(R_b + R_c)}{R_a + R_b + R_c} = R_2 + R_3,$$

$$R_{ca} = \frac{R_b(R_c + R_a)}{R_a + R_b + R_c} = R_1 + R_3.$$



Solving the above three equations yields:

1) $\Delta \rightarrow$ **Y**

$$R_1 = \frac{R_b R_c}{R_a + R_b + R_c}$$

$$R_2 = \frac{R_c R_a}{R_a + R_b + R_c}$$

$$R_3 = \frac{R_a R_b}{R_a + R_b + R_c}$$

2) Y→ Δ

$$R_a = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_1}$$

$$R_b = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_2}$$

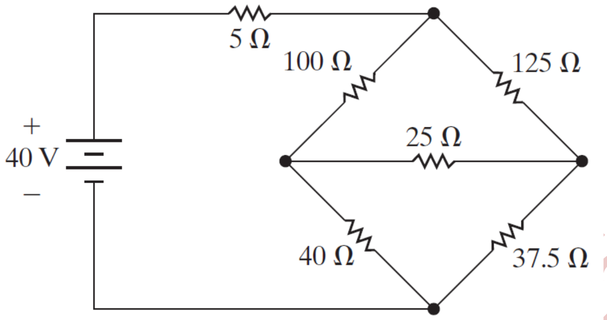
$$R_c = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_3}$$

If $R_1 = R_2 = R_3 \leftrightarrow R_a = R_b = R_c$ then:

$\boldsymbol{R_Y = \frac{R_\Delta}{3}}$ and $\boldsymbol{R_\Delta = 3R_Y}$

Example # 6:

Find the current and power supplied by the 40 V source in the circuit shown using Δ → Y.



Solution:

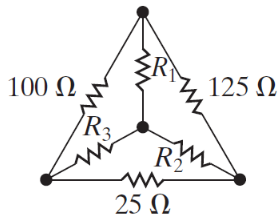
Either replace either the upper Δ (100, 125, 25Ω) or the lower Δ (40, 25, 37.5Ω) with its equivalent Y.

Converting the upper Δ (100, 125, 25Ω) to an equivalent Y, yields:

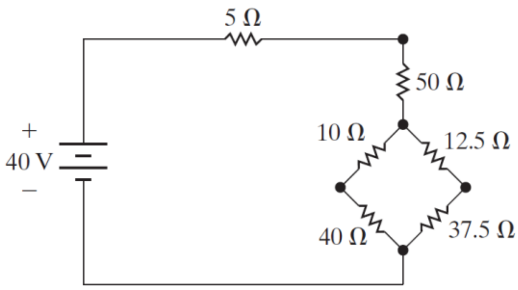
$$R_1 = \frac{100 \times 125}{250} = 50\ \Omega,$$

$$R_2 = \frac{125 \times 25}{250} = 12.5\ \Omega,$$

$$R_3 = \frac{100 \times 25}{250} = 10\ \Omega.$$



The resulting circuit is as show in the figure next:

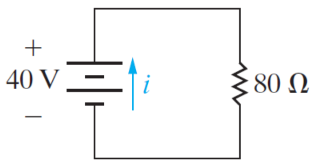


The resistance across the terminals of the 40 V source can be found by series-parallel simplifications:

$$R_{eq} = 55 + \frac{(50)(50)}{100} = 80\ \Omega$$

An 80Ω resistor across a 40V source, therefore the current is:

$$i = \frac{v}{R_{eq}} = \frac{40}{80} = 0.5A$$



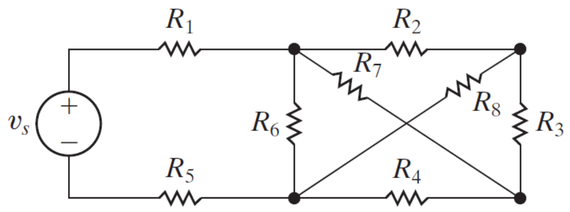
The source delivers a power of:

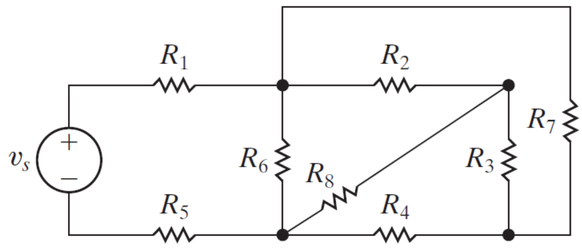
$$p = vi = 40(0.5) = 20W$$

Chapter 4

Techniques of Circuits' Analysis

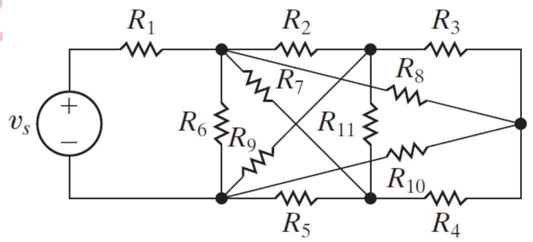
A planar circuit: is a circuit that can be redrawn on a plane with no crossing branches.





A nonplanar circuit: is a circuit that cannot be redrawn in such a way that all nodes' connections are maintained and no branches overlap.

An example of **nonplanar** circuit is shown in the figure next.



Notes: 1) Node-voltage method is applicable to both planar and nonplanar circuits

2) Whereas the mesh current method is applicable to planar circuits only

An essential node (n_e): is a node where three or more circuit elements join.

An essential branch (b_e): is a path which connects two essential nodes without passing through an essential node.

Notes:

1- The number of independent equations obtained by using KCL equals $(n_e - 1)$

2- The number of independent equations obtained by using KVL equals $(b_e - (n_e - 1))$

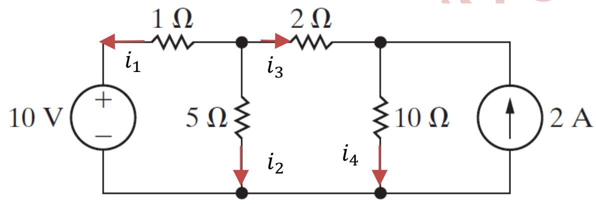
4.1 Node-Voltage Method (Nodal Analysis)

- It relies on finding the node voltages and applies KCL at the essential nodes.
- The first step is choosing a reference node, which is assumed to have a zero voltage; the node which is connected to most branches.
- Find the node voltages $(n_e - 1)$; the voltage rise from the reference node to the nonreference node.

4.1.1 Node-Voltage and Independent Sources

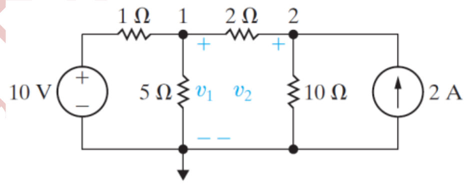
Example # 1

Find the branch currents using nodal analysis for the circuit shown in the figure next.



Solution:

- There are 3 essential nodes. Thus the number of needed equations is $(n_e - 1) = 2$
- Select the bottom node as the reference node.
- Name the upper nodes' voltages; v_1 & v_2
- Apply KCL to the upper nodes;



<u>KCL at node 1</u>

$$\frac{v_1 - 10}{1} + \frac{v_1}{5} + \frac{v_1 - v_2}{2} = 0$$

$$10v_1 - 100 + 2v_1 + 5v_1 - 5v_2 = 0$$

$$\mathbf{17v_1 - 5v_2 = 100} \qquad (1)$$

<u>KCL at node 2</u>

$$\frac{v_2 - v_1}{2} + \frac{v_2}{10} - 2 = 0$$

$$5v_2 - 5v_1 + v_2 - 20 = 0$$

$$\mathbf{-5v_1 + 6v_2 = 20} \qquad (2)$$

Solving eqs. (1) and (2) yields:

$$v_1 = \frac{100}{11} = 9.09 \text{ V}$$

$$v_2 = \frac{120}{11} = 10.91 \text{ V}$$

Now all the branches' currents can be found!

$$i_1 = \frac{v_1 - 10}{1} = \frac{9.91 - 10}{1} = -0.91A$$

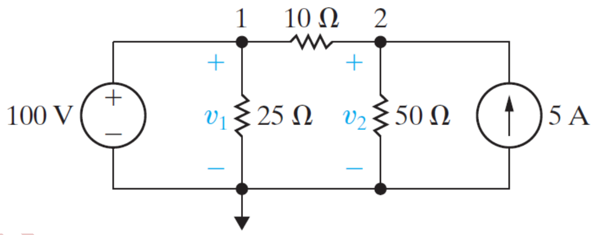
$$i_2 = \frac{v_1}{5} = \frac{9.91}{5} = 1.818A$$

$$i_3 = \frac{v_1 - v_2}{2} = \frac{9.09 - 10.91}{2} = -0.91A$$

$$i_4 = \frac{v_2}{10} = \frac{10.91}{10} = 1.091A$$

Note: *When there are voltage sources connected directly between essential nodes, the number of unknown node voltages is reduced, that is because the voltage sources constrain the difference between the node voltages at these nodes to equal the voltage of the source.*

Example # 2: Find the node voltages in the circuit shown in the following figure.



Solution:

- There are 3 essential nodes.
- Take the bottom node as a reference node with 0V.
- The number of needed equations is 3-1 =2 equations.
- But, since the 100V is connected between two essential nodes, then $v_1 = 100V$.
- Therefore, only one node voltage is unknown, which is v_2
- Applying KCL at node 2 yields:

$$\frac{v_2 - v_1}{10} + \frac{v_2}{50} - 5 = 0$$

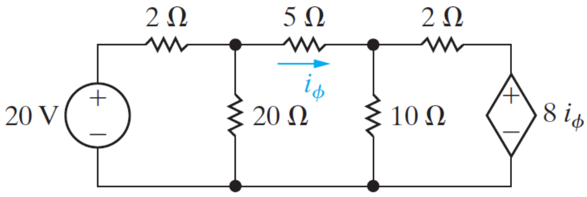
Substituting for $v_1 = 100V$ in the above equation and solving for v_2 yield:

$$v_2 = 125 \text{ V}$$

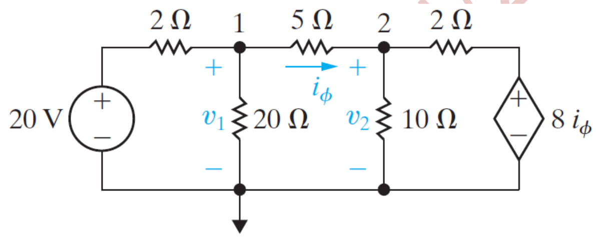
The branches' currents can now be found!

4.1.2 Node-Voltage Method with Dependent Sources

Example # 3: Use the node-voltage method to find the power dissipated in the 5Ω resistor in the circuit shown.



- There are 3 essential nodes. Thus the number of needed equations is: $(n_e - 1) = 2$
- Select the bottom node as the reference node.
- Name the upper nodes' voltages; v_1 & v_2
- Apply KCL to the upper nodes;



<u>KCL at node 1</u>

$$\frac{v_1 - 20}{2} + \frac{v_1}{20} + \frac{v_1 - v_2}{5} = 0$$

$$\mathbf{15v_1 - 4v_2 = 200} \qquad (1)$$

<u>KCL at node 2</u>

$$\frac{v_2 - v_1}{5} + \frac{v_2}{10} + \frac{v_2 - 8i_\phi}{2} = 0$$

$$2v_2 - 2v_1 + v_2 + 5v_2 - 40i_\emptyset = 0 \qquad (2)$$

But, $i_\emptyset = \frac{v_1 - v_2}{5}$

Substituting for $i_\emptyset$ in eq. (2) yields:

$$\mathbf{-v_1 + 1.6v_2 = 0} \qquad (3)$$

Solving eqs. (1) and (3) yields:

$v_1 = 16V$ and $v_2 = 10V$

$$i_\emptyset = \frac{v_1 - v_2}{5} = \frac{16-10}{5} = 1.2A$$

and $P_{5\Omega} = Ri_\emptyset{}^2 = 5(1.2)^2 = \mathbf{7.2W}$

Cramer's Rule

It is used to solve a set on linear equations.

The k^th unknown x_k is:

$$x_k = \frac{N_k}{\Delta}$$

where Δ is the characteristic determinant, and N_k is the numerator determinant for the k^th unknown.

For example: Assume that, a set of 3 currents equations is:

$$21i_1 - 9i_2 - 12i_3 = -33,$$

$$-3i_1 + 6i_2 - 2i_3 = 3,$$

$$-8i_1 - 4i_2 + 22i_3 = 50.$$

The first step is to determine the characteristic determinant, Δ, as:

$$\Delta = \begin{vmatrix} 21 & -9 & -12 \\ -3 & 6 & -2 \\ -8 & -4 & 22 \end{vmatrix}$$

$$\Delta = 21(1)\begin{vmatrix} 6 & -2 \\ -4 & 22 \end{vmatrix} - 3(-1)\begin{vmatrix} -9 & -12 \\ -4 & 22 \end{vmatrix} - 8(1)\begin{vmatrix} -9 & -12 \\ 6 & -2 \end{vmatrix}$$

$$\Delta = 21(132 - 8) + 3(-198 - 48) - 8(18 + 72)$$

$$= 2604 - 738 - 720 = 1146.$$

The second step is forming the numerator determinant N_k from the characteristic determinant by replacing the k^th column in the characteristic determinant with the column of values appearing on the right-hand side of the equations.

$$N_1 = \begin{vmatrix} -33 & -9 & -12 \\ 3 & 6 & -2 \\ 50 & -4 & 22 \end{vmatrix}$$

$$N_1 = 1146,$$

$$N_2 = \begin{vmatrix} 21 & -33 & -12 \\ -3 & 3 & -2 \\ -8 & 50 & 22 \end{vmatrix}$$

$$N_2 = 2292,$$

$$N_3 = \begin{vmatrix} 21 & -9 & -33 \\ -3 & 6 & 3 \\ -8 & -4 & 50 \end{vmatrix}$$

$$N_3 = 3438.$$

Thus, the solutions for the currents are:

$$i_1 = \frac{N_1}{\Delta} = 1\text{ A},$$

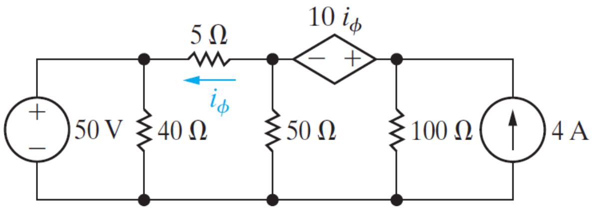
$$i_2 = \frac{N_2}{\Delta} = 2\text{ A},$$

And

$$i_3 = \frac{N_3}{\Delta} = 3\text{ A}.$$

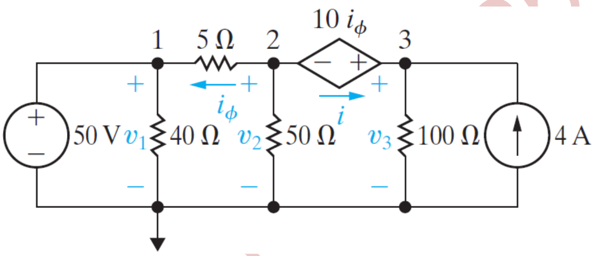
4.1.3 Node-Voltage Method and Supernode

Example # 4: Find the node voltages in the circuit shown in the figure below.



Solution:

- Assigning node voltages, and the current in the dependent source, i, as it cannot be expressed in terms of node voltages, yield the figure next.



- Note that, $v_1 = 50V$
- Choose the bottom node as the reference node (0V)
- Applying KCL at node 2 yields:

$$\frac{v_2 - v_1}{5} + \frac{v_2}{50} + i = 0$$

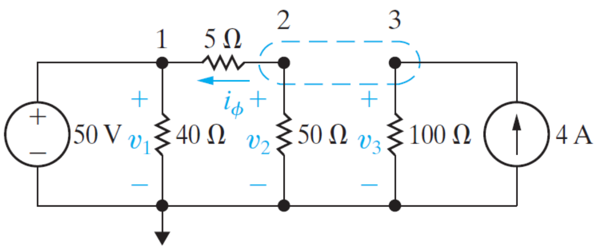
- Applying KCL at node 3 yields:

$$\frac{v_3}{100} - i - 4 = 0$$

- Adding the latter two equations to eliminate i yields:

$$\frac{v_2 - v_1}{5} + \frac{v_2}{50} + \frac{v_3}{100} - 4 = 0$$

The last equation can be obtained directly if the two essential nodes, including the dependent voltage source, were combined in a one node called **supernode**, as illustrated in the figure next, and then apply KCL at the super node as:



$$\frac{v_2 - v_1}{5} + \frac{v_2}{50} + \frac{v_3}{100} - 4 = 0 \qquad (1)$$

***Supernode:** is a node combining two essential nodes with a voltage source between them.*

Another equation can be obtained from the dependent voltage source constraint equation as:

$$v_3 = v_2 + 10i_\phi \tag{2}$$

where $$i_\phi = \frac{v_2 - 50}{5} \tag{3}$$

Substituting eq. (3) in (2) yields a formula for v_3 , then substituting for v_3 & $v_1 = 50V$ in eq. (1) yields:

$$v_2\left(\frac{1}{50} + \frac{1}{5} + \frac{1}{100} + \frac{10}{500}\right) = 10 + 4 + 1$$

$$v_2(0.25) = 15,$$

$$v_2 = 60 \text{ V}$$

From eq. (3), $$i_\phi = \frac{60 - 50}{5} = 2 \text{ A}$$

and from eq. (2), $$v_3 = 60 + 20 = 80 \text{ V}$$

4.2 Mesh-Current Method

- **A Mesh current:** is the current that exists in the perimeter of a mesh, not necessarily the branch current.
- *A **Mesh** is a loop with no other loops inside it.*
- It is applicable to planar circuits only.
- It describes the circuit by $b_e - (n_e - 1)$ equations,

  where, b_e is the number of essential branches with unknown currents, and n_e is the number of essential nodes
- Assign mesh currents on the circuit
- ***Apply KVL for each mesh***

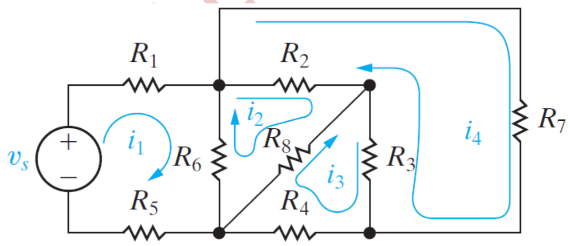
For example, for the circuit shown in the figure next,

$$n_e = 4$$

$$b_e = 7$$

$$\# \; of \; equations = b_e - (n_e - 1) = 4 \; eqs.$$

The 4 meshes are marked on the figure.

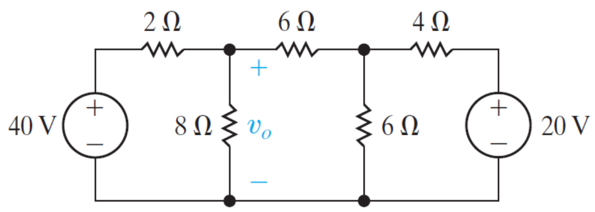


4.2.1 Mesh Analysis with Independent Sources

Example # 5: For the circuit shown in the figure, use the mesh-current method to:

a) Determine the power associated with each voltage source

b) Calculate the voltage, v_o, across the resistor



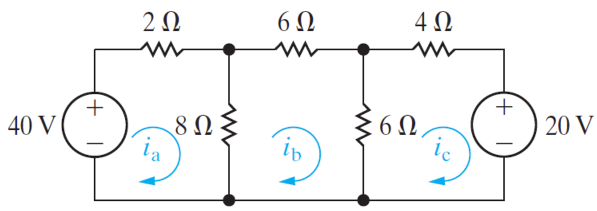
Solution:

$$n_e = 3$$

$$b_e = 5$$

$$\# \; of \; equations = b_e - (n_e - 1)$$

$$= 5 - (3 - 1) = 3$$



Apply KVL for mesh a:

$$-40 + 2i_a + 8(i_a - i_b) = 0$$

Apply KVL for mesh b:

$$8(i_b - i_a) + 6i_b + 6(i_b - i_c) = 0$$

Apply KVL for mesh c:

$$6(i_c - i_b) + 4i_c + 20 = 0$$

Rearranging the latter 3 equations yields:

$$10i_a - 8i_b + 0i_c = 40;$$

$$-8i_a + 20i_b - 6i_c = 0;$$

$$0i_a - 6i_b + 10i_c = -20.$$

Solving these equations yield:

$$i_a = 5.6 \text{ A},$$

$$i_b = 2.0 \text{ A},$$

$$i_c = -0.80 \text{ A}.$$

a) The current in the 40V-source is the same as the mesh current "i_a".

The power consumed by the 40V-source is:

$$p_{40V} = -40i_a = -224 \text{ W}$$

The ***minus sign of the power*** means that this source is delivering (supplying) power to the network.

The current in the 20V-source is identical to the mesh current "i_c", therefore, the power consumed by 20V-source is:

$$p_{20V} = 20i_c = -16 \text{ W}.$$

The ***minus sign of the power*** means that this source also is delivering (supplying) power to the network.

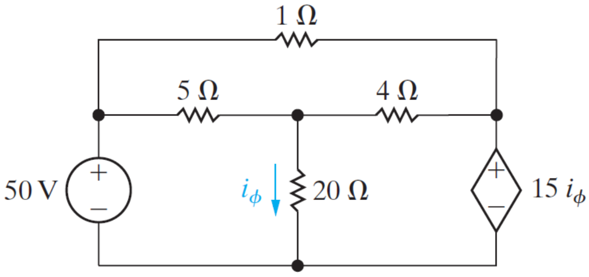
b) The branch current in the 8Ω resistor in the direction of the voltage drop, v_o, is $(i_a - i_b)$.

Therefore,

$$v_o = 8(i_a - i_b) = 8(3.6) = 28.8 \text{ V}$$

4.2.2 Mesh-Current Method with Dependent Sources

Example # 6: Use the mesh-current method of circuit analysis to determine the power dissipated in the resistor 4Ω in the circuit shown in the figure.



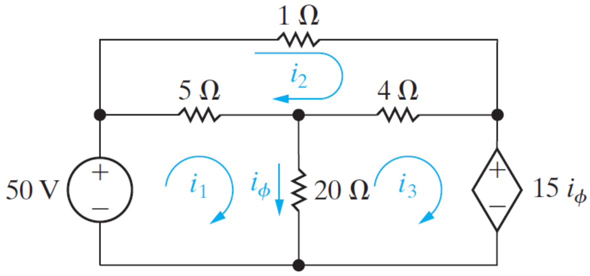
Solution:

$n_e = 4$

$b_e = 6$

$\#\ of\ equations = b_e - (n_e - 1)$

$= 6 - (4 - 1) = 3$



Apply KVL for mesh 1:

$$50 = 5(i_1 - i_2) + 20(i_1 - i_3), \qquad (1)$$

Apply KVL for mesh 2:

$$0 = 5(i_2 - i_1) + 1i_2 + 4(i_2 - i_3), \qquad (2)$$

Apply KVL for mesh 3:

$$0 = 20(i_3 - i_1) + 4(i_3 - i_2) + 15i_\phi \qquad (3)$$

But the branch current $i_\emptyset$ can be expressed as:

$$i_\phi = i_1 - i_3$$

Substituting for $i_\emptyset$ in eq. (3) yields:

$$0 = -5i_1 - 4i_2 + 9i_3. \qquad (3')$$

Rearranging eqs. (1), (2) and (3') yields:

$$50 = 25i_1 - 5i_2 - 20i_3,$$

$$0 = -5i_1 + 10i_2 - 4i_3,$$

$$0 = -5i_1 - 4i_2 + 9i_3.$$

Since the question is concerned with the power in the 4Ω resistor it is enough to find i_2 & i_3;

$$i_2 = 26 \text{ A},$$

$$i_3 = 28 \text{ A}.$$

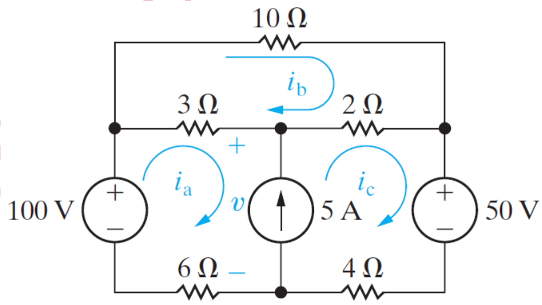
and

$$p_{4\Omega} = (i_3 - i_2)^2(4) = (2)^2(4) = 16 \text{ W}.$$

Note that, if the node voltage method was used, it reduces the problem to finding one unknown node voltage because of the presence of two voltage sources between essential nodes.

4.2.3 Mesh-Current Method and Supermesh

Example # 7: For the circuit shown in the figure next find the mesh currents.



Solution:

$n_e = 4$

There are 6 essential branches, but the unknown branch currents are 5; so $b_e = 5$

$$\#\ of\ equations = b_e - (n_e - 1)$$
$$= 5 - (4 - 1) = 2 \text{ are needed!}$$

Assign the mesh current on the figure as was shown.

Apply KVL for mesh a:

$$100 = 3(i_a - i_b) + v + 6i_a \quad (1)$$

The current source has an unknown voltage across it; v!

Apply KVL for mesh c:

$$-50 = 4i_c - v + 2(i_c - i_b) \quad (2)$$

Adding eqs. (1) and (2) yields:

$$50 = 9i_a - 5i_b + 6i_c \quad (3)$$

Apply KVL for mesh b:

$$0 = 3(i_b - i_a) + 10i_b + 2(i_b - i_c) \quad (4)$$

The constraint equation of the current source is:

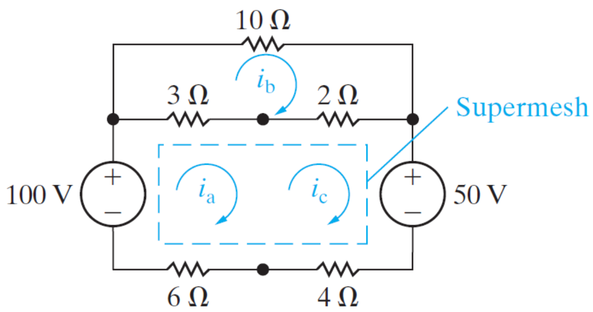
$$i_c - i_a = 5 \quad (5)$$

Simplification and solving of eqs. (3), (4) and (5) yield:

$$i_a = 1.75 \text{ A}, \quad i_b = 1.25 \text{ A}, \quad \text{and} \quad i_c = 6.75 \text{ A}.$$

Concept of Supermesh:

Equation (3) can be derived directly if KVL is applied to the ***supermesh*** shown in the figure next, without the need to include the voltage across the current source, v.



The supermesh is a mesh which a voids the current source!

Applying KVL to the supermesh yields:

$$-100 + 3(i_a - i_b) + 2(i_c - i_b) + 50 + 4i_c + 6i_a = 0$$

which reduces to

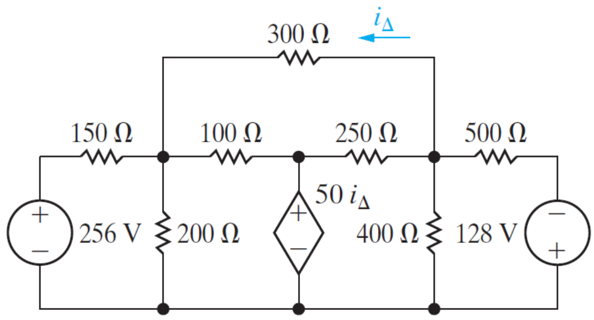
$$50 = 9i_a - 5i_b + 6i_c$$

which is the same equation as (3)!

4.3 Nodal Analysis versus Mesh Analysis

1- Use the method which results in a fewer simultaneous equation
2- If there is a super node use Nodal analysis
3- If there is a supermesh use Mesh analysis
4- If there is a method which gives the requested answer by solving a portion of the circuit only, then use that method.

Example # 8: Find the power dissipated in the 300Ω resistor in the circuit shown in the figure.



Solution:

$n_e = 4$

$b_e = 8$

Using Nodal analysis,

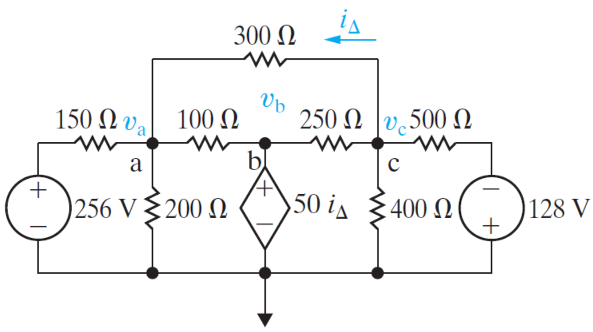
$$\#\ of\ eqs. = n_e - 1 = 4 - 1 = 3$$

Using Mesh analysis,

$$\#\ of\ eqs. = b_e - (n_e - 1) = 8 - (4 - 1) = 5$$

Thus, use Node method, since it needs less equations.

Choose the bottom node as the reference node, and assign the other node voltages as shown in the figure next.



KCL at node "a" yields:

$$\frac{v_a}{200} + \frac{v_a - 256}{150} + \frac{v_a - v_b}{100} + \frac{v_a - v_c}{300} = 0$$

KCL at node "c" yields:

$$\frac{v_c}{400} + \frac{v_c + 128}{500} + \frac{v_c - v_b}{250} + \frac{v_c - v_a}{300} = 0$$

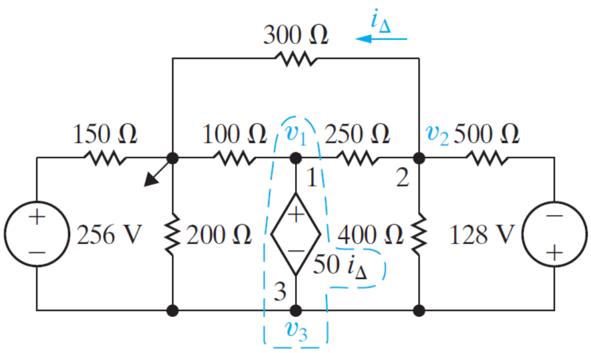
But,

$$v_b = 50 i_\Delta = \frac{50(v_c - v_a)}{300} = \frac{v_c - v_a}{6}$$

Rearrange and solve the above equations to find v_a& v_c!

The power in 300Ω is: $p_{300\Omega} = \frac{(v_c - v_a)^2}{300} = 16.57W$ (prove it!)

Derive the equations for Nodal analysis if the reference was taken at node “a”, then a supernode will include the dependent source, as shown in the figure next!

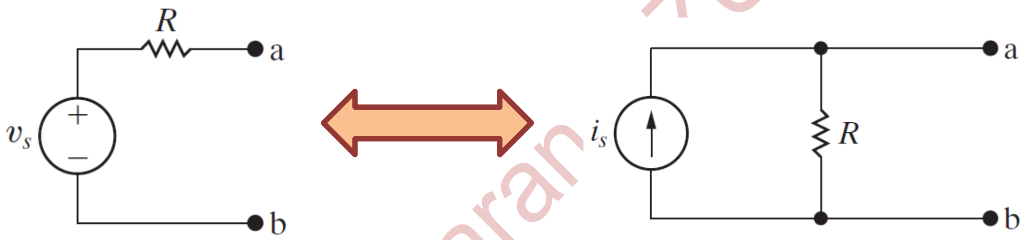


Study Example 4-6 (of the text book,10th ed.)

Study other examples from the text book!

4.4 Source Transformation

It allows a **voltage source in series with a resistor** to be replaces by a **current source in parallel with the same resistor,** and vice versa.



(a) (b)

If a load resistor R_L was connected across the terminals a-b, in both circuits, then the current in figure (a) is:

$$i_L = \frac{v_s}{R + R_L}$$

Whilst, the current in figure (b), by current divider, is:

$$i_L = \frac{R}{R + R_L} i_s$$

If the two circuits are equivalent, then the two load currents must be the same:

$$\frac{v_s}{R+R_L} = \frac{R i_s}{R+R_L}$$

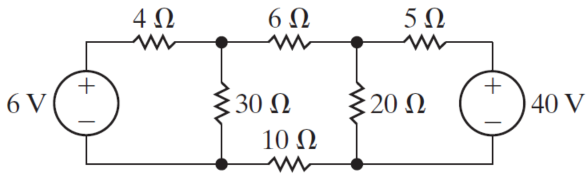
This implies that, $\boldsymbol{v_s = R i_s}$ or $\boldsymbol{i_s = \frac{v_s}{R}}$

Note the polarity of $\boldsymbol{v_s}$ and direction of $\boldsymbol{i_s}$!

Example # 9: ***Use Source Transformation***

a) For the circuit shown in figure below, find the power associated with the 6 V source.

b) State whether the 6V source is absorbing or delivering the power calculated in (a).



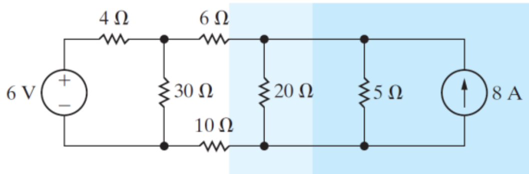
Solution:

Note that,

- 3 node equations or 3 mesh equations have to be solved.
- The 6V source must be preserved when conducting any source transformation.

1- Transform 40V and 5Ω

$$i_{s1} = \frac{v_s}{R} = \frac{40}{5} = 8A \text{ in parallel with } 5\Omega!$$

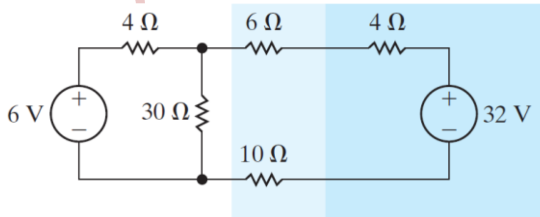


2- 5Ω//20Ω

$$R_{eq1} = \frac{5(20)}{5+20} = 4\Omega$$

3- Transform 8A in parallel with 4Ω

$$v_{s1} = R_{eq1} i_{s1} = 4(8) = 32V$$

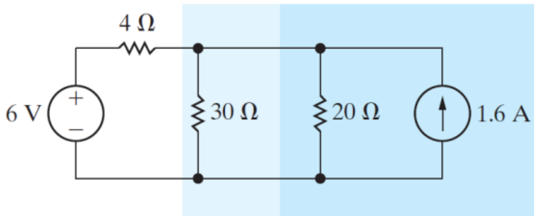


4- 4Ω in series with 10Ω and 6Ω

$$R_{eq2} = 4 + 10 + 6 = 20\Omega$$

5- Transform 32V and 20Ω

$$i_{s2} = \frac{v_{s1}}{R_{eq2}} = \frac{32}{20} = 1.6A \text{ in parallel with } 20\Omega!$$

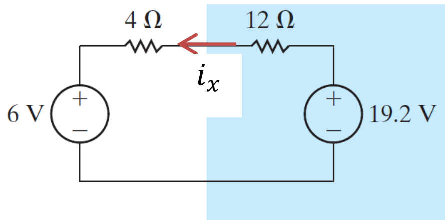


6- 30Ω//20Ω

$$R_{eq3} = \frac{30(20)}{30+20} = 12\Omega$$

7- Transform 1.6A in parallel with 12Ω

$$v_{s2} = R_{eq3} i_{s2} = 12(1.6) = 19.2V$$



8- The current i_x

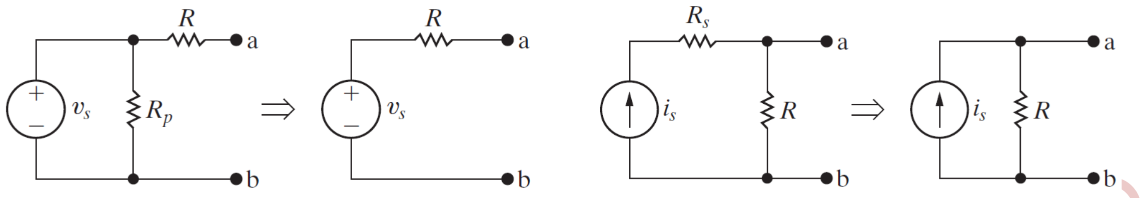
$$i_x = \frac{v_{s2} - 6}{4+12} = \frac{19.2-6}{16} = 0.825A$$

9- The power p_{6V} is:

$$p_{6V} = v i_x = 6(0.825) = 4.95W$$

The power is absorbed by the 6V-source, as the current is entering the positive terminal of the 6V-source.

Note that, a resistor in parallel with the voltage source has no effect on the transformation (figure (a)), whereas a resistor in series with a current source has no effect on the transformation (figure (b)).



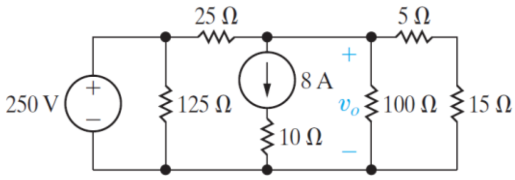
(a) (b)

Example # 10:

a) Use Source Transformation to find the voltage v_o

b) Find the power developed by the 250V source

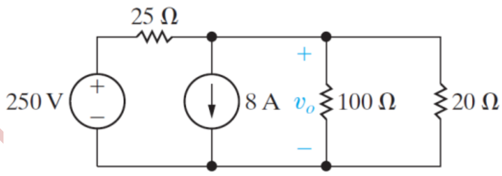
c) Find the power developed by the 8A source



Solution:

a) The 125Ω resistor across the voltage source has no effect, so it can neglected/removed!

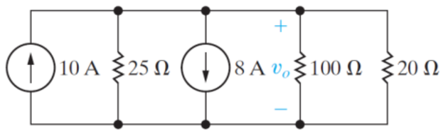
Also the 10Ω resistor in series with the current source has no effect, so it can neglected/removed!



1- 5Ω in series with 15Ω → $R_{eq1} = 20\Omega$

2- Transform 250V and 25Ω

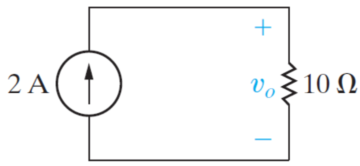
$i_{s1} = \frac{v_s}{R} = \frac{250}{25} = 10A$; in parallel with 25Ω!



3- 25Ω//20Ω//100 Ω

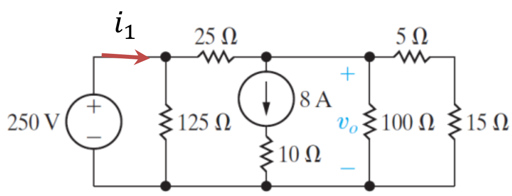
$R_{eq2} = 10\Omega$

4- The current sources can be combined together producing $i_{s2} = 2A$ source (upwards); in parallel with 10Ω



$v_o = R_{eq2} i_{s2} = 10(2) = 20V$

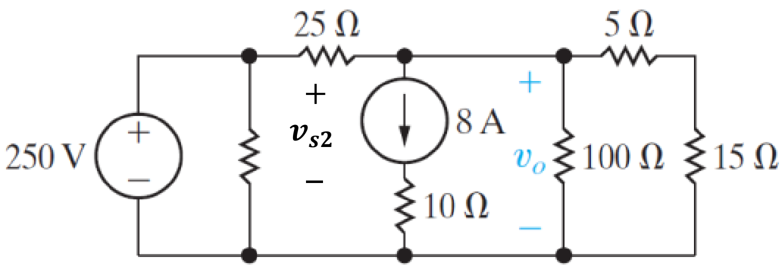
b) Refer to the original circuit, the current out of the 250V source, i_1, is the sum of the current in the 125Ω and the current in the 25Ω;



$$i_1 = \frac{v_s}{125} + \frac{v_s - v_o}{25} = \frac{250}{125} + \frac{250-20}{25} = 11.2A$$

$p_{250V} = v_s i_1 = 250(11.2) = 2800W$ (supplied by the source)

c) Refer to the original circuit, the power consumed by the 8A source is:



$$p_{8A} = v_{s2} i_{8A}$$

The voltage of 8A source, v_{s2}, is (by KVL):

$$-v_{s2} + v_o - 10(8) = 0$$

$$v_{s2} = 20 - 10(8)$$

$$v_{s2} = -60V$$

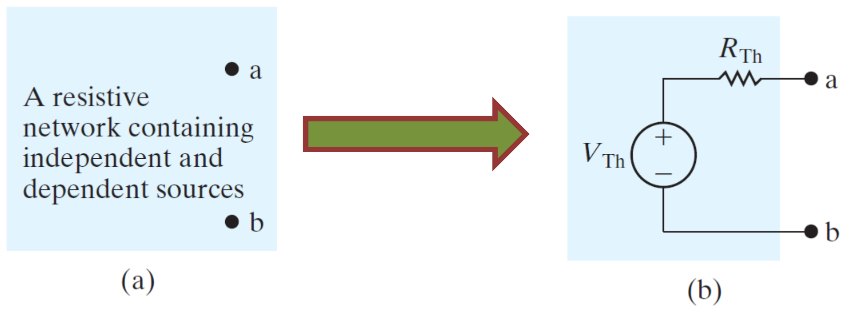
$$p_{8A} = -60(8) = -480W \text{ (consumed)}$$

$$p_{8A} = 480W \text{ (developed by or supplied by the source)}$$

Note that, the 125Ω and 10Ω resistors do not affect the value of v_o, but do affect the power calculations.

4.5 Thevenin's and Norton's Theorems

Thevenin's Theorem: it is possible to replace the entire network, exclusive of the load, by an equivalent circuit that contains only an independent voltage source ***(V_{Th}) in series with a resistor (R_{Th}),*** in such a way that the current-voltage relationships at the load are unchanged.



Norton's Theorem: it is possible to replace the entire network, exclusive of the load, by an equivalent circuit that contains only an independent current source ***(I_N) in Parallel with a resistor (R_N),*** in such a way that the current-voltage relationships at the load are unchanged.

Such that,

$\boldsymbol{V_{Th}} = \boldsymbol{V_{oc}}$ **,** where V_{oc} the open circuit voltage at terminals a-b

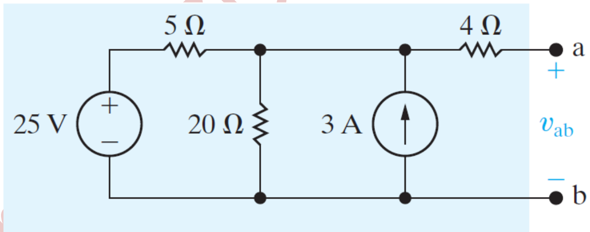
$\boldsymbol{I_N} = \boldsymbol{i_{sc}}$, where i_{sc} the short circuit current at terminals a-b

$\boldsymbol{R_{Th}} = \boldsymbol{R_N} = \frac{\boldsymbol{V_{oc}}}{\boldsymbol{i_{sc}}}$**;** in general (for cases 1 and 2), but it can be found by various methods depending on the types of sources in the network!

4.5.1 Case 1: Independent Sources Only

R_{Th} or (R_N) can be found by **looking into the open circuit terminals**; a-b, after setting all sources to zero; current sources are set to zero current (replaced by an open circuit), and the voltage sources are also set to zero voltage (replaced by a short circuit).

Example # 11: Find the Thevenin's and Norton's equivalent of the circuit in the figure below.



Solution:

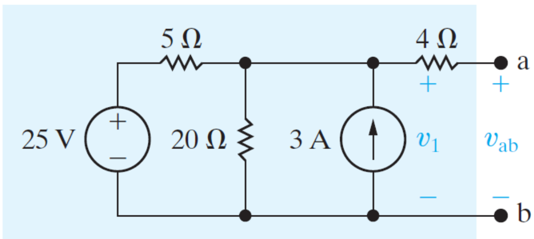
1- The Thevenin's voltage, $V_{Th} = V_{oc}$:

Note that, when the terminals a-b are open, there is no current in the 4Ω resistor.

Therefore the open-circuit voltage is identical to the voltage across the 3 A current source, labelled v_1.

Using Node-method, taking the lower node as a reference node, and applying KCL at the upper node yield:

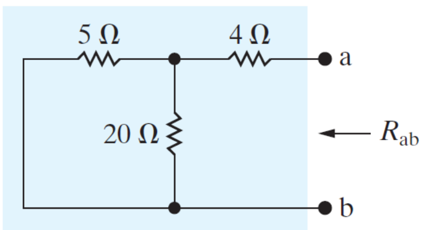
$$\frac{v_1 - 25}{5} + \frac{v_1}{20} - 3 = 0$$



Solving yields, $v_1 = 32V$

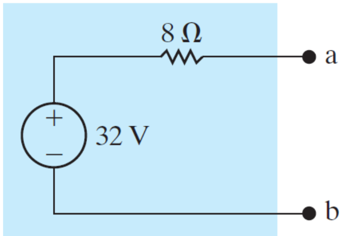
Therefore, $V_{Th} = V_{oc} = V_{ab} = v_1 = 32V$

2- Thevenin's resistance, this is case 1, so deactivate (get rid of) all independent sources; the current source is replaced by an open circuit and the voltage source is replaced by a short circuit, then look into the terminals a-b;

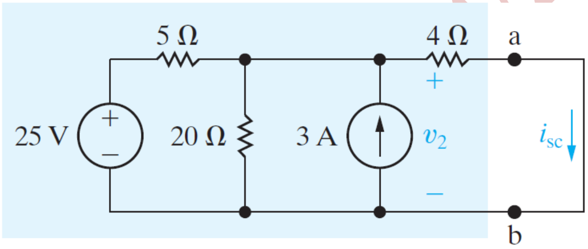


$$R_{ab} = R_{Th} = 4 + \frac{5 \times 20}{25} = 8\ \Omega$$

Therefore, the equivalent Thevenin's circuit is:



3- To find $\boldsymbol{I_N = i_{sc}}$



Using Node-method, taking the lower node as a reference node, and applying KCL at the upper node yield:

$$\frac{v_2 - 25}{5} + \frac{v_2}{20} - 3 + \frac{v_2}{4} = 0$$

Solving yields, $v_2 = 16V$

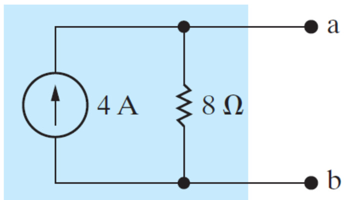
Thus, $i_{sc} = \frac{v_2}{4} = \frac{16}{4} = 4A = I_N$

4- To find the Norton's equivalent

$$R_N = R_{Th} = 8\Omega$$

It can also be found as: $R_N = \frac{V_{oc}}{i_{sc}} = \frac{32}{4} = 8\Omega$;

The equivalent Norton's circuit is shown in the figure next.



Note that,

- Thevenin's and Norton's equivalent circuits are the source transformation of each other.
- *The current in the short circuit i_{sc} must be assumed to flow from the positive to the negative terminal of V_{oc}.*

4.5.2 Case 2: Independent and Dependent Sources

In this case,

$\boldsymbol{V_{Th} = V_{oc}}$, where V_{oc} the open circuit voltage at terminals a-b

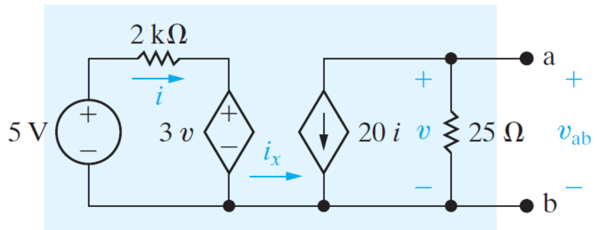
$\boldsymbol{I_N = i_{sc}}$, where i_{sc} the short circuit current at terminals a-b

$\boldsymbol{R_{Th}}$ & $\boldsymbol{R_N}$ can be found either by:

1- $\boldsymbol{R_{Th} = R_N = \frac{V_{oc}}{i_{sc}}}$, as explained before, or

2- $\boldsymbol{R_{Th} = R_N = \frac{v_T}{i_T}}$, where v_T & i_T are the Test voltage and current, respectively, applied at the network terminals

Example # 12: Find the Thevenin's and Norton's equivalent circuits at the terminals a-b for the circuit in the figure shown next.



Solution:

- This is case 2; the network has dependent and independent sources.
- Note that, i_x is zero as there is no return path!
- To find $\boldsymbol{V_{Th} = V_{oc} = V_{ab} = v}$,

$$V_{ab} = -(20i)25 = -500i \qquad (1)$$

But, the current i, controlling the current-controlled current source, is:

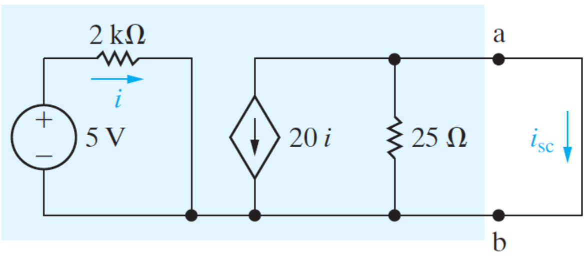
$$i = \frac{5 - 3v}{2000} = \frac{5 - 3V_{Th}}{2000} \qquad (2)$$

Solving eqs. (1) and (2) yields:

$$\boldsymbol{V_{Th} = -5V}$$

- To find $\boldsymbol{I_N = i_{sc}}$

Place a short circuit link at the terminals a-b, and assign i_{sc} on it, as shown in the figure next.



Since, there is short circuit across a-b, the voltage across a-b is zero $\boldsymbol{v} = 0$, and therefore the voltage-controlled voltage source is zero; it is replaced by a short circuit.

Note that, with a short circuit at a-b, *all the current of the current-controlled current-source flows in the short circuit*;

$$i_{sc} = -20i$$

But, the current, i, controlling the current-controlled current source is:

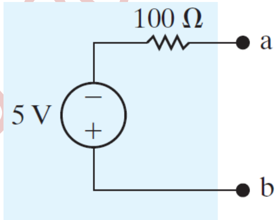
$$i = \frac{5}{2000} = 2.5 \text{ mA}$$

Therefore, $i_{sc} = -20(0.0025) = -50mA \rightarrow \boldsymbol{I_N = -50mA}$

Thus,

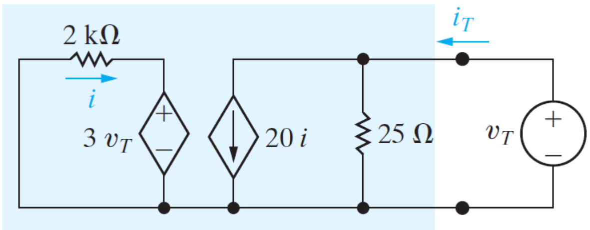
$$R_{Th} = R_N = \frac{V_{oc}}{i_{sc}} = \frac{-5}{-0.05} = \mathbf{100\Omega}$$

The Thevenin's equivalent circuit is shown next.



To find Thevenin's Resistance using the Test Source:

Deactivate all independent sources and connect a Test source at the terminals a-b of the original circuit, whose current is i_T and voltage is v_T, as shown in the figure below.



Apply KCL at the output node:

$$i_T = \frac{v_T}{25} + 20i$$

But, $$i = \frac{-3v_T}{2} \text{ mA}$$

Combining the latter two equations yields:

$$i_T = \frac{v_T}{25} - \frac{60v_T}{2000}$$

Rearranging the latter equation, yields:

$$\frac{i_T}{v_T} = \frac{1}{25} - \frac{6}{200} = \frac{50}{5000} = \frac{1}{100}$$

Therefore, $$R_{\text{Th}} = \frac{v_T}{i_T} = 100\ \Omega$$

4.5.3 Case 3: Dependent Sources Only (Dead Circuits)

Since there is no independent source, $V_{Th} = 0$ & $I_N = 0$.

$R_{Th} = R_N = \frac{v_T}{i_T}$, where v_T & i_T are the Test voltage and current, respectively, applied at the network terminals

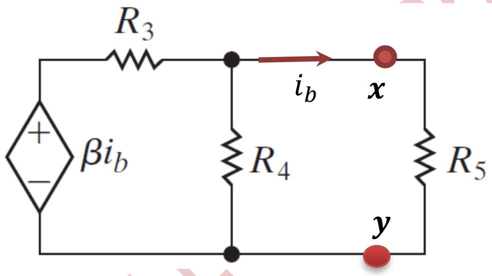
Example # 13: Find the Thevenin's and Norton's equivalent circuits at the terminals x-y for the circuit in the figure shown next, where:

$\beta = 1.5$

$R_3 = 3\Omega$

$R_4 = 2\Omega$

$R_5 = 2\Omega = R_L$

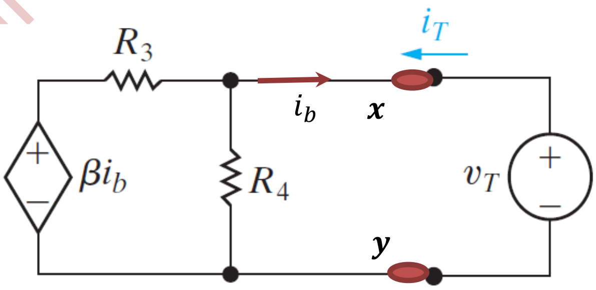


Solution:

Since there is no independent source, $V_{Th} = 0$ & $I_N = 0$

To find R_{Th} or R_N, apply a test voltage v_T at x-y, and find i_T, then $R_{Th} = R_N = \frac{v_T}{i_T}$

KCL at the upper node, and taking the bottom node as a reference yield:

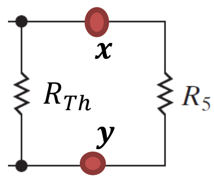


$$\frac{v_T - \beta i_b}{R_3} + \frac{v_T}{R_4} - i_T = 0$$

But, $i_b = -i_T$

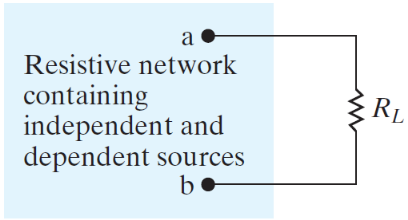
Combining the latter two equations, substituting values and rearranging yield:

$$\frac{v_T}{i_T} = 0.6\Omega \rightarrow R_{Th} = R_N = 0.6\Omega$$



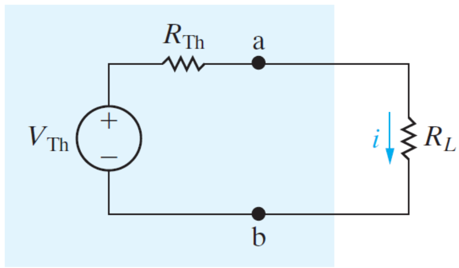
4.6 Maximum Power Transfer

To calculate the value of the load resistor that draws the maximum power from a network, replace the whole network by its equivalent Thevenin's or Norton's circuit!



The load power is:

$$p = i^2 R_L = \left(\frac{V_{Th}}{R_{Th} + R_L}\right)^2 R_L$$



To find the value of R_L that maximizes the power transferred to the load,

1- derive the power with respect to R_L:

$$\frac{dp}{dR_L} = V_{Th}^2\left[\frac{(R_{Th} + R_L)^2 - R_L \cdot 2(R_{Th} + R_L)}{(R_{Th} + R_L)^4}\right]$$

2- set the derivative equal to zero:

$$(R_{Th} + R_L)^2 = 2R_L(R_{Th} + R_L)$$

3- The power is maximum when:

$$\boldsymbol{R_L = R_{Th}}$$

4- The maximum power is obtained by substituting for $R_L = R_{Th}$ in the power equation:

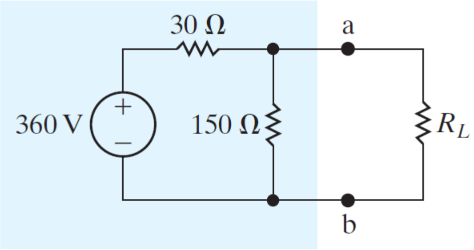
$$p_{max} = \left(\frac{V_{Th}}{R_{Th}+R_{Th}}\right)^2 R_{Th} = \left(\frac{V_{Th}}{2R_{Th}}\right)^2 R_{Th}$$

Thus,

$$\boldsymbol{p_{max} = \frac{(V_{Th})^2}{4R_{Th}}} \quad \textbf{or} \quad \boldsymbol{p_{max} = \frac{R_N(I_N)^2}{4}}$$

Example # 14:

a) For the circuit shown in figure next, find the value of R_L that results in maximum power being transferred to R_L



b) Calculate the maximum power that can be delivered to R_L

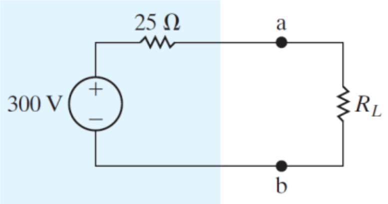
c) When R_L is adjusted for maximum power transfer, what percentage of the power delivered by 360 V source reaches R_L?

Solution:

a) The Thévenin's voltage for the circuit to the left of the terminals a-b is:

$$V_{\text{Th}} = \frac{150}{180}(360) = 300 \text{ V}$$

The Thevenin's resistor is found by looking into the terminals after deactivating the voltage source, as it is case 1:



$$R_{\text{Th}} = \frac{(150)(30)}{180} = 25 \ \Omega$$

Therefore, $R_L = R_{Th} = \mathbf{25\Omega}$

b) The maximum power that can be delivered to R_L is:

$$p_{max} = \frac{(V_{Th})^2}{4R_{Th}} = \frac{(300)^2}{4(25)} = 900W$$

c) When $R_L = 25\Omega$, the voltage v_{ab} is:

$$v_{\text{ab}} = \left(\frac{300}{50}\right)(25) = 150 \text{ V}$$

From the original circuits, the source current flowing out of the positive terminal is:

$$i_s = \frac{v_s - v_{ab}}{30} = \frac{360-150}{30} = \frac{210}{30} = 7A$$

The power delivered by the source to the circuit is:

$$p_s = v_s i_s = 360(7) = 2520W$$

The percentage of the source power delivered to the load is:

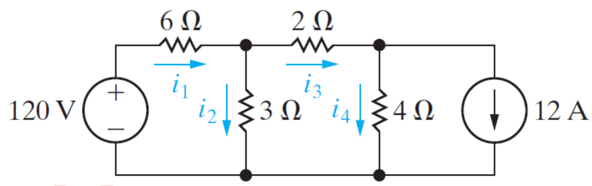
$$\frac{900}{2520} \times 100 = 35.71\%$$

4.7 Superposition

- Superposition states that whenever a linear system is excited, or driven, by *more than one independent source* of energy, the *total response is the sum of the individual responses*. An individual response is the result of an independent source acting alone.
- Only one independent source is left in the circuit, all the other independent sources are deactivated; a *current source is replaced by an open circuit*, whilst a *voltage source is replaced by a short circuit*.
- All dependent sources stay in the circuit.
- Linearity requires additivity and homogeneity (scaling).
- The response to one independent source is superimposed on the responses of the other independent sources.

Example # 15:

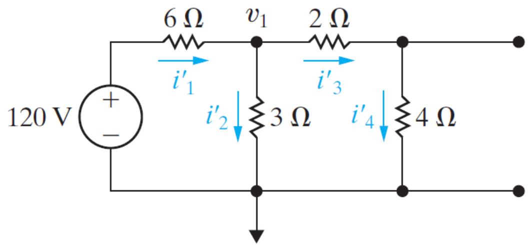
For the circuit shown in the figure next, use superposition to find the branch currents.



Solution:

1- The response to 120V source acting alone (12A source is replaced by an open circuit)

Use Nodal analysis, and apply KCL at node 1



$$\frac{v_1 - 120}{6} + \frac{v_1}{3} + \frac{v_1}{2 + 4} = 0$$

Arranging and solving yields:

$$v_1 = 30 \text{ V}$$

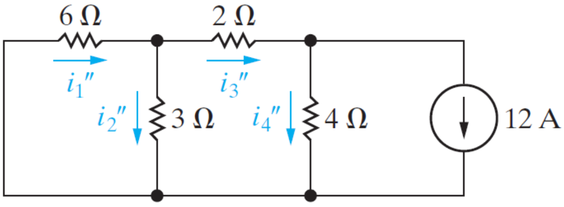
The branch currents in response to the 120V source are:

$$i'_1 = \frac{120 - 30}{6} = 15 \text{ A},$$

$$i'_2 = \frac{30}{3} = 10 \text{ A},$$

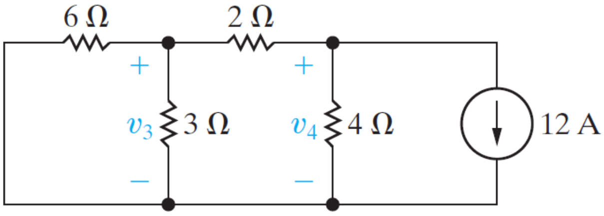
$$i'_3 = i'_4 = \frac{30}{6} = 5 \text{ A}.$$

2- The response to 12A source acting alone (120V source is replaced by a short circuit)



Use nodal analysis to find the node voltages v_3 & v_4, with the bottom node as a reference:

<u>KCL at Node 3:</u>



$$\frac{v_3}{3} + \frac{v_3}{6} + \frac{v_3 - v_4}{2} = 0$$

<u>KCL at Node 4:</u>

$$\frac{v_4 - v_3}{2} + \frac{v_4}{4} + 12 = 0$$

Arranging and solving the latter two equations yield:

$$v_3 = -12 \text{ V},$$

$$v_4 = -24 \text{ V}.$$

The branch currents in response to the 12A source are:

$$i_1'' = \frac{-v_3}{6} = \frac{12}{6} = 2 \text{ A},$$

$$i_2'' = \frac{v_3}{3} = \frac{-12}{3} = -4 \text{ A},$$

$$i_3'' = \frac{v_3 - v_4}{2} = \frac{-12 + 24}{2} = 6 \text{ A},$$

$$i_4'' = \frac{v_4}{4} = \frac{-24}{4} = -6 \text{ A}.$$

3- The branch current in the original circuit is the algebraic sum of individual contributions of each individual source acting alone:

$$i_1 = i_1' + i_1'' = 15 + 2 = 17 \text{ A},$$

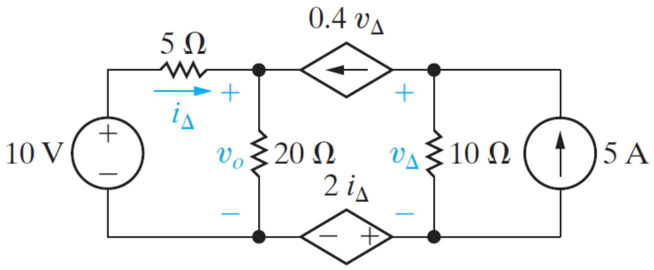
$$i_2 = i_2' + i_2'' = 10 - 4 = 6 \text{ A},$$

$$i_3 = i_3' + i_3'' = 5 + 6 = 11 \text{ A},$$

$$i_4 = i_4' + i_4'' = 5 - 6 = -1 \text{ A}.$$

Example # 16:

For the circuit shown in the figure next, use superposition to find the output voltage, v_o.



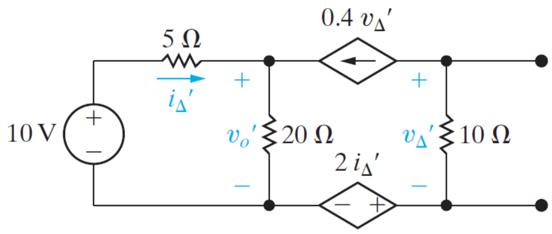
Solution:

1- The response to 10V source acting alone (5A source is replaced by an open circuit)

The right hand loop, has a current of $0.4v'_\Delta$;

$$v'_\Delta = -10(0.4v'_\Delta)$$

$$v'_\Delta = -4v'_\Delta \rightarrow v'_\Delta = 0$$



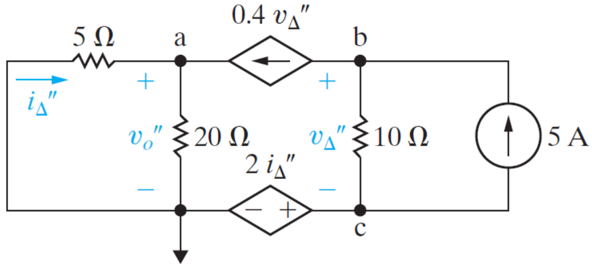
No current flows in the right hand branch, which means that the branch containing the two dependent sources is open circuit.

Therefore, v_o is obtained by voltage divider:

$$v'_o = \frac{20}{25}(10) = 8 \text{ V}$$

2- The response to 5A source acting alone (10V source is replaced by a short circuit)

Use nodal analysis, and assign nodes



KCL at Node a:

$$\frac{v''_o}{20} + \frac{v''_o}{5} - 0.4v''_\Delta = 0, \quad \text{or} \quad 5v''_o - 8v''_\Delta = 0 \tag{1}$$

KCL at Node b:

$$0.4v''_\Delta + \frac{v_b - 2i''_\Delta}{10} - 5 = 0, \text{ or}$$

$$4v''_\Delta + v_b - 2i''_\Delta = 50. \tag{2}$$

But, $v_b = v_c + v_\Delta^{"}$

$$v_b = 2i_\Delta'' + v_\Delta''$$

Substituting for v_b in eq. (2) yields:

$$5v_\Delta'' = 50, \quad \text{or} \quad v_\Delta'' = 10 \text{ V}$$

Substituting for $v_\Delta^{"}$ in eq. (1) yields:

$$5v_0'' = 80, \quad \text{or} \quad v_0'' = 16 \text{ V}$$

3- The output voltage in the original circuit is the algebraic sum of individual contributions of each individual source acting alone:

$$v_0 = v_o' + v_o^{"}$$

$$v_0 = 8 + 16 = 24V$$

Chapter 5

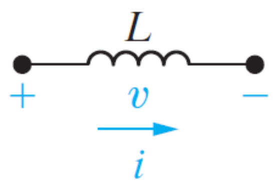
Inductance and Capacitance

5.1 Inductance and Inductors

Inductance: relates the induced voltage (due to the time varying magnetic field) to the current.

An inductor: is a passive circuit element, which is composed of a coil of wire wound around a supporting core, whose material may be magnetic of nonmagnetic. It *stores energy in its magnetic field.*

- Examples of inductors are the coils of wires used to make an electro-magnet, or the windings of wire in an electric motor.
- It is measured in Henrys (H); μH to ten's of Henrys
- It is represented graphically as a coiled wire.



- The voltage across the inductor is proportional to the rate of change of the current through it as:

$$v = L\frac{di}{dt}$$

where v is measured in Volts, L in Henrys, i in Amperes, and t in seconds.

Note that,

1- If the inductor current is constant, then $v = 0$; the inductor is a short circuit

2- The current cannot change instantaneously in an inductor

The inductor's current is:

$$v\,dt = L\,di \quad \rightarrow \quad L\int_{i(t_0)}^{i(t)} dx = \int_{t_0}^{t} v\,d\tau$$

where $i(t_0)$ is the initial inductor's current at time t_0.

$$i(t) = \frac{1}{L}\int_{t_0}^{t} v\,d\tau + i(t_0),$$

5.1.1 Power and Energy in the Inductor

The power of the inductor is:

$$p = iv$$

But, $v = L\frac{di}{dt}$, therefore,

$$p = Li\frac{di}{dt} \quad \text{or} \quad p = v\left[\frac{1}{L}\int_{t_0}^{t} v\,d\tau + i(t_0)\right]$$

The energy and the power are related by:

$$p = \frac{dw}{dt} = Li\frac{di}{dt}$$

where w is the energy (or work)

$$dw = Li\,di$$

If the reference of zero energy corresponds to zero current, changing variables of integration and integrating yields:

$$\int_0^w dx = L\int_0^i y\,dy$$

Thus, the energy of an inductor is:

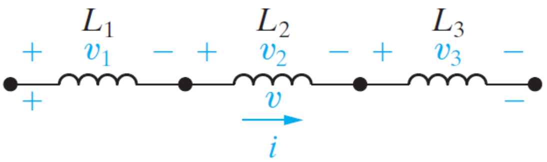
$$w = \frac{1}{2}Li^2$$

Energy is measured in Joules.

5.1.2 Series-Parallel Combination of Inductors

A) Series Inductors:

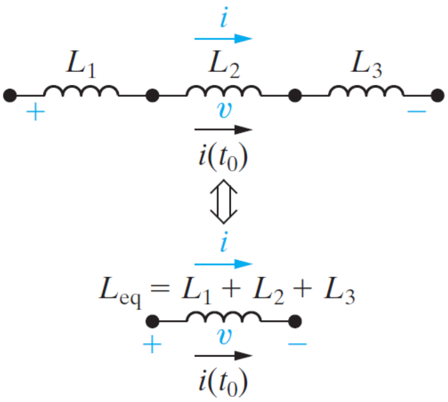
They carry the same current, i.



The voltage across each inductor is:

$$v_1 = L_1 \frac{di}{dt}, v_2 = L_2 \frac{di}{dt}, \text{ and } v_3 = L_3 \frac{di}{dt}$$

But, the total voltage, v:



$$v = v_1 + v_2 + v_3$$

$$v = L_1 \frac{di}{dt} + L_2 \frac{di}{dt} + L_3 \frac{di}{dt}$$

$$v = (L_1 + L_2 + L_3) \frac{di}{dt}$$

$$v = L_{eq} \frac{di}{dt}$$

where L_{eq} is the equivalent inductance such that,

$$L_{eq} = L_1 + L_2 + L_3$$

For N series inductors, the equivalent inductor is:

$$L_{eq} = L_1 + L_2 + L_3 + \cdots + L_n$$

Note that, the equivalent inductor carries also the same initial current, $i(t_0)$, as the series inductors.

B) Parallel Inductors:

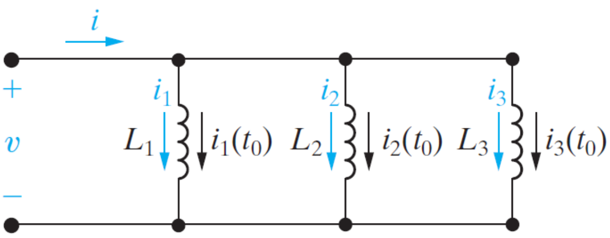
Parallel inductors have the same voltage, v.

The currents in the inductors are:

$$i_1 = \frac{1}{L_1} \int_{t_0}^{t} v \, d\tau + i_1(t_0)$$

$$i_2 = \frac{1}{L_2} \int_{t_0}^{t} v \, d\tau + i_2(t_0)$$

$$i_3 = \frac{1}{L_3} \int_{t_0}^{t} v \, d\tau + i_3(t_0)$$



KCL at the upper node yields:

$$i = i_1 + i_2 + i_3$$

Substituting for each current and arranging terms yield:

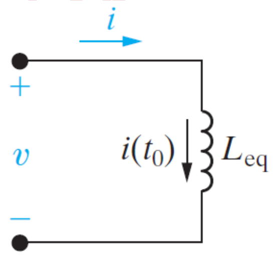
$$i = \left(\frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3}\right)\int_{t_0}^{t} v \, d\tau + i_1(t_0) + i_2(t_0) + i_3(t_0)$$

$$i = \frac{1}{L_{eq}}\int_{t_0}^{t} v \, d\tau + i(t_0)$$

Such that,

$$\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3}$$

$$i(t_0) = i_1(t_0) + i_2(t_0) + i_3(t_0)$$



Thus, for N parallel inductors, the equivalent inductor is:

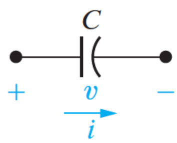
$$\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} + \cdots + \frac{1}{L_n}$$

and the equivalent initial current is the sum of individual initial currents;

$$i(t_0) = i_1(t_0) + i_2(t_0) + \cdots + i_n(t_0)$$

5.2 Capacitance and Capacitors

A Capacitor (C):



- It is a passive circuit element composed of two conducting plates separated by a layer of an insulating or a dielectric material ($C = \frac{\varepsilon A}{d}$; ε is the permittivity of the dielectric, “A” is the area of the plates, and “d” is the separating distance between plates).
- It is measured in Farads (pF … mF)
- It stores energy in its electric field.
- *As the voltage varies with time, the displacement of charge also varies with time, causing what is known as the displacement current.*

➢ The “displacement current” current of a capacitor is proportional to the rate of change of the voltage across its terminals, with respect to time, as:

$$i = C\frac{dv}{dt},$$

where i is measured in Amperes, C in Farads, v in Volts, and t in seconds.

➢ Thus, the capacitor’s voltage can be found by arranging and integrating as:

$$i\,dt = C\,dv \qquad \text{or} \qquad \int_{v(t_0)}^{v(t)} dx = \frac{1}{C}\int_{t_0}^{t} i\,d\tau.$$

$$v(t) = \frac{1}{C}\int_{t_0}^{t} i\,d\tau + v(t_0)$$

where $v(t_0)$ is the initial capacitor’s voltage at time t_0.

5.2.1 Power and Energy in the Capacitor

The power of the capacitor is:

$$p = vi$$

But, $i = C\frac{dv}{dt}$, therefore,

$$p = Cv\frac{dv}{dt} \quad \text{or} \quad p = i\left[\frac{1}{C}\int_{t_0}^{t} i\,d\tau + v(t_0)\right]$$

The energy (w) of the capacitor can be found as:

$$p = \frac{dw}{dt} = Cv\frac{dv}{dt}$$

Therefore, $dw = Cvdv$

Integrating both side yields:

$$\int_0^w dx = C\int_0^v y\,dy$$

Thus,

$$w = \frac{1}{2}Cv^2$$

The reference for zero energy corresponds to zero voltage.

5.2.2 Series-Parallel Combination of Capacitors

A) Series Capacitors:

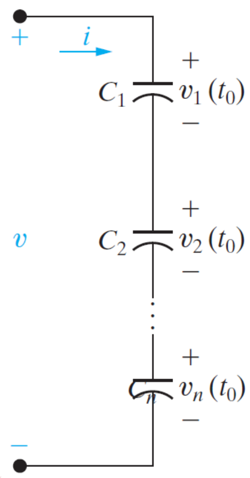
Applying KVL:

$$v = v_1 + v_2 + \cdots + v_n$$

But,

$$v(t) = \frac{1}{C}\int_{t_0}^{t} i\,d\tau + v(t_0)$$

Thus,



$$v = \left(\frac{1}{C_1} + \frac{1}{C_2} + \ldots + \frac{1}{C_n}\right)\int_{t_0}^{t} i d\tau + \{\, v_1(t_0) + v_2(t_0) + \cdots + v_n(t_0)\}$$

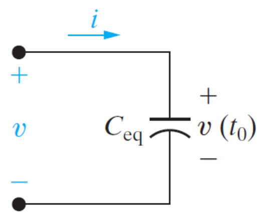
$$v = \left(\frac{1}{C_{eq}}\right)\int_{t_0}^{t} i d\tau + \ v(t_0)$$

where, is the equivalent capacitor is:

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}$$

and the initial voltage is:

$$v(t_0) = v_1(t_0) + v_2(t_0) + v_3(t_0)$$



In general, For N series capacitors, the equivalent capacitor is:

$$\boldsymbol{\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_n}}$$

And the initial voltage at the equivalent capacitor is:

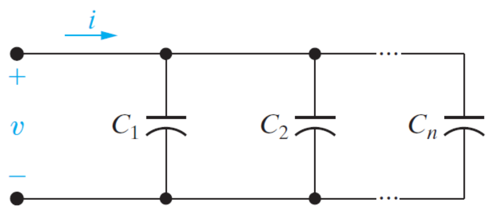
$$\boldsymbol{v(t_0) = v_1(t_0) + v_2(t_0) + \cdots + v_n(t_0)}$$

B) Parallel Capacitors:

Applying KCL at the upper node:

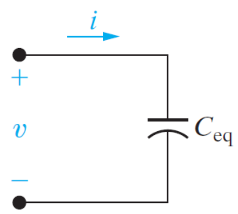
$$i = i_1 + i_2 + \cdots + i_n$$

But, $i_j = C_j \frac{dv}{dt}$, the above equation becomes:



$$C_{eq}\frac{dv}{dt} = C_1\frac{dv}{dt} + C_2\frac{dv}{dt} + \cdots + C_n\frac{dv}{dt}$$

$$C_{eq}\frac{dv}{dt} = (C_1 + C_2 + \cdots + C_n)\frac{dv}{dt}$$



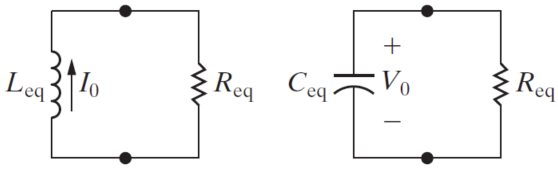
Therefore, $\boldsymbol{C_{eq} = C_1 + C_2 + \cdots + C_n}$

Note that, the equivalent capacitor has the same initial voltage as the parallel capacitors, $v(t_0)$.

Chapter 6

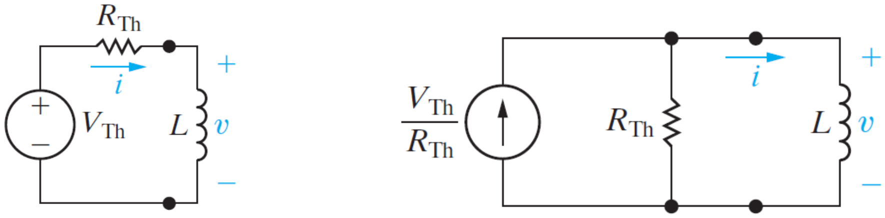
Response of First-Order RL or RC Circuits

- The analysis will be carried out to determine the response (the currents and voltages) that arise when energy is either released or acquired by an inductor or capacitor in response to an abrupt change in a DC voltage or current source.
- The response due to releasing the energy stored by an inductor or capacitor into a resistive network is known as *the natural response*, to emphasize that the nature of the circuit itself, *not external sources of excitation*, determines its behaviour.
- The equivalent circuit for *the natural response* could be represented as shown in the figure below.

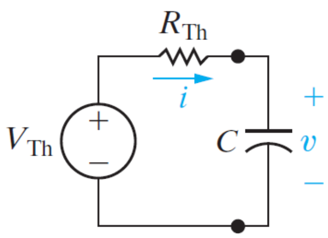


(a) (b)

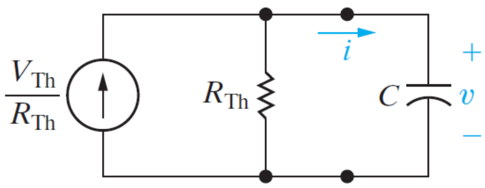
- RL (resistor-inductor) and RC (resistor-capacitor) circuits are also known as first-order circuits, because their voltages and currents are described by first-order differential equations.
- No matter how complex a circuit may appear, if it can be reduced to a Thévenin or Norton equivalent circuit connected to the terminals of an equivalent inductor or capacitor, it is a first-order circuit.



(a) (b)



(c)



(d)

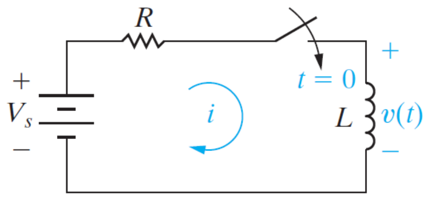
6.1 The Step Response of an RL and an RC Circuit

It shows how a circuit responds when energy is being released/stored in an inductor or a capacitor.

6.1.1 The Step Response of an RL Circuit

Assume that the switch in the figure is closed at time $t = 0s$.

Applying KVL yields:



$$L\frac{di(t)}{dt} + Ri(t) = V_s$$

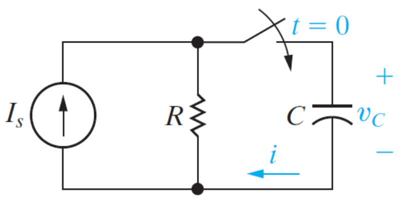
Dividing both sides by L yields:

$$\frac{di(t)}{dt} + \frac{R}{L}i(t) = \frac{V_s}{L}$$

6.1.2 The Step Response of an RC Circuit

Assume that the switch in the figure is closed at time $t = 0s$.

Applying KCL yields:



$$C\frac{dv(t)}{dt} + \frac{1}{R}v(t) = I_s$$

Dividing both sides by C yields:

$$\frac{dv(t)}{dt} + \frac{1}{RC}v(t) = \frac{I_s}{C}$$

6.1.3 Solution to the Differential Equation

Both networks can be described by a single differential equation in the form:

$$\frac{dX(t)}{dt} + aX(t) = f(t)$$

where $X(t)$ represents a current or a voltage in the network; $X(t)$is the solution to the differential equation and equals:

$$X(t) = X_p(t) + X_c(t)$$

where $X_p(t)$ is the particular integral solution or forced steady state response, such that:

$$\frac{dX_p(t)}{dt} + aX_p(t) = f(t)$$

$X_p(t)$ consists of functional forms of $f(t)$ and its first derivatives, except for $f(t) = Ae^{-at}$ where "a" in $f(t)$ is the same as "a" in the equation.

Some forms of $f(t)$ and their particular solutions are shown in the table below.

$\boldsymbol{f(t)}$	$\boldsymbol{X_p(t)}$
$\boldsymbol{A}$	k_1
$\boldsymbol{at^n}$	$b_0t^n + b_1t^{n-1} + b_2t^{n-2} + \cdots + b_n$
$\boldsymbol{Ae^{-rt}}$	k_1e^{-rt}
$\boldsymbol{A\cos\omega t}$	$D\cos\omega t + E\sin\omega t$
$\boldsymbol{B\sin\omega t}$	

Whilst, $X_c(t)$ is the complementary solution or natural (transient) response, and it is any solution to the homogeneous equation:

$$\frac{dX_c(t)}{dt} + aX_c(t) = 0$$

$X_c(t)$ and its derivative have the same time varying form, therefore,

$$\boldsymbol{X_c(t) = k_2e^{-at}}$$

If $f(t)$ is a constant, A, as it is the case with DC source excitation, then the general solution to the differential equation

$$\frac{dX(t)}{dt} + aX(t) = A$$

is:

$$X(t) = X_p(t) + X_c(t)$$

$$\rightarrow X(t) = k_1 + k_2 e^{-at}$$

As $t \rightarrow \infty$, $k_2 e^{-at} \rightarrow 0$, the remaining term k_1 is referred to as a steady state solution (was called a particular solution).

If $X(t)$ is writen in the form:

$$X(t) = k_1 + k_2 e^{-\frac{t}{\tau}}$$

Then the term τ is called the time constant of the network, and $\tau = \frac{1}{a}$.

Time Constant (τ):

It determines the rate at which the voltage or current reaches its steady state value.

For RL network, $= \frac{L_{eq}}{R_{eq}}$, (or $\tau = \frac{L_{eq}}{R_{Th}}$ if the circuit has dependent sources)

For RC network, $\tau = R_{eq}C_{eq}$, (or $\tau = R_{Th}C_{eq}$ if the circuit has dependent sources)

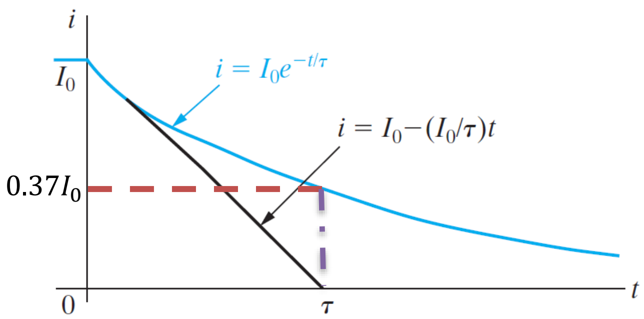
For example, if

$$i(t) = I_0 e^{-t/\tau}, \quad t \geq 0$$

after 1τ: $i(t) = 0.37I_0$

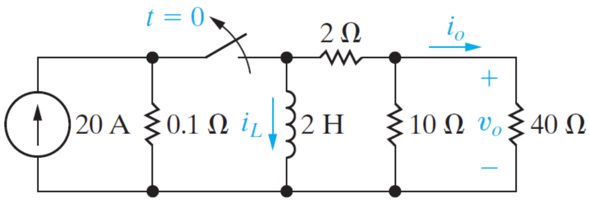
after 5τ: $i(t) = 0.01I_0$

Which means that during the first 1τ, the function has 63% change of its final value.



Example # 1: Natural response of an RL circuit

The switch has been closed for a long time before it is opened at $t = 0$. Find:



a) i_L(t) for $t \geq 0$

b) i_o(t) for $t \geq 0^+$

c) $v_o(t)$ for $t \geq 0^+$

d) the percentage of the total energy stored in the $2H$ inductor that is dissipated in the 10Ω resistor

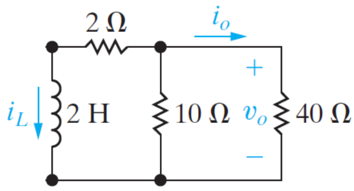
Solution:

a) **At $t = 0^-$,**

$i_L(0^-) = 20A$; all the source current passes in the inductor as the circuit was in steady state, and the inductor acts like a short circuit; *for a long time means reached steady state!*

$i_L(0^+) = i_L(0^-) = 20A$; no instantaneous change in the inductor's current.

For $t > 0$,



The circuit has no excitation, and can be described by a homogeneous differential equation, $(f(t) = 0)$; it has transient (natural) response only, and no particular response; steady state response is zero.

Thus,

$$i_L(t) = k_2 e^{-\frac{t}{\tau}}$$

The equivalent resistor connected with the inductor is:

$$R_{eq} = 2 + (40 \parallel 10) = 10\,\Omega$$

The time constant (τ) is:

$$\tau = \frac{L_{eq}}{R_{eq}} = \frac{2}{10} = 0.2s$$

Therefore,

$$i_L(t) = k_2 e^{-\frac{t}{0.2}} = k_2 e^{-5t}$$

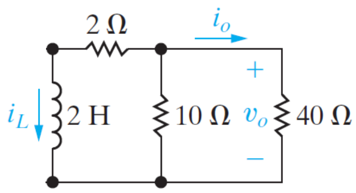
But the inductor current at $t = 0^+$ is:

$$i_L(0^+) = 20A = k_2 e^{-5(0)}$$

$$\therefore k_2 = i_L(0^+) = 20A$$

$$i_L(t) = 20\, e^{-5t}\ \text{A};\ t \geq 0$$

b) The current in 40Ω resistor, $i_o(t)$, can be found by current divider:



$i_o(t) = \frac{10(-i_L(t))}{10+40}$; valid for $t \geq 0^+$, because the current in the resistor is not a continuous function; $i_o(0^-) = 0A$

$i_o(t) = -4e^{-5t}$ A ; $t \geq 0^+$

c) The voltage across the 40Ω $v_o(t)$ can be found by Ohm's Law:

$v_o(t) = 40(-4e^{-5t})$; $t \geq 0^+$

$v_o(t) = -160e^{-5t}$ V ; $t \geq 0^+$

d) The power in 10Ω resistor:

$$p_{10\Omega}(t) = \frac{v_o^2}{10} = 2560e^{-10t} \text{ W}, \quad t \geq 0^+$$

The total energy dissipated in the 10 Ω resistor is

$$w_{10\Omega}(t) = \int_0^\infty 2560e^{-10t}\,dt = 256 \text{ J}.$$

The initial energy stored in the 2 H inductor is

$$w(0) = \frac{1}{2}Li^2(0) = \frac{1}{2}(2)(400) = 400 \text{ J}.$$

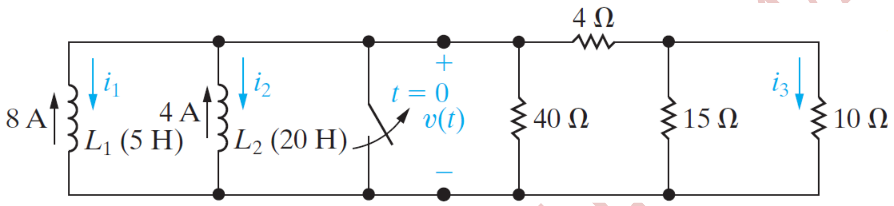
The percentage of the inductor's energy dissipated in the 10Ω resistor is:

$$\frac{256}{400}(100) = 64\%$$

Example # 2: Natural response of an RL circuit with parallel inductors

In the circuit shown, the initial currents in inductors L_1& L_2 and have been established by sources not shown. The switch is opened at $t = 0$:

- **a)** Find i_1, i_2 and i_3 for $t \geq 0$
- **b)** Calculate the initial energy stored in the parallel inductors
- **c)** Determine how much energy is stored in the inductors as $t \to \infty$
- **d)** Show that the total energy delivered to the resistive network equals the difference between the results obtained in b) and c)



Solution:

Note that, the initial currents in the inductors are given, so no need to analyse the circuit for $t = 0^-$!

For $t \geq 0$,

The two inductors are in parallel, the equivalent inductor is:

$$L_{eq} = \frac{L_1 L_2}{L_1 + L_2} = \frac{5(20)}{5+20} = 4H$$

The initial current of each individual inductor is:

$$i_{L1}(0^+) = i_{L1}(0^-) = 8A \qquad \text{and} \qquad i_{L2}(0^+) = i_{L2}(0^-) = 4A$$

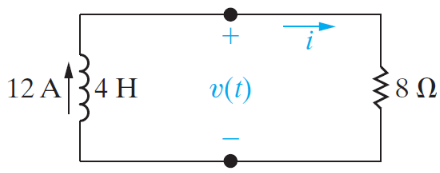
The initial current of the equivalent inductor is the algebraic sum of the initial currents in individual inductors:

$$i_L(0^+) = i_{L1}(0^+) + i_{L2}(0^+) = 8 + 4 = 12A$$

And the equivalent resistor is:

$$R_{eq} = 40||(4 + (15||10)) = 8\Omega$$

the circuit can be reduced to the equivalent circuit shown in the figure next.



The time constant (τ) is:

$$\tau = \frac{L_{eq}}{R_{eq}} = \frac{4}{8} = 0.5s$$

The circuit has no excitation, thus the current is the inductor's current is:

$$i(t) = k_2 e^{-\frac{t}{\tau}}$$

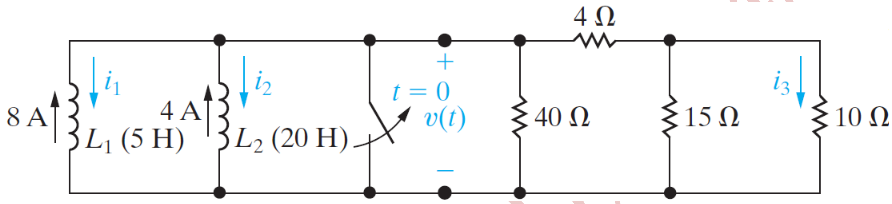
$$i(t) = i_L(0^-)e^{-\frac{t}{0.5}}$$

$$i(t) = 12e^{-2t}\ \text{A};\ t \geq 0$$

Now $v(t)$ is simply the product $8i$, so

$$v(t) = 96e^{-2t}\ \text{V}, \quad t \geq 0^+.$$ Because $v(0^-) = 0V$

a) Refer to the original circuit to be able to calculate the currents i_1, i_2 and i_3:



$$i_1 = \frac{1}{5}\int_0^t 96e^{-2x}\,dx - 8$$

$$= 1.6 - 9.6e^{-2t}\ \text{A}, \quad t \geq 0,$$

$$i_2 = \frac{1}{20}\int_0^t 96e^{-2x}\,dx - 4$$

$$= -1.6 - 2.4e^{-2t}\ \text{A}, \quad t \geq 0,$$

$$i_3 = \frac{v(t)}{10}\frac{15}{25} = 5.76e^{-2t}\ \text{A}, \quad t \geq 0^+.$$

b) The initial stored energy in the inductors is:

$$w = \frac{1}{2}(5)(64) + \frac{1}{2}(20)(16) = 320\ \text{J}$$

c) As $t \to \infty$, $i_1 \to 1.6$ A and $i_2 \to -1.6$ A

Thus, the energy stored in the two inductors is:

$$w = \frac{1}{2}(5)(1.6)^2 + \frac{1}{2}(20)(-1.6)^2 = 32\ \text{J}.$$

d) The energy delivered to the resistive network can be found by integrating the instantaneous power function from zero to infinity, where:

$$p(t) = \frac{v(t)^2}{R_{eq}} = \frac{(96e^{-2t})^2}{8} = 1152e^{-4t}$$

Therefore,

$$w = \int_0^{\infty} pdt = \int_0^{\infty} 1152e^{-4t}dt$$

$$w = \frac{1152e^{-4t}}{-4}\bigg|_0^{\infty} = \mathbf{288J}$$

The difference in the inductor's energy is:

$$\Delta w = w_{initial} - w_{final}$$

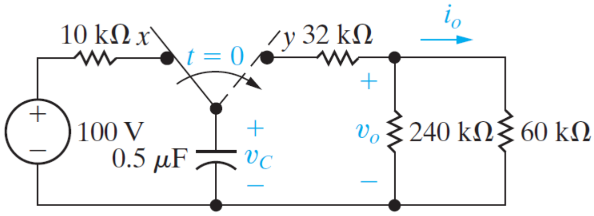
$\Delta w = 320 - 32 = \mathbf{288J}$, which is the same as the energy delivered to the resistive network!

Example # 3: Natural response of an RC circuit

The switch in the circuit shown in the figure has been in position "x" for a long time.

At $t = 0$ the switch moves instantaneously to position "y". Find:

a) $v_C(t)$ for $t \geq 0$,
b) $v_o(t)$ for $t \geq 0^+$,
c) $i_o(t)$ for $t \geq 0^+$, and
d) the total energy dissipated in the 60 kΩ resistor.



Solution:

For $t < 0$,

The switch was in its position for a long time, that means the circuit reached its steady state, and the capacitor was fully charged to its final value; $100V$.

The charging current is zero (the source voltage is constant), and the capacitor is represented as open circuit.

$v_c(0^-) = 100V = v_c(0^+)$; the voltage across the capacitor is continuous

For $t \geq 0$s,

The equivalent resistor connected with the capacitor is:

$$R_{eq} = 32k + (240k||60k) = 80k\Omega$$

And the time constant is:

$$\tau = R_{eq}C = 80k(0.5\mu) = 40ms$$

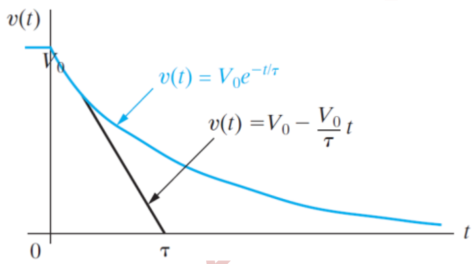
a) Since there is no excitation in the circuit for $t \geq 0$s, the capacitor's voltage is:

$$v_c(t) = k_2 e^{-\frac{t}{\tau}} \;, t \geq 0\text{s}$$

But, $\quad v_c(0^-) = v_c(0^+) = k_2 \rightarrow k_2 = 100V$

$$v_c(t) = v_c(0^-)e^{-\frac{t}{40m}}$$

$$v_c(t) = 100e^{-25t}\text{ V}; t \geq 0$$



b) $v_o(t)$ can be found by voltage divider of $v_c(t)$ between $32k\ and\ (240k||60k)$;

$$R_{eq2} = (240k||60k) = 48k$$

By voltage divider,

$$v_o(t) = \frac{48\text{k}(v_c(t))}{48\text{k}+32\text{k}}$$

$$v_o(t) = 60e^{-25t}V \;, t \geq 0^+$$

Note that, $v_o(t)$ is valid for $\geq 0^+$, because $v_0(0^-) = 0$!

c) The output current is found by Ohm's law:

$$i_o(t) = \frac{v_o(t)}{60\text{k}} = e^{-25t}\, mA \;, t \geq 0^+$$

d) The instantaneous power of 60kΩ resistor is:

$$p_{60k\Omega}(t) = (60k)i_o(t)^2$$

$$p_{60k\Omega}(t) = 60e^{-50t}\, mW \;, t \geq 0^+$$

The total energy of the 60kΩ resistor is found by integrating the instantaneous power function from zero to infinity;

$$w_{60\text{k}\Omega} = \int_0^{\infty} i_o^2(t)(60 \times 10^3)\, dt = 1.2 \text{ mJ}.$$

6.2 A General Solution for Step and Natural Responses

In general, the step response (for DC sources and switching at $t = 0$) is in the form:

$$X(t) = k_1 + k_2 e^{-\frac{t}{\tau}}$$

Such that, For RL network, $\tau = \frac{L_{eq}}{R_{eq}}$

For RC network, $\tau = R_{eq} C_{eq}$

Note that, if the circuit has dependent sources, R_{Th} or R_N has to be found, and

$R_{eq} = R_{Th} = R_N$

To find the constants k_1 & k_2:

$$X(0^+) = k_1 + k_2 \quad (1)$$

$$X(\infty) = k_1 \quad (2)$$

Expressing k_1 & k_2 in terms of $X(0^+)$ & $X(\infty)$ yields:

$$k_1 = X(\infty)$$

And

$$k_2 = X(0^+) - X(\infty)$$

Therefore, the solution $X(t) = k_1 + k_2 e^{-\frac{t}{\tau}}$ will have the form:

$$\boldsymbol{X(t) = X(\infty) + \left(X(0^+) - X(\infty)\right)e^{-\frac{t}{\tau}}} \quad ; t > 0$$

i.e.,

$$v(t) = v(\infty) + \left(v(0^+) - v(\infty)\right)e^{-\frac{t}{\tau}} \quad ; t > 0$$

$$i(t) = i(\infty) + \left(i(0^+) - i(\infty)\right)e^{-\frac{t}{\tau}} \quad ; t > 0$$

If the switching occurs at $t = t_0$, then:

$$\boldsymbol{X(t) = X(\infty) + \left(X(t_0{}^+) - X(\infty)\right)e^{-\frac{t-t_0}{\tau}}} \quad ; t > t_0$$

To find $X(t_0{}^+)$:

1- Draw the network for $t = t_0{}^+$

a) The capacitor is replaced by a voltage source in series with a short circuit; the value of the voltage source is $\boldsymbol{v_c(t_0{}^-)}$.

b) The inductor is replaced by a current source in parallel with an open circuit; the value of the current source is $\boldsymbol{i_L(t_0{}^-)}$.

2- Then use DC analysis to find $X(t_0{}^+)$

To find $X(\infty)$:

1- Draw the network for $t = \infty$

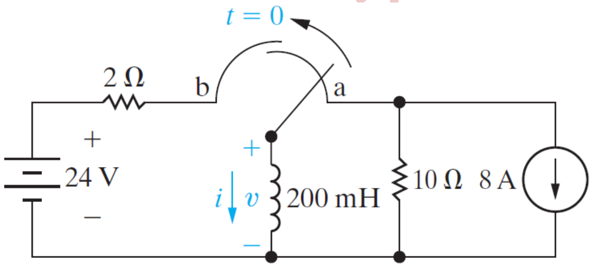
a) *The capacitor is replaced by an open circuit* (precisely, a voltage source in series with an open circuit; the value of the voltage source is $v_c({t_0}^-)$).

b) *The inductor is replaced by a short circuit* (precisely, a current source in parallel with a short circuit; the value of the current source is $i_L({t_0}^-)$).

2- Then use DC analysis to find $X(\infty)$

Example # 4:

The switch in the circuit shown in the figure has been in position a for a long time. At $t = 0$, the switch moves from position "a" to position "b". The switch is a make-before-break type; that is, the connection at position "b" is established before the connection at position "a" is broken, so there is no interruption of current through the inductor.



a) Find the expression for $i(t)$ for $t > 0$

b) What is the initial voltage across the inductor just after the switch has been moved to position "b"?

c) How many milliseconds after the switch has been moved does the inductor voltage equal 24 V?

d) Does this initial voltage make sense in terms of circuit behavior?

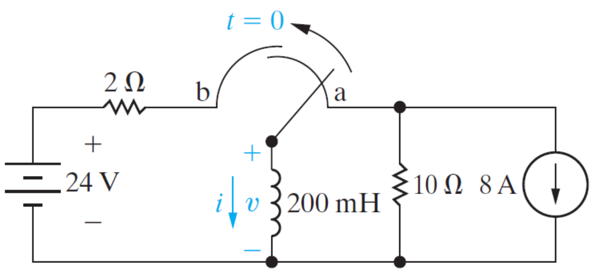
e) Plot both $i(t)$ and $v(t)$ versus t.

Solution:

The general solution is: $X(t) = X(\infty) + \left(X(0^+) - X(\infty)\right)e^{-\frac{t}{\tau}}$; $t > 0$

<u>For $t = 0^-$,</u>

Since the switch is in position "a" for a long time, that means steady state is reached and the inductor is a short circuit and has a current of $-8A$; $i(0^-) = -8A$

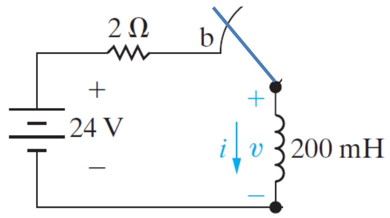


$\therefore\ i(0^+) = i(0^-) = -8A$

For $t > 0$,

The time constant (τ) is:

$$\tau = \frac{L}{R} = \frac{0.2}{2} = 0.1s$$

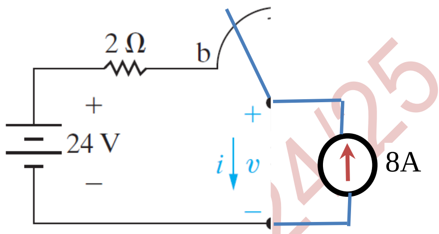


For $t = 0^+$,

The inductor is represented by a current source (of -8A) in parallel with an open circuit;

$$i(0^+) = i(0^-) = -8A$$

$$v(0^+) = 8(2) + 24 = 40V$$

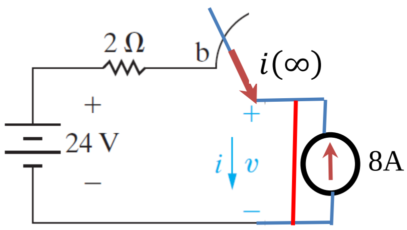


For $t = \infty$,

The inductor is represented by a current source (of -8A) in parallel with a short circuit; the current at the terminal of the inductor is found by Ohm's law;

$$i(\infty) = \frac{24}{2} = 12A$$

$$v(\infty) = 0V$$



a) Therefore,

$$i(t) = i(\infty) + \left(i(0^+) - i(\infty)\right)e^{-\frac{t}{\tau}} \quad ; t > 0$$

$$i(t) = 12 + (-8 - 12)e^{-\frac{t}{0.1}} \quad ; t \geq 0$$

$$i(t) = 12 - 20e^{-10t} \quad ; t \geq 0$$

b) The voltage across the inductor is:

$$v = L\frac{di}{dt}$$

$$= 0.2(200e^{-10t})$$

$$= 40e^{-10t}\,\text{V}, \quad t \geq 0^+.$$

The initial inductor voltage is

$$v(0^+) = 40\ \text{V}.$$

c) The time for the voltage to be equal to 24V is:

$$24 = 40e^{-10t}$$

for t:

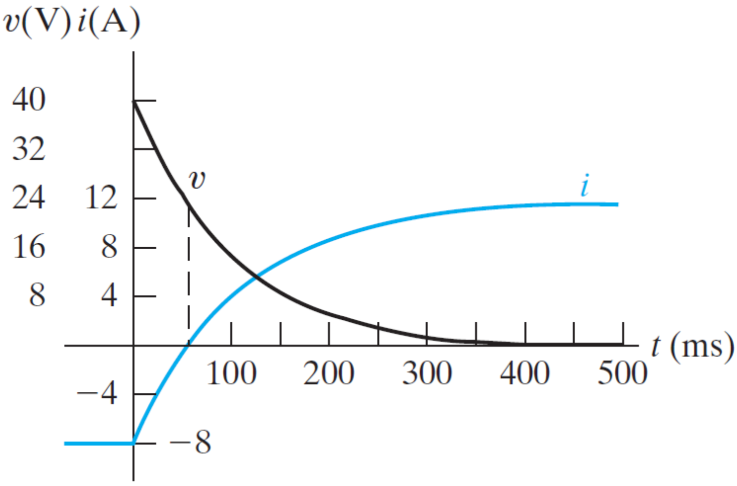
$$t = \frac{1}{10}\ln\frac{40}{24}$$

$$= 51.08 \times 10^{-3}$$

$$= 51.08 \text{ ms.}$$

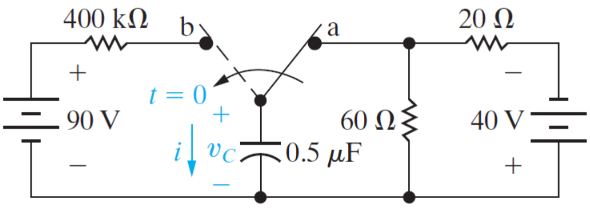
d) Yes; in the instant after the switch has been moved to position b, the inductor sustains a current of 8 A counterclockwise around the newly formed closed path. This current causes a 16 V drop across the resistor. This voltage drop adds to the source voltage, producing a 40 V across the inductor.

e) Plots of $i(t)$ and $v(t)$ versus t are shown below:



Example # 5:

The switch in the circuit shown in the figure has been in position "a" for a long time. At $t = 0$, the switch is moved to position "b".



a) What is the initial value of v_C?

b) What is the final value of v_C?

c) What is the time constant of the circuit when the switch is in position "b"?

d) What is the expression for $v_C(t)$ when ; $t \geq 0$?

e) What is the expression for $i(t)$ when $t \geq 0^+$?

f) How long after the switch is in position "b" does the capacitor voltage equal zero?

g) Plot $v_C(t)$ and $i(t)$ versus t.

Solution:

a) Initial voltage at the capacitor

- The first step is to find the initial voltage at the capacitor before moving the switch.
- The switch has been in position "a" for a long time, so the capacitor looks like an open circuit.
- Therefore, the voltage across the capacitor is the voltage across the 60Ω resistor.
- From the voltage divider rule, the voltage across the resistor 60Ω is:

$$V_C(0^-) = \frac{60}{60+20}(-40) = -30V$$

Thus, $V_C(0^+) = V_C(0^-) = -30V$

b) The final voltage at the capacitor is the same as the source voltage as the capacitor becomes open circuit at $t = \infty$; $V_C(\infty) = 90V$

c) The time constant is

$$\tau = RC$$
$$= (400 \times 10^3)(0.5 \times 10^{-6})$$
$$= 0.2 \text{ s.}$$

d) The capacitor voltage is:

$$v_C(t) = v_C(\infty) + \big(v_C(0^+) - v_C(\infty)\big)e^{-\frac{t}{\tau}}$$

$$\therefore v_C(t) = 90 + (-30 - 90)e^{-\frac{t}{0.2}}$$

$$\rightarrow v_C(t) = 90 - 120e^{-5t} \quad , t \geq 0$$

e) The capacitor current $i(t)$ when $t \geq 0^+$ is:

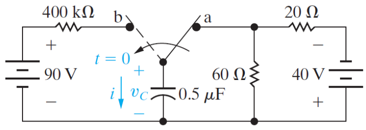
$$i_C = C\frac{dv_C}{dt}$$

$$i_C(t) = (0.5\mu)(-120(-5)e^{-5t}), \quad t \geq 0^+$$

$$i_C(t) = 300e^{-5t}\ \mu A; \quad t \geq 0^+$$

or find $i(0^+)$; the capacitor is replaced by a battery of -30V

$$i(0^+) = \frac{90--30}{400k} = 300\mu A$$



And find $i(\infty)$; the capacitor is replaced by an open circuit:

$$i(\infty) = 0$$

Note that, the time constant is the same as before.

Therefore,

$$i(t) = i(\infty) + \big(i(0^+) - i(\infty)\big)e^{-\frac{t}{\tau}} \quad ; t \geq 0^+$$

$$\therefore i(t) = 0 + (300\mu - 0)e^{-\frac{t}{0.2}} \quad ; t \geq 0^+$$

$$\rightarrow i(t) = 300e^{-5t} \quad \mu A\ ; t \geq 0^+$$

f) The time needed for the voltage at the capacitor to become zero is obtained by equating the capacitor voltage with zero.

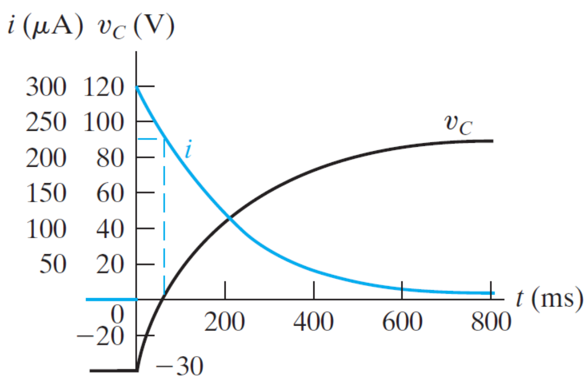
$$v_C(t) = 90 - 120e^{-5t} = 0$$

$$\rightarrow \quad 90 = 120e^{-5t} \rightarrow \quad e^{-5t} = \frac{90}{120}$$

$$-5t = ln\frac{3}{4} \quad \rightarrow \quad t = \frac{1}{5}ln\frac{4}{3}$$

$$\rightarrow t = 57.54ms$$

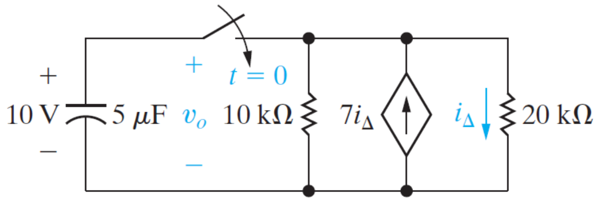
g) Plots of $v_C(t)$ and $i(t)$ versus t are:



6.3 Unbounded Response

Example # 6:

a) When the switch is closed in the circuit shown in the figure next, the voltage on the capacitor is 10 V. Find the expression for $v_o(t)$ for $t \geq 0$

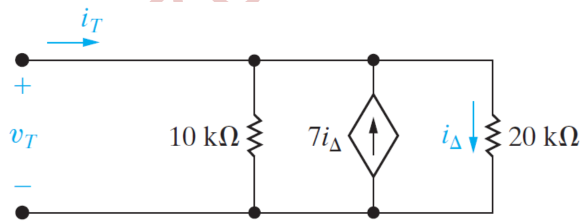


b) Assume that the capacitor short-circuits when its terminal voltage reaches 150 V. How many milliseconds elapse before the capacitor short circuits?

Solution:

a) To find the Thévenin's equivalent resistance with respect to the capacitor terminals, use the test source method (case 3).

The circuit becomes as shown next.



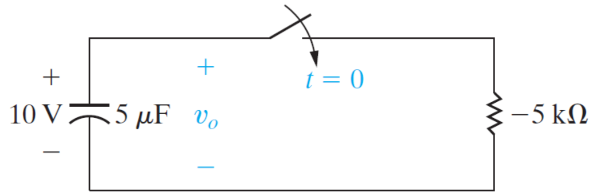
Applying KCL at the upper node yields:

$$i_T = \frac{v_T}{10} - 7\left(\frac{v_T}{20}\right) + \frac{v_T}{20}\ \text{mA}$$

Rearranging yields:

$$R_{\text{Th}} = \frac{v_T}{i_T} = -5\ \text{k}\Omega$$

The equivalent circuit is shown next.



Since R_{Th} is negative, the formula $\left(v_C(t) = v_C(\infty) + \left(v_C(0^+) - v_C(\infty)\right)e^{-\frac{t}{\tau}}\right)$ cannot be used, because at $t = \infty$, the exponential does not decay and becomes significant. The aforementioned formula is based on $v_C(\infty) = k_1$.

Thus, find the differential equation describing the circuit.

$$i_c + i_{R_{Th}} = 0$$

$$(5 \times 10^{-6})\frac{dv_o}{dt} - \frac{v_o}{5} \times 10^{-3} = 0.$$

Dividing by the coefficient of the first derivative yields:

$$\frac{dv_o}{dt} - 40v_o = 0.$$

The solution to the homogeneous differential equation is:

$$v_o(t) = k_2 e^{40t}$$

But, at $t = 0, v_o(0) = 10V$.

Thus,

$$v_o(0) = k_2 e^{40(0)} = 10$$

$$\rightarrow k_2 = 10V$$

$$v_o(t) = 10e^{40t} \ V\ ; \ \ t \geq 0$$

b) When $v_o(t) = 150V$, the time t_x is:

$$v_o(t_x) = 10e^{40t_x} = 150$$

$$e^{40t_x} = \frac{150}{10}$$

$$40t_x = \ln 15$$

$$t_x = 67.7ms$$

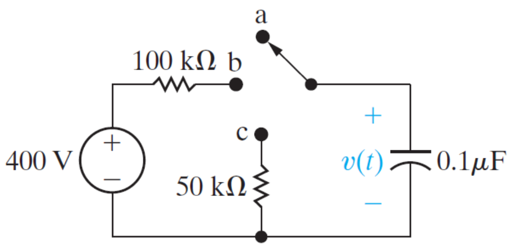
This type of response is called unbounded response, which may grow rather than decay, R_{Th} is negative, and the time constant is also negative; $\tau = \frac{1}{a} = \frac{1}{-40}$; resulting in an exponential increase!

6.4 Sequential Switching

Whenever switching occurs more than once in a circuit, the switching is called **sequential switching**.

Example # 7:

The uncharged capacitor in the circuit shown in the figure next is initially switched to terminal "a" of the three-position switch. At $t = 0$, the switch is moved to position "b", where it remains for 15ms. After the 15ms delay, the switch is moved to position "c", where it remains indefinitely.

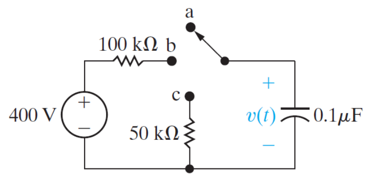


a) Derive the numerical expression for the voltage across the capacitor.

b) Plot the capacitor voltage versus time.

c) When will the voltage on the capacitor equal 200 V?

Solution:



a) The capacitor voltage

For $0 \leq t \leq 15ms$,

Initially discharged capacitor

$$v(0^+) = v(0^-) = 0;$$

If the switch was left indefinitely in position "b", the capacitor would charge to $v(\infty') = 400V$

The time constant for the circuit during this period is:

$$\tau = RC = 100k(0.1\mu) = 10ms$$

Therefore,

$$v(t) = v(\infty') + \left(v(0^+) - v(\infty')\right)e^{-\frac{t}{\tau}}$$

$$v(t) = 400 + (0 - 400)e^{-\frac{t}{0.01}}$$

$$\therefore v(t) = 400(1 - e^{-100t}) \text{ V}; \quad 0 \leq t \leq 15ms$$

After the switch has been in this position for 15ms, the voltage on the capacitor will be:

$$v(t = 15ms^-) = 400\left(1 - e^{-100(15m)}\right) = 310.75\text{V} = v(t = 15ms^+)$$

Thus, the initial voltage at the capacitor for the second period is $v_C(t = 15ms^+) = 310.75V$

For $t \geq 15ms$; $t_0 = 15ms$!

$$v(t = 15ms^+) = 310.75V$$

If the switch stays in position "c" indefinitely, then the capacitor will discharge through the $50k\Omega$ resistor, hence, $v(\infty) = 0V$

The time constant for the circuit during the second period is:

$$\tau_2 = R_2C = 50k(0.1\mu) = 5ms$$

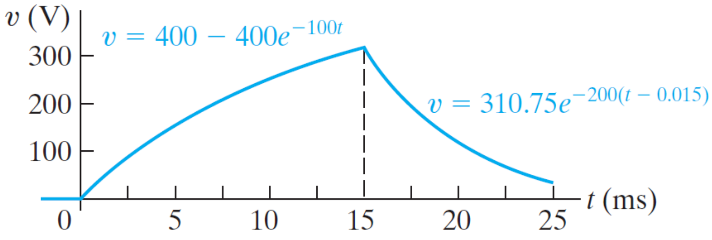
Therefore,

$$v(t) = v(\infty) + \left(v(15ms^+) - v(\infty)\right)e^{-\frac{(t-t_0)}{\tau_2}}$$

$$v(t) = 0 + (310.75 - 0)e^{-\frac{(t-0.015)}{0.005}}$$

$$\therefore v(t) = 310.75e^{-200(t-0.015)} \text{ V}; \quad 15ms \leq t$$

b) $v(t)$ plot versus time is shown in the figure below:



c) The plot in the figure above reveals that the capacitor voltage will equal 200V at two different times; once in the interval between 0 and 15ms and once after 15ms.

The first time can be found by solving:

$$200 = 400 - 400e^{-100t_1},$$

$$t_1 = 6.93 \text{ ms}.$$

The second time can be found by solving:

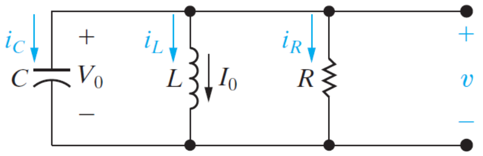
$$200 = 310.75e^{-200(t_2 - 0.015)};$$

$$t_2 = 17.20 \text{ ms}.$$

Chapter 7

Natural and Step Responses of RLC Circuits

7.1 Natural Response of Parallel RLC Circuits



- The voltage v is common to all parallel elements.
- Assuming that, the initial voltage at the capacitor is V_0 and the initial current in the inductor is I_0, and applying KCL at the top node yield:

$$\frac{v}{R} + \frac{1}{L}\int_0^t v\,d\tau + I_0 + C\frac{dv}{dt} = 0$$

Differentiating with respect to time, t, and noting that I_0 is constant, yield:

$$\frac{1}{R}\frac{dv}{dt} + \frac{v}{L} + C\frac{d^2v}{dt^2} = 0$$

Dividing the latter equation by the capacitance C and arranging the derivatives in descending order yields:

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{v}{LC} = 0$$

which is an ordinary, second-order differential equation with constant coefficients.

- The circuit is called a second order circuit; it contains both an inductor and a capacitor.
- Assuming that the solution is of the form:

$$v = Ae^{st}$$

The first derivative of the solution is:

$$\frac{dv}{dt} = Ase^{st}$$

The second derivative of the solution is:

$$\frac{d^2v}{dt^2} = As^2e^{st}$$

- Substituting the solution and its derivatives in the second order differential equation yields:

$$As^2e^{st} + \frac{As}{RC}e^{st} + \frac{Ae^{st}}{LC} = 0$$

and taking Ae^{st} as a common factor to all terms;

$$Ae^{st}\left(s^2 + \frac{s}{RC} + \frac{1}{LC}\right) = 0$$

Noting that, A cannot be zero, otherwise there is no solution, or e^{st} cannot be zero for any finite value of st, thus the parenthetical term, which is called a *characteristic equation*, is zero;

$$s^2 + \frac{s}{RC} + \frac{1}{LC} = 0$$

- The roots of this quadratic equation determine the mathematical character of $v(t)$, the two roots are:

$$s = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = -\frac{b}{2a} \pm \sqrt{\frac{b^2}{4a^2} - \frac{c}{a}}$$

$$s_1 = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

$$s_2 = -\frac{1}{2RC} - \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

- Therefore, the solutions to the differential equations, the voltages, are:

$$v_1 = A_1e^{s_1t}$$

$$v_2 = A_2e^{s_2t}$$

- The sum of both solutions is also a, general, solution;

$$v = v_1 + v_2 = A_1e^{s_1t} + A_2e^{s_2t}$$

$$v = A_1e^{s_1t} + A_2e^{s_2t} \qquad t \geq 0$$

- The initial conditions I_0 & V_0 determine the values of the constants A_1 & A_2.
- s_1 & s_2 are called the complex frequencies measured in rad/s or Hz.

- Let

$\boldsymbol{\alpha = \frac{1}{2RC}}$: The *neper frequency* [rad/s]

and $\boldsymbol{\omega_0 = \frac{1}{\sqrt{LC}}}$: The *resonant radian frequency* [rad/s]

Thus, the complex frequencies are:

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

Three types of natural response are possible:

1- Overdamped response

$\omega_0^2 < \alpha^2$: The roots, s_1& s_2, are real, negative and distinct

2- Underdamped response

$\omega_0^2 > \alpha^2$: The roots, s_1& s_2, are complex conjugates

3- Critically damped response

$\omega_0^2 = \alpha^2$: The roots, s_1& s_2, are real and equal

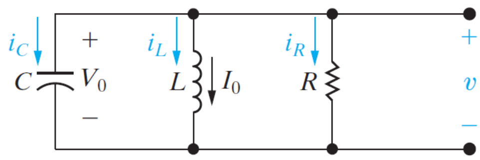
7.1.1 Overdamped Natural Response

- In this case, $\omega_0^2 < \alpha^2$, the roots, s_1& s_2, are real, negative and distinct;

$|s_1| < |s_2|$;

because s_2 is more negative than s_1!

$\tau_1 = \frac{1}{|s_1|}$ and $\tau_2 = \frac{1}{|s_2|}$



- Thus, $\tau_1 > \tau_2$, and the second part of the solution, $A_2e^{s_2t}$, dies faster than the first part , $A_1e^{s_1t}$.
- The response is:

$$v(t) = A_1e^{s_1t} + A_2e^{s_2t} \quad , \; t \geq 0s$$

- At $t = 0^+s$, $v(0^+) = A_1 + A_2 = V_0$

where V_0 is the initial voltage at the capacitor.

- Since $i_c(t) = C\frac{dv(t)}{dt}$

The current in the capacitor at $t = 0^+$ is:

$$i_c(t) = C(A_1s_1e^{s_1t} + A_2s_2e^{s_2t})$$

$$\Rightarrow \quad i_c(0^+) = C(A_1s_1 + A_2s_2)$$

- But, from KCL at the upper node at $t = 0^+$,

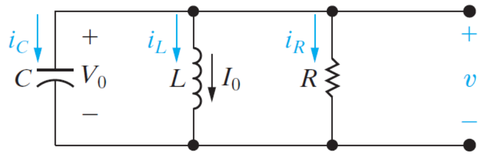
$$i_c(0^+) = C\frac{dv(0^+)}{dt} = C(A_1 s_1 + A_2 s_2) = -I_0 - \frac{V_0}{R}$$

Therefore, $\frac{dv(0^+)}{dt} = A_1 s_1 + A_2 s_2$

$$\frac{dv(0^+)}{dt} = \frac{1}{C}\left(-I_0 - \frac{V_0}{R}\right)$$

7.1.2 Underdamped Natural Response

In this case, $\omega_0^2 > \alpha^2$, the roots of the characteristic equation, s_1 & s_2, are complex conjugates;



$$s_1 = -\alpha + \sqrt{-(\omega_0^2 - \alpha^2)}$$

$$= -\alpha + j\sqrt{\omega_0^2 - \alpha^2}$$

$$= -\alpha + j\omega_d$$

$$s_2 = -\alpha - j\omega_d,$$

where $$\omega_d = \sqrt{\omega_0^2 - \alpha^2}$$

ω_d is called the ***damped radian frequency***

Again $\omega_0 = \frac{1}{\sqrt{LC}}$ is the resonant frequency

α is called the ***damping factor or damping coefficient***, $\alpha = \frac{1}{2RC}$, it shows how quickly the oscillations subside.

- The response is:

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \quad , \; t \geq 0s$$

$$v(t) = A_1 e^{(-\alpha + j\omega_d)t} + A_2 e^{-(\alpha + j\omega_d)t} \quad , \; t \geq 0s$$

$$= A_1 e^{-\alpha t} e^{j\omega_d t} + A_2 e^{-\alpha t} e^{-j\omega_d t}$$

Using Euler's identity

$$e^{\pm j\theta} = \cos\theta \pm j\sin\theta$$

Thus, $$v(t) = e^{-\alpha t}(A_1 \cos\omega_d t + jA_1 \sin\omega_d t + A_2 \cos\omega_d t - jA_2 \sin\omega_d t)$$

$$= e^{-\alpha t}[(A_1 + A_2)\cos\omega_d t + j(A_1 - A_2)\sin\omega_d t].$$

And $v(t)$ can be expressed as:

$$v(t) = e^{-\alpha t}(B_1 \cos \omega_d t + B_2 \sin \omega_d t)$$

$$= B_1 e^{-\alpha t} \cos \omega_d t + B_2 e^{-\alpha t} \sin \omega_d t$$

Note that, B_1 & B_2 are real, not complex, because $v(t)$ is a real function

A_1 & A_2 are complex conjugates

To be able to plot the response, the voltage $v(t)$ can be re-expressed in terms of a sine (or a cosine) function only;

$$\boldsymbol{v(t) = e^{-\alpha t}(E \sin(\omega_d + \theta))}$$

where $E = \sqrt{B_1^2 + B_2^2}$ and $\theta = \tan^{-1} \frac{B_1}{B_2}$

- From the initial condition, at $t = 0^+ s$,

$$v(0^+) = B_1 = V_0$$

where V_0 is the initial voltage at the capacitor.

- Since $i_c(t) = C \frac{dv(t)}{dt}$

The current in the capacitor at $t = 0^+$ is:

$$i_c(0^+) = C \frac{dv(0^+)}{dt} = C(-\alpha B_1 + \omega_d B_2)$$

Therefore,

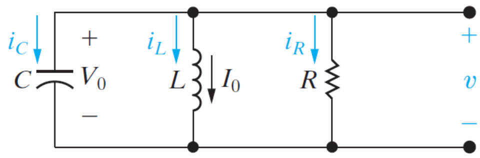
$$\frac{i_c(0^+)}{C} = \frac{dv(0^+)}{dt} = -\alpha B_1 + \omega_d B_2$$

$$\frac{dv(0^+)}{dt} = \frac{1}{C}\left(-I_0 - \frac{V_0}{R}\right)$$

Note that, since $= \frac{1}{2RC}$**, if $R = \infty$, then $\alpha = 0$**, there will be no damping in the circuit and the oscillations (ringing) will be sustained, the circuit will be called ***lossless circuit***.

7.1.3 Critically Damped Natural Response

In this case, $\omega_0^2 = \alpha^2$, or $\omega_0 = \alpha$, the response is on the verge of oscillations, the roots of the characteristic equation, $s_1 \& s_2$, are equal;



$$s_1 = s_2 = -\alpha = -\frac{1}{2RC}$$

The response is:

$$v(t) = D_1 te^{-\alpha t} + D_2 e^{-\alpha t} \quad , \ t \geq 0s$$

- From the initial condition, at $t = 0^+ s$,

$$v(0^+) = D_2 = V_0 \quad ; \qquad V_0 \text{ is the initial voltage at the capacitor.}$$

- Since,

$$i_c(t) = C\frac{dv(t)}{dt}$$

The current in the capacitor at $t = 0^+$:

$$i_c(0^+) = C\frac{dv(0^+)}{dt} = C(D_1 - \alpha D_2)$$

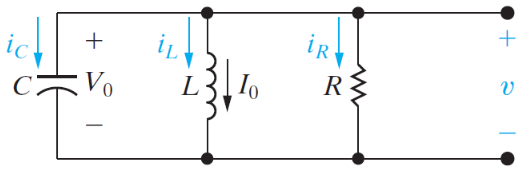
Therefore,

$$\frac{i_c(0^+)}{C} = \frac{dv(0^+)}{dt} = D_1 - \alpha D_2$$

$$\frac{dv(0^+)}{dt} = \frac{1}{C}\left(-I_0 - \frac{V_0}{R}\right)$$

Example # 1:

a) Find the roots of the characteristic equation that govern the transient behaviour of the voltage shown in the figure if $R = 200\Omega, L = 50mH$ and $C = 0.2\mu F$



b) Will the response be overdamped, underdamped, or critically damped?

c) Repeat (a) and (b) for $R = 312.5\Omega$

d) What value of R causes the response to be critically damped?

Solution:

a) For the given values of R, L, and C,

$$\alpha = \frac{1}{2RC} = \frac{10^6}{(400)(0.2)} = 1.25 \times 10^4 \text{ rad/s},$$

$$\omega_0^2 = \frac{1}{LC} = \frac{(10^3)(10^6)}{(50)(0.2)} = 10^8 \text{ rad}^2/\text{s}^2.$$

The roots of the characteristic equation are:

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

Therefore,

$$s_1 = -1.25 \times 10^4 + \sqrt{1.5625 \times 10^8 - 10^8}$$

$$= -12{,}500 + 7500 = -5000 \text{ rad/s},$$

$$s_2 = -1.25 \times 10^4 - \sqrt{1.5625 \times 10^8 - 10^8}$$

$$s_2 = -20{,}000\, rad/s$$

b) The voltage response is overdamped because $\omega_0^2 < \alpha^2$.

c) For $R = 312.5\ \Omega$,

$$\alpha = \frac{10^6}{(625)(0.2)} = 8000 \text{ rad/s},$$

$$\alpha^2 = 64 \times 10^6 = 0.64 \times 10^8 \text{ rad}^2/\text{s}^2.$$

As ω_0^2 remains at 10^8 rad^2/s^2,

$$s_1 = -8000 + j6000 \text{ rad/s},$$

$$s_2 = -8000 - j6000 \text{ rad/s}.$$

The voltage response is underdamped because $\omega_0^2 > \alpha^2$

d) For critical damping, $\alpha^2 = \omega_0^2$, so

$$\left(\frac{1}{2RC}\right)^2 = \frac{1}{LC} = 10^8,$$

or

$$\frac{1}{2RC} = 10^4,$$

and

$$R = \frac{10^6}{(2 \times 10^4)(0.2)} = 250\ \Omega.$$

Example # 2:

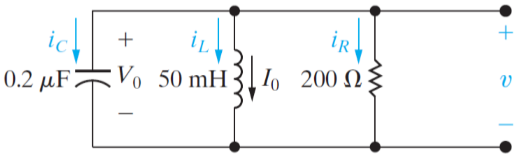
For the circuit shown in the figure, if $v(0^+) = 12V$ and $i_L(0^+) = 30mA$, then

a) Find the initial current in each branch of the circuit.

b) Find the initial value of dv/dt.

c) Find the expression for $v(t)$.

d) Sketch $v(t)$ in the interval $0 \le t \le 250$ ms.



Solution:

a) The inductor's current is continuous;

$$i_L(0^-) = i_L(0) = i_L(0^+) = 30 \text{ mA}.$$

The initial current in the resistor is;

$$i_R(0^+) = \frac{v(0^+)}{R} = \frac{12}{200} = 60mA$$

The initial current in the capacitor can be found by applying KCL at the upper node;

$$i_C(0^+) = -i_L(0^+) - i_R(0^+)$$
$$= -90 \text{ mA}.$$

b) The capacitor's current is:

$$i_c(t) = C\frac{dv(t)}{dt}$$

$$i_c(0^+) = C\frac{dv(0^+)}{dt}$$

$$\frac{dv(0^+)}{dt} = \frac{i_c(0^+)}{C}$$

$$\frac{dv(0^+)}{dt} = \frac{-90 \times 10^{-3}}{0.2 \times 10^{-6}} = -450 \text{ kV/s}.$$

c) The roots of the characteristic equation are:

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

where

$$\alpha = \frac{1}{2RC} = \frac{10^6}{(400)(0.2)} = 1.25 \times 10^4 \text{ rad/s},$$

$$\omega_0^2 = \frac{1}{LC} = \frac{(10^3)(10^6)}{(50)(0.2)} = 10^8 \text{ rad}^2/\text{s}^2.$$

Thus,

$$s_1 = -1.25 \times 10^4 + \sqrt{1.5625 \times 10^8 - 10^8}$$

$$= -12{,}500 + 7500 = -5000 \text{ rad/s},$$

$$s_2 = -1.25 \times 10^4 - \sqrt{1.5625 \times 10^8 - 10^8}$$

$$= -12{,}500 - 7500 = -20{,}000 \text{ rad/s}.$$

The response

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \quad , \ t \geq 0s$$

$$v(t) = A_1 e^{-5{,}000t} + A_2 e^{-20{,}000t} \quad , \ t \geq 0s$$

But $\quad v(0^+) = A_1 + A_2 = V_0 \qquad (1)$

And $\quad \frac{dv(0^+)}{dt} = A_1 s_1 + A_2 s_2$

$$-450k = A_1(-5{,}000) + A_2(-20{,}000) \qquad (2)$$

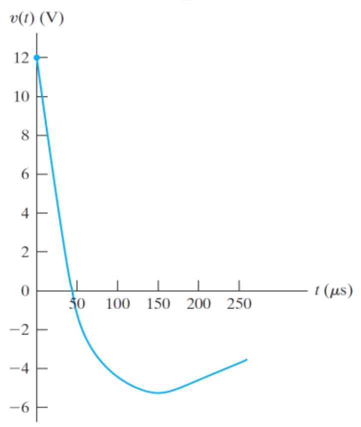
Solving equations (1) and (2) yields:

$$A_1 = -14 \text{ V and } A_2 = 26 \text{ V}$$

Therefore,

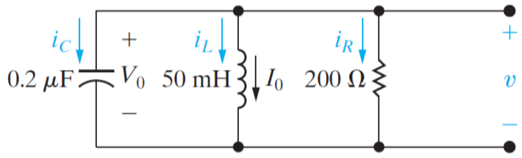
$$v(t) = (-14e^{-5000t} + 26e^{-20{,}000t}) \text{ V}, \quad t \geq 0.$$

d) The plot of $v(t)$ for $0s \leq t \leq 250ms$



Example # 3:

Find the branch currents i_R, i_L, and i_C for $t \geq 0s$ in Example 2.



Solution:

The resistor's current is:

$$i_R(t) = \frac{v(t)}{R}$$

$$i_R(t) = \frac{v(t)}{200} = (-70e^{-5000t} + 130e^{-20,000t})\text{ mA}, \quad t \geq 0.$$

The inductor's current can be found either from the integral

$$i_L(t) = \frac{1}{L}\int_0^t v_L(x)\,dx + I_0$$

or from KCL

$$i_R + i_L + i_C = 0$$

where

$$i_C(t) = C\frac{dv}{dt}$$

$$= 0.2 \times 10^{-6}(70,000e^{-5000t} - 520,000e^{-20,000t})$$

$$= (14e^{-5000t} - 104e^{-20,000t})\text{ mA}, \quad t \geq 0^+.$$

Therefore,

$$i_L(t) = -i_R(t) - i_C(t)$$

$$= (56e^{-5000t} - 26e^{-20,000t})\text{ mA}, \quad t \geq 0.$$

Example # 4:

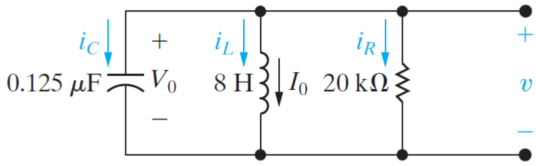
In the circuit shown in the figure, $V_0 = 0$, and $I_0 = -12.25$ mA.

a) Calculate the roots of the characteristic equation.

b) Calculate v and dv/dt at $t = 0^+$.

c) Calculate the voltage response for $t \geq 0$.

d) Plot $v(t)$ versus t for the time interval $0 \leq t \leq 11$ ms.



Solution

a)

$$\alpha = \frac{1}{2RC} = \frac{10^6}{2(20)10^3(0.125)} = 200 \text{ rad/s},$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \sqrt{\frac{10^6}{(8)(0.125)}} = 10^3 \text{ rad/s},$$

Since $\omega_0^2 > \alpha^2$,, the response is underdamped;

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} = \sqrt{10^6 - 4 \times 10^4} = 100\sqrt{96}$$

$$= 979.80 \text{ rad/s},$$

$$s_1 = -\alpha + j\omega_d = -200 + j979.80 \text{ rad/s},$$

$$s_2 = -\alpha - j\omega_d = -200 - j979.80 \text{ rad/s}.$$

b) From initial conditions,

$$v(0) = v(0^+) = V_0 = 0$$

Since $V_0 = 0$, then $i_R(0^+) = 0$, and $i_C(0^+) = -i_L(0^+)$;

$$i_C(0^+) = -(-12.25) = 12.25 \text{ mA}$$

and

$$\frac{dv(0^+)}{dt} = \frac{i_c(0^+)}{C}$$

$$\frac{dv(0^+)}{dt} = \frac{(12.25)(10^{-3})}{(0.125)(10^{-6})} = 98{,}000 \text{ V/s}$$

c) To find the coefficients,

$$v(0^+) = B_1 = V_0 = 0$$

and

$$\frac{dv(0^+)}{dt} = -\alpha B_1 + \omega_d B_2$$

$$98{,}000 = -\alpha(0) + \omega_d B_2$$

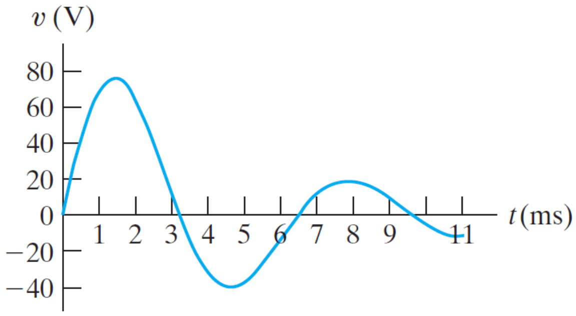
Thus,

$$B_2 = \frac{98{,}000}{\omega_d} \approx 100 \text{ V}$$

The natural underdamped response is:

$$v(t) = 100e^{-200t} \sin 979.80t \text{ V}, \quad t \geq 0.$$

d) The plot the response for the first 11ms, eventually it approaches its final value which is zero.

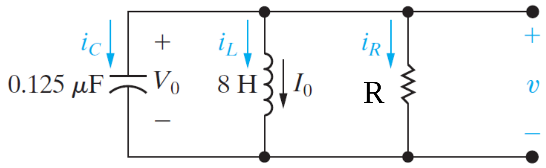


Example # 5:

a) For the circuit in the figure shown next, find the value of R that results in a critically damped voltage response.

b) Calculate $v(t)$ for $t \geq 0$.

c) Plot $v(t)$ versus t for $0 \leq t \leq 7$ ms.



Solution:

a) It was calculated before, the resonant frequency is:

$$\omega_0^2 = 10^6$$

But, for critically damped response, $\omega_0 = \alpha$, therefore

$$\alpha = 10^3 = \frac{1}{2RC}$$

$$R = \frac{10^6}{(2000)(0.125)} = 4000\ \Omega$$

b) From the previous example,

$$v(0^+) = 0 \quad \text{and} \quad dv(0^+)/dt = 98{,}000\ \text{V/s}$$

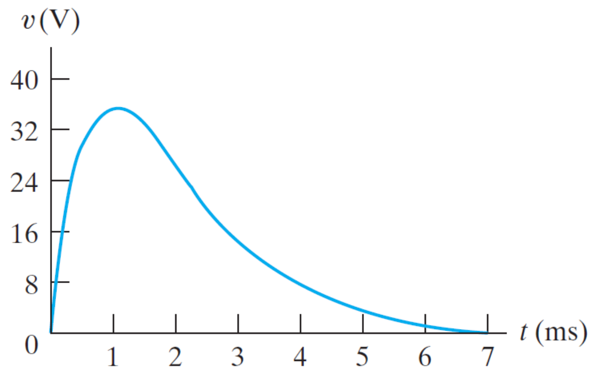
yielding

$$D_2 = 0 \text{ and } D_1 = 98{,}000\ \text{V/s}$$

Thus the natural response is:

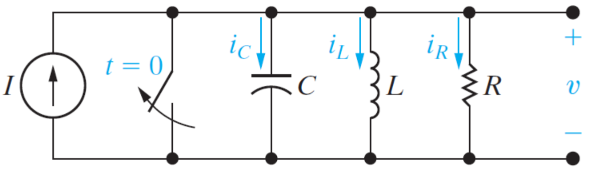
$$v(t) = 98{,}000te^{-1000t}\ \text{V}, \quad t \geq 0.$$

c) The plot of the response for $0 \leq t \leq 7ms$ is shown in the figure below.



7.2 The Step Response of a Parallel RLC Circuit

Consider the circuit shown in the figure next



Applying KCL at the upper node yields:

$$i_L + i_R + i_C = I$$

or

$$i_L + \frac{v}{R} + C\frac{dv}{dt} = I$$

But,

$$v = L\frac{di_L}{dt}$$

and

$$\frac{dv}{dt} = L\frac{d^2 i_L}{dt^2}$$

Substituting for v and its first derivative in the differential equation yields,

$$i_L + \frac{L}{R}\frac{di_L}{dt} + LC\frac{d^2 i_L}{dt^2} = I$$

Rearranging yields second order nonhomogeneous differential equation;

$$\frac{d^2 i_L}{dt^2} + \frac{1}{RC}\frac{di_L}{dt} + \frac{i_L}{LC} = \frac{I}{LC}$$

In general, the solution of the second order nonhomogeneous differential equation is of the form:

$$\boldsymbol{x(t) = X_f + x_n(t)}$$

$x(t)$ could be: $v(t) = V_f + v_n(t)$

or $i(t) = I_f + i_n(t)$

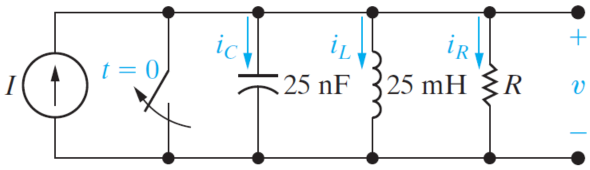
where $\boldsymbol{X_f}$ ***is the final value*** of the variable and is called the forced response;

For DC circuits, *to find the final value X_f, set the inductor as a short circuit and the capacitor as an open circuit.*

Whilst, *$X_n(t)$ is a function of the same form as **the natural response*** when the sources are deactivated (an independent current source is open circuit, and an independent voltage source is short circuit).

Example # 6:

The initial energy stored in the circuit in the figure is zero. At $t = 0$, a dc current source of 24 mA is applied to the circuit. The value of the resistor is 400 Ω.



a) What is the initial value of i_L?

b) What is the initial value of di_L/dt?

c) What are the roots of the characteristic equation?

d) What is the numerical expression for $i_L(t)$ when $t \geq 0$?

Solution:

a) Since the initial stored energy is zero, the inductor's current is:

$$i_L(0^+) = i_L(0^-) = 0A$$

b) Since $v_C(0^+) = v_C(0^-) = 0V = v(0^+)$

and because

$$v = L di_L/dt$$

$$\frac{di_L}{dt}(0^+) = 0$$

c) To find the roots, calculate

$$\omega_0^2 = \frac{1}{LC} = \frac{10^{12}}{(25)(25)} = 16 \times 10^8,$$

$$\alpha = \frac{1}{2RC} = \frac{10^9}{(2)(400)(25)} = 5 \times 10^4 \text{ rad/s},$$

or

$$\alpha^2 = 25 \times 10^8$$

Because $\omega_0^2 < \alpha^2$, the roots, s_1 & s_2, are real, negative and distinct (and the response is overdamped);

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

$$s_1 = -5 \times 10^4 + 3 \times 10^4 = -20{,}000 \text{ rad/s}$$

$$s_2 = -5 \times 10^4 - 3 \times 10^4 = -80{,}000 \text{ rad/s}$$

d) i_L is overdamped and has the form:

$$i_L = I_f + A_1' e^{s_1 t} + A_2' e^{s_2 t}$$

To find A_1' and A_2' solve the following equations obtained satisfying the initial conditions;

$$i_L(0) = I_f + A_1' + A_2' = 0,$$
$$\frac{di_L}{dt}(0) = s_1 A_1' + s_2 A_2' = 0.$$

which yields,

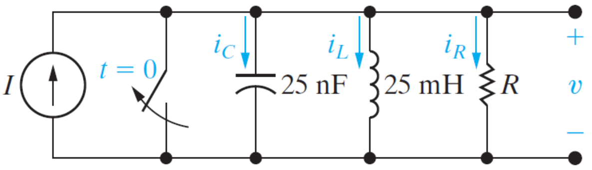
$$A_1' = -32 \text{ mA} \quad \text{and} \quad A_2' = 8 \text{ mA}$$

Therefore, the inductor's current is:

$$i_L(t) = (24 - 32e^{-20{,}000t} + 8e^{-80{,}000t}) \text{ mA}, \quad t \geq 0$$

Example # 7:

If the resistor in Example # 6 was increased to 625Ω then find the inductor's current $i_L(t)$ for $t \geq 0s$.



Solution:

Since L & C are the same as in Example # 6, then the resonant frequency does not change;

$$\omega_0^2 = 16 \times 10^8$$

But α, $\alpha = \frac{1}{2RC}$, decreases with increasing R to 3.2×10^4 rad/s

Thus the roots of the characteristic equation are:

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

$$s_1 = -3.2 \times 10^4 + j2.4 \times 10^4 \text{ rad/s}$$

$$s_2 = -3.2 \times 10^4 - j2.4 \times 10^4 \text{ rad/s}$$

The response is underdamped and $i_L(t)$ is:

$$i_L(t) = I_f + B_1' e^{-\alpha t} \cos \omega_d t + B_2' e^{-\alpha t} \sin \omega_d t.$$

Here, α is 32,000 rad/s, ω_d is 24,000 rad/s, and I_f is 24 mA.

The constants can be found from initial condition equations for underdamped response:

$$i_L(0) = I_f + B_1' = 0,$$

$$\frac{di_L}{dt}(0) = \omega_d B_2' - \alpha B_1' = 0.$$

Then,

$$B_1' = -24 \text{ mA}$$

and

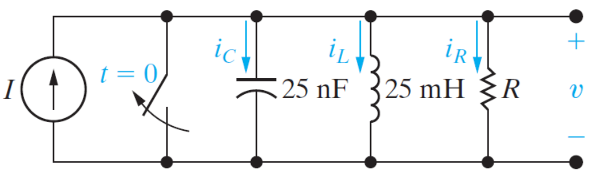
$$B_2' = -32 \text{ mA}.$$

The numerical solution for $i_L(t)$ is

$$i_L(t) = (24 - 24e^{-32{,}000t} \cos 24{,}000t - 32e^{-32{,}000t} \sin 24{,}000t) \text{ mA}, \quad t \geq 0.$$

Example # 8:

If the resistor in Example # 6 was increased to 500Ω then find the inductor's current $i_L(t)$ for $t \geq 0s$.



Solution:

Since L & C are the same as in Example # 6, the resonant radian frequency does not change;

$$\omega_0^2 = 16 \times 10^8$$

But α, $\alpha = \frac{1}{2RC}$, then it becomes $4 \times 10^4 \text{ s}^{-1}$

Since $\omega_0^2 = \alpha^2$, the roots, s_1 & s_2, are real and equal, and the response is critically damped.

The response form for a critically damped $i_L(t)$ is:

$$i_L(t) = I_f + D_1' t e^{-\alpha t} + D_2' e^{-\alpha t}.$$

The constants can be found from initial condition equations for critically damped response;

$$i_L(0) = I_f + D_2' = 0,$$

$$\frac{di_L}{dt}(0) = D_1' - \alpha D_2' = 0.$$

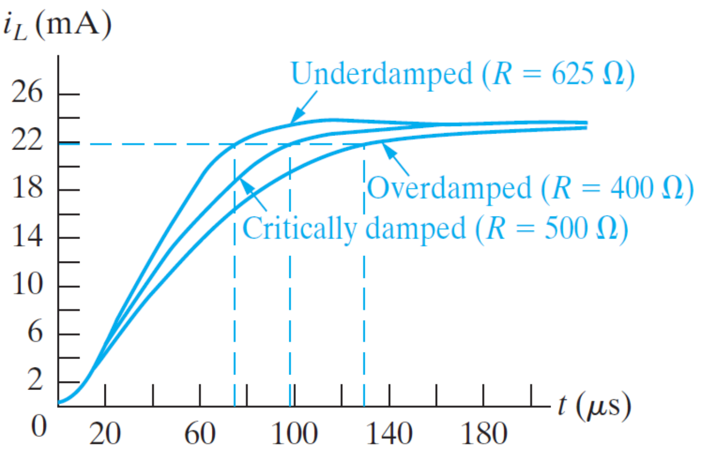
Thus,

$$D_1' = -960{,}000 \text{ mA/s} \quad \text{and} \quad D_2' = -24 \text{ mA}.$$

The numerical expression for $i_L(t)$ is

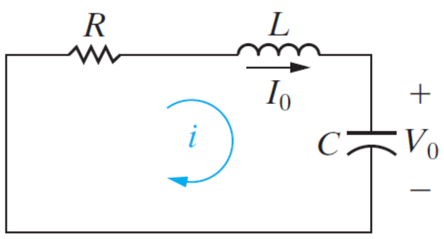
$$i_L(t) = (24 - 960{,}000te^{-40{,}000t} - 24e^{-40{,}000t}) \text{ mA}, \; t \geq 0.$$

A plot of the three types of response is shown in the figure below



7.3 The Natural and Step Responses of Series RLC Circuits

For the series RLC circuit shown in the Figure next, apply KVL,



$$Ri + L\frac{di}{dt} + \frac{1}{C}\int_0^t i\,d\tau + V_0 = 0$$

Differentiating with respect to time, t, yields:

$$R\frac{di}{dt} + L\frac{d^2i}{dt^2} + \frac{i}{C} = 0$$

Rearranging yields:

$$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0$$

The roots of the characteristic equation are

$$s_{1,2} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}},$$

or

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}.$$

The neper frequency (α) for the series RLC circuit is

$$\alpha = \frac{R}{2L} \text{ rad/s},$$

and the expression for the resonant radian frequency is

$$\omega_0 = \frac{1}{\sqrt{LC}} \text{ rad/s}.$$

Three types of natural response are possible:

4- Overdamped response

$\omega_0^2 < \alpha^2$: The roots, s_1& s_2, are real, negative and distinct

5- Underdamped response

$\omega_0^2 > \alpha^2$: The roots, s_1& s_2, are complex conjugates

6- Critically damped response

$\omega_0^2 = \alpha^2$: The roots, s_1& s_2, are real and equal

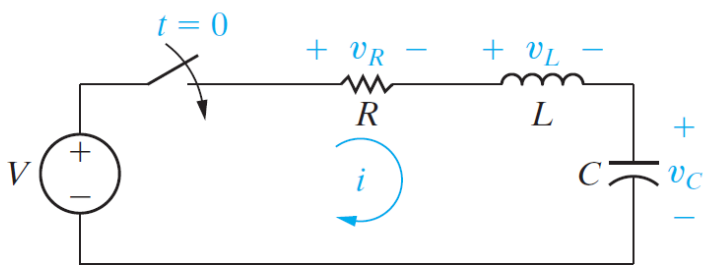
The ***three possible solutions*** for the current:

$$i(t) = A_1e^{s_1t} + A_2e^{s_2t} \quad \text{(overdamped)},$$

$$i(t) = B_1e^{-\alpha t}\cos\omega_d t + B_2e^{-\alpha t}\sin\omega_d t \quad \text{(underdamped)},$$

$$i(t) = D_1te^{-\alpha t} + D_2e^{-\alpha t} \quad \text{(critically damped)}.$$

The step response is also as that of parallel RLC circuits:



Applying KVL for the loop, after switching, yields:

$$V = Ri + L\frac{di}{dt} + v_C.$$

The current (i) is related to the capacitor voltage (v_C) by

$$i = C\frac{dv_C}{dt},$$

from which

$$\frac{di}{dt} = C\frac{d^2v_C}{dt^2}$$

Substituting for the current and its derivative in the KVL equation yields:

$$\frac{d^2 v_C}{dt^2} + \frac{R}{L}\frac{dv_C}{dt} + \frac{v_C}{LC} = \frac{V}{LC}$$

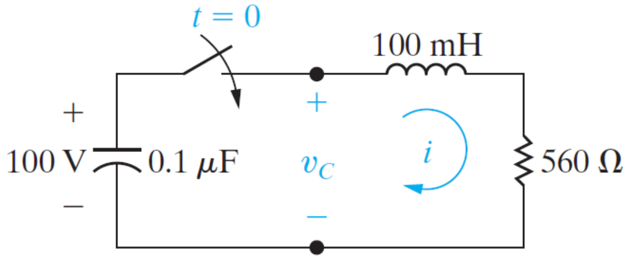
For the voltage across the capacitor, the response has one of the following forms:

$$v_C = V_f + A_1' e^{s_1 t} + A_2' e^{s_2 t} \text{ (overdamped)},$$
$$v_C = V_f + B_1' e^{-\alpha t}\cos\omega_d t + B_2' e^{-\alpha t}\sin\omega_d t \text{ (underdamped)},$$
$$v_C = V_f + D_1' t e^{-\alpha t} + D_2' e^{-\alpha t} \text{ (critically damped)},$$

where V_f is the final value of the voltage across the capacitor, and it is for this circuit the same as the DC source; V.

Example # 9:

The $0.1\mu F$ capacitor in the circuit shown in the figure next is charged to 100V. At $t = 0s$, the capacitor is discharged through a series combination of a 100mH inductor and a 560Ω resistor.



a) Find $i(t)$ for $t \geq 0s$

b) Find $v_C(t)$ for $t \geq 0s$

Solution:

a) The first step to find $i(t)$ is to calculate the roots of the characteristic equation; the resonant radian frequency:

$$\omega_0^2 = \frac{1}{LC} = \frac{(10^3)(10^6)}{(100)(0.1)} = 10^8,$$

and the neper frequency:

$$\alpha = \frac{R}{2L} = \frac{560}{2(100)} \times 10^3 = 2800 \text{ rad/s}.$$

Thus,

$$\alpha^2 = 7.84 \times 10^6$$
$$= 0.0784 \times 10^8.$$

Since $\omega_0^2 > \alpha^2$, $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$ and the response in underdamped and has the form:

$$i(t) = B_1 e^{-\alpha t} \cos \omega_d t + B_2 e^{-\alpha t} \sin \omega_d t,$$

where $\alpha = 2800$ rad/s and $\omega_d = 9600$ rad/s.

Note that, the response is natural response, as the excitation in the circuit is zero.

The values of the constants can be found from the initial conditions;

When the switch was open, the inductors current was zero;

$$i(0^-) = 0 = i(0^+)$$

Therefore,

$$i(0) = 0 = B_1$$

Immediately after closing the switch, the voltage drop across the resistor is zero (because the current is zero), thus,

$$v_L(0^+) = L\frac{di(0^+)}{dt} = v_C(0^+)$$

But,

$$v_C(0^+) = v_C(0^-) = 100V$$

Therefore,

$$L\frac{di(0^+)}{dt} = 100$$

$$\frac{di(0^+)}{dt} = \frac{100}{0.1} = 1000A/s$$

Because $B_1 = 0$,

$$\frac{di}{dt} = 400B_2 e^{-2800t}(24 \cos 9600t - 7 \sin 9600t)$$

Thus,

$$\frac{di(0^+)}{dt} = 9600B_2,$$

$$B_2 = \frac{1000}{9600} \approx 0.1042 \text{ A}$$

The solution for $i(t)$ is

$$i(t) = 0.1042e^{-2800t} \sin 9600t \text{ A}, \quad t \geq 0$$

b) $v_C(t)$ can be found either by

$$v_C = -\frac{1}{C}\int_0^t i\,d\tau + 100 \text{ or}$$

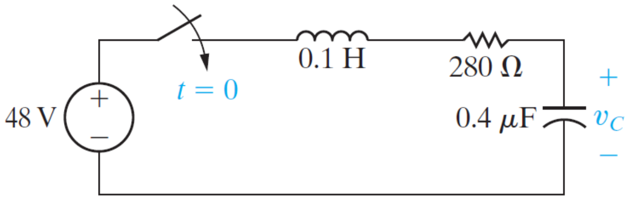
$$v_C = iR + L\frac{di}{dt}.$$

The latter formula is easier to implement;

$$v_C(t) = (100 \cos 9600t + 29.17 \sin 9600t)e^{-2800t} \text{ V}, \quad t \geq 0.$$

Example # 10:

No energy is stored in the 100mH inductor or the $0.4\mu F$ capacitor when the switch in the circuit shown in the figure next is closed. Find $v_C(t)$ for $t \geq 0s$.



Solution:

The neper frequency (α) for the series RLC circuit is

$$\alpha = \frac{R}{2L} \text{ rad/s},$$

and the expression for the resonant radian frequency is

$$\omega_0 = \frac{1}{\sqrt{LC}} \text{ rad/s}.$$

The roots of the characteristic equation are:

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}.$$

$$s_1 = -\frac{280}{0.2} + \sqrt{\left(\frac{280}{0.2}\right)^2 - \frac{10^6}{(0.1)(0.4)}}$$

$$= (-1400 + j4800) \text{ rad/s},$$

$$s_2 = (-1400 - j4800) \text{ rad/s}.$$

Since the roots are complex, the voltage response is underdamped and has the form:

$$v_C = V_f + B_1' e^{-\alpha t} \cos \omega_d t + B_2' e^{-\alpha t} \sin \omega_d t$$

where $V_f = 48V$ and $\omega_d = 4800$ rad/s

$$v_C(t) = 48 + B_1' e^{-1400t} \cos 4800t + B_2' e^{-1400t} \sin 4800t, \quad t \geq 0.$$

Since no energy is stored in the circuit;

$$v_C(0^-) = 0 = v_C(0^+)$$

$$i_L(0^+) = i_L(0^-) = 0A$$

Therefore, $$i_C(0^+) = C\frac{dv_C(0^+)}{dt} = i_L(0^+)$$

Hence, $$C\frac{dv_C(0^+)}{dt} = 0$$

$$\frac{dv_C(0^+)}{dt} = 0$$

Thus,

$$v_C(0) = 0 = 48 + B_1',$$

$$\frac{dv_C(0^+)}{dt} = 0 = 4800B_2' - 1400B_1'.$$

Solving for B_1' and B_2' yields

$$B_1' = -48 \text{ V}$$

$$B_2' = -14 \text{ V}$$

Therefore, the solution for $v_C(t)$ is

$$v_C(t) = (48 - 48e^{-1400t} \cos 4800t - 14e^{-1400t} \sin 4800t) \text{ V}, \quad t \geq 0.$$

Chapter 8

Sinusoidal Steady-State Analysis

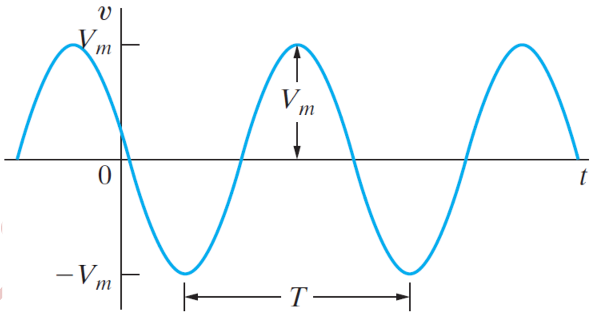
8.1 Sinusoidal Source

A sinusoidal voltage source (independent or dependent) produces a voltage that varies sinusoidally with time.

A sinusoidal current source (independent or dependent) produces a current that varies sinusoidally with time.

The sinusoidally varying function is either the sine function or the cosine function; here *the cosine function will be used*.

$$v = V_m \cos(\omega t + \phi)$$



The period (T): is the length of time required for the sinusoidal function to pass through all its possible values; measured in seconds.

Sinusoidal function is a periodic function; it repeats itself every period.

The frequency(f): is the number of cycles per second; measured in Hertz (Hz). It is the reciprocal of the period.

$$f = \frac{1}{T}$$

Omega (ω) is the **angular frequency** of the sinusoidal function, is measured in radians/second (rad/s)

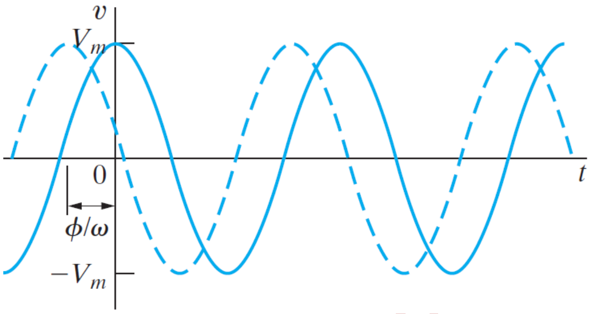
$$\omega = 2\pi f = 2\pi/T \text{ (radians/second)}$$

V_m is the maximum **amplitude** of the sinusoidal voltage.

$\emptyset$ is the **phase angle** of the sinusoidal voltage, it shift the sinusoidal function along the t-axis.

for $v = V_m \cos(\omega t + \phi)$

If $\emptyset$ is positive the function shifts to the left, and vice versa.



RMS Value

The **rms value** of a periodic function is defined as the square **r**oot of the **m**ean value of the **s**quared function.

For a **sinusoidal voltage**, the rms value is:

$$V_{\text{rms}} = \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} V_m^2 \cos^2(\omega t + \phi)\, dt}.$$

where t_0 is an arbitrary time

$$V_{\text{rms}} = \frac{V_m}{\sqrt{2}}$$

Example # 1:

A sinusoidal current has a maximum amplitude of 20 A. The current passes through one complete cycle in 1 ms. The magnitude of the current at zero time is 10 A.

a) What is the frequency of the current in hertz?

b) What is the frequency in radians per second?

c) Write the expression for $i(t)$ using the cosine function. Express ϕ in degrees.

d) What is the rms value of the current?

Solution

a) From the statement of the problem, $T = 1$ ms; hence $f = 1/T = 1000$ Hz.

b) $\omega = 2\pi f = 2000\pi$ rad/s.

c) We have $i(t) = I_m \cos(\omega t + \phi) = 20\cos(2000\pi t + \phi)$, but $i(0) = 10$ A. Therefore $10 = 20\cos\phi$ and $\phi = 60^\circ$. Thus the expression for $i(t)$ becomes

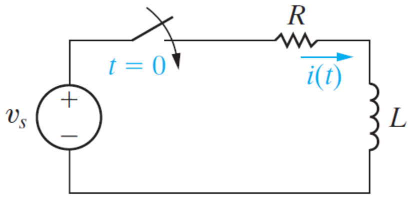
$$i(t) = 20\cos(2000\pi t + 60^\circ).$$

d) From the derivation of Eq. 9.6, the rms value of a sinusoidal current is $I_m/\sqrt{2}$. Therefore the rms value is $20/\sqrt{2}$, or 14.14 A.

8.2 The Sinusoidal Response

Example # 2:

Find the total response of the circuit (natural (or transient) and forced (steady state) responses), $i(t)$ if:



$$v_s = V_m \cos(\omega t + \phi)$$

Assuming that the initial current is zero; $i(0^-) = 0A$.

Solution:

Applying KVL for the mesh yields:

$$L\frac{di}{dt} + Ri = V_m \cos(\omega t + \phi)$$

The solution of this ordinary first order differential equation (whose $f(t)$ is a cosine function) is:

$$i = \frac{-V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos(\phi - \theta) e^{-(R/L)t} + \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos(\omega t + \phi - \theta)$$

where $\theta = \tan^{-1}\left(\frac{\omega L}{R}\right)$

The solution has two terms, the first is the **transient component;** it becomes so small as time elapses:

$$\frac{-V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos(\phi - \theta) e^{-(R/L)t}$$

Whilst, the second term is the **steady state component;** it lasts as long as the switch is closed:

$$\frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos(\omega t + \phi - \theta).$$

Note that, *the steady state component has the same frequency* (or ω) *as the source frequency.*

In this chapter we are interested in finding the steady state response, which will be found using ***Phasor*** method.

8.3 The Phasor

The **phasor** is a complex number that carries the ***amplitude and phase angle*** information of a sinusoidal function. It is based on Euler's identity.

Euler's Identity: it relates exponential function to trigonometric functions;

$$e^{\pm j\theta} = \cos\theta \pm j\sin\theta$$

where the real part of the exponential function is:

$$\cos\theta = \Re\{e^{j\theta}\}$$

and the imaginary part of the exponential function is:

$$\sin\theta = \Im\{e^{j\theta}\}$$

Therefore, the source voltage can be expressed as:

$$v = V_m \cos(\omega t + \phi)$$

$$= V_m \Re\{e^{j(\omega t + \phi)}\}$$

$$= V_m \Re\{e^{j\omega t} e^{j\phi}\}.$$

- Such that, $\boldsymbol{V_m e^{j\emptyset}}$ is a complex number that carries the amplitude and the phase angle of a sinusoidal function.
- This complex number $(\boldsymbol{V_m e^{j\emptyset}})$ is by definition **the phasor representation, or phasor transform**, of the given sinusoidal function. Such that, the **polar (exponential) form** of the phasor is:

$$\mathbf{V} = V_m e^{j\phi} = \mathcal{P}\{V_m \cos(\omega t + \phi)\}$$

- The phasor transform transfers the sinusoidal function **from the time domain to** the **complex-number domain**, which is also called the **frequency domain**.
- The **rectangular form** of the phasor is:

$$\mathbf{V} = V_m \cos\phi + jV_m \sin\phi$$

- The angle notation of unity phasor is:

$$1\angle\phi^\circ \equiv 1e^{j\phi}$$

Inverse Phasor Transform:

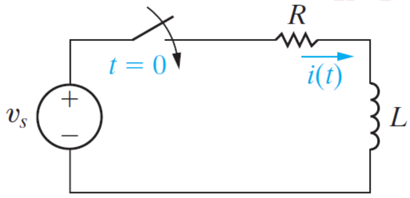
To reverse the process. That is, for a phasor the expression for the sinusoidal function can be written as for:

$$\mathbf{V} = 100\angle -26^\circ, \text{ the expression for } v \text{ is } 100\cos(\omega t - 26^\circ)$$

or

$$\mathcal{P}^{-1}\{V_m e^{j\phi}\} = \Re\{V_m e^{j\phi} e^{j\omega t}\}$$

For the series RL circuit, the steady-state solution for the current $i(t)$ is of the form:



$$i_{ss}(t) = \Re\{I_m e^{j\beta} e^{j\omega t}\}$$

Substituting for $i_{ss}(t)$ and $\frac{di_{ss}(t)}{dt}$ in the differential equation

$$L\frac{di}{dt} + Ri = V_m \cos(\omega t + \phi)$$

yields:

$$\Re\{j\omega L I_m e^{j\beta} e^{j\omega t}\} + \Re\{R I_m e^{j\beta} e^{j\omega t}\} = \Re\{V_m e^{j\phi} e^{j\omega t}\}$$

But, the sum of real numbers is the real number of the sum;

$$\Re\{(j\omega L + R) I_m e^{j\beta} e^{j\omega t}\} = \Re\{V_m e^{j\phi} e^{j\omega t}\}$$

Eliminating, $\boldsymbol{e^{j\omega t}}$ from both sides, yields:

$$(j\omega L + R) I_m e^{j\beta} = V_m e^{j\phi}$$

Therefore,

$$I_m e^{j\beta} = \frac{V_m e^{j\phi}}{R + j\omega L}$$

where, the phase angle of the current is:

$$\beta = \emptyset - \tan^{-1}\left(\frac{\omega L}{R}\right)$$

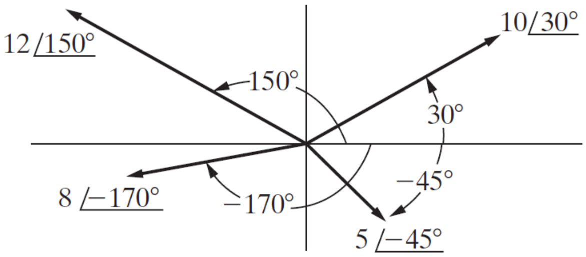
And the magnitude of the steady state current is:

$$I_m = \frac{V_m}{\sqrt{R^2 + (\omega L)^2}}$$

8.4 Phasor Diagrams

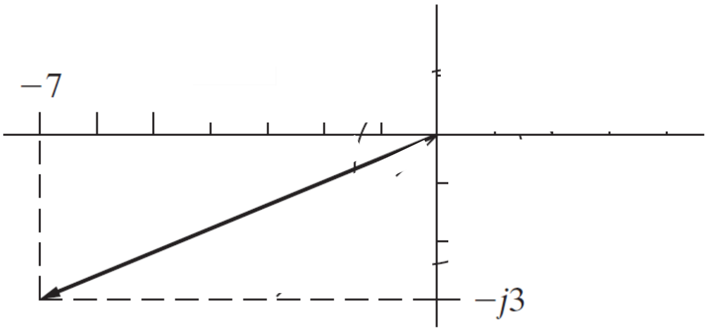
- **A phasor diagram** shows the magnitude and phase angle of each phasor quantity in the complex-number plane.
- Phase angles are measured counterclockwise from the positive real axis, and magnitudes are measured from the origin of the axes.
- The phasor diagrams of the following quantities are shown in the figure below.

$10\angle 30^\circ$, $12\angle 150^\circ$, $5\angle -45^\circ$, and $8\angle -170^\circ$.



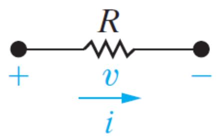
- To draw the phasor diagram of a quantity given in the rectangular form, convert it into the polar form first.
- For example; $\boldsymbol{X} = -7 - j3$

$$\boldsymbol{X} = -7 - j3 = 7.62\angle -156.80^\circ$$



8.5 Passive Circuit Elements in Frequency Domain

8.5.1 The Resistor



If the current is:

$$i = I_m \cos(\omega t + \theta_i)$$

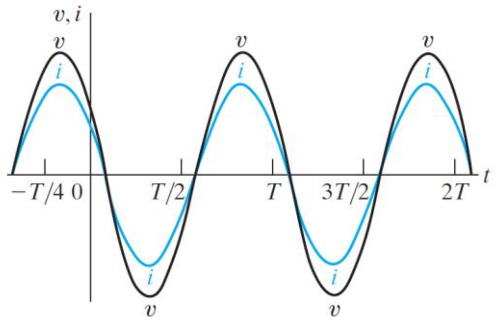
where I_m is the amplitude of the current, and θ_i is the phase angle of the current.

Then the voltage by Ohm's Law is:

$$v = R[I_m \cos(\omega t + \theta_i)]$$

$$= RI_m[\cos(\omega t + \theta_i)]$$

In time domain, the waveforms are shown next.



The phasor transformation of this voltage is:

$$\mathbf{V} = RI_m e^{j\theta_i} = RI_m \angle \theta_i$$

Note that, the *phase angle of the voltage is the same as the phase angle of current*. Both are in phase.

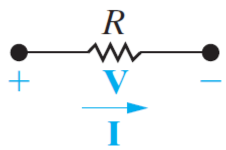
If the phasor representation of the current is:

$$I_m \angle \theta_i$$

Then, the voltage is:

$$\mathbf{V} = R\mathbf{I},$$

Therefore, the representation of the resistor in frequency domain is the same as it is in the time domain.



8.5.2 The Inductor

If the current is:

$$i = I_m \cos(\omega t + \theta_i)$$

Then, the voltage across the inductor is:

$$v = L\frac{di}{dt} = -\omega L I_m \sin(\omega t + \theta_i)$$

Using the cosine function, the inductor's voltage can be written as:

$$v = -\omega L I_m \cos(\omega t + \theta_i - 90^\circ)$$

The phasor representation of the inductor's voltage is:

$$\mathbf{V} = -\omega L I_m e^{j(\theta_i - 90^\circ)}$$

$$= -\omega L I_m e^{j\theta_i} e^{-j90^\circ}$$

Using the identity

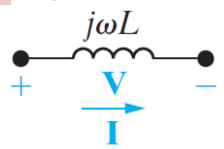
$$e^{-j90^\circ} = \cos 90^\circ - j\sin 90^\circ = -j$$

yields **V**:

$$= j\omega L I_m e^{j\theta_i}$$

$$= j\omega L\mathbf{I}.$$

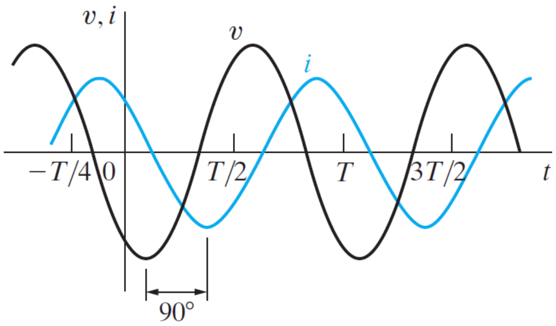
Therefore, the representation of the inductor in frequency domain is:



The voltage can be rewritten as:

$$\mathbf{V} = (\omega L \angle 90^\circ) I_m \angle \theta_i$$

$$= \omega L I_m \angle (\theta_i + 90)^\circ$$



The inductor's voltage leads its current by 90°, or the inductor's current lags its voltage by 90°.

Note that, the mutual inductance is also represented by $j\omega M$.

8.5.3 The Capacitor

The capacitor's current is related to its voltage in time domain by:

$$i = C\frac{dv}{dt}$$

If the voltage is

$$v = V_m \cos(\omega t + \theta_v)$$

then, the phasor representation of the capacitor's current is:

$$\mathbf{I} = j\omega C\mathbf{V}$$

The phasor representation of the capacitor's voltage is:

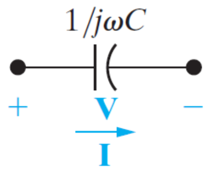
$$\mathbf{V} = \frac{\mathbf{1}}{\mathbf{j\omega C}}\mathbf{I}.$$

$$\mathbf{V} = \frac{1}{\omega C}\angle -90^\circ I_m \angle \theta_i^\circ$$

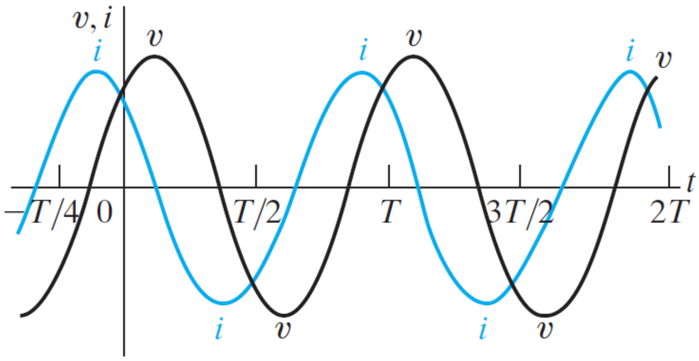
$$= \frac{I_m}{\omega C}\angle (\theta_i - 90)^\circ.$$

Note that, the capacitors voltage lags its current 90°, or the capacitor's current leads its voltage by 90°.

The representation of the capacitor in frequency domain is:



In time domain, plots of the capacitor's current and voltage are shown in the figure below.



8.5.4 Impedance and Reactance

The impedance ($\boldsymbol{Z}$) is a complex number, but not a phasor, which is defined as:

$$\boldsymbol{Z} = \frac{\boldsymbol{V}}{\boldsymbol{I}} = R + jX$$

- For a resistor, the impedance $\boldsymbol{Z} = R$
- For an inductor, the impedance is $\boldsymbol{Z} = j\omega L$
- For a capacitor, the impedance is $\boldsymbol{Z} = \frac{1}{j\omega C} = \frac{-j}{\omega C}$
- For mutual inductance, the impedance is $\boldsymbol{Z} = j\omega M$

Note that, all phasors are complex numbers, but not all complex numbers are phasors, because the phase is a coefficient $e^{j\omega t}$*!*

The Reactance: is the imaginary part of the impedance, X.

8.6 Kirchhoff's Voltage and Current Laws in Frequency Domain

KVL: The algebraic sum of phasor voltages in any closed path is zero;

$$\mathbf{V}_1 + \mathbf{V}_2 + \cdots + \mathbf{V}_n = 0,$$

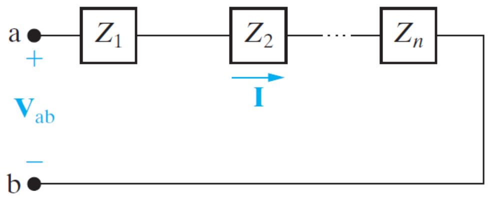
KCL: The algebraic sum of phasor currents at any node is zero;

$$\mathbf{I}_1 + \mathbf{I}_2 + \cdots + \mathbf{I}_n = 0,$$

8.7 Series, Parallel, and Delta-to-Wye Simplifications

Series Impedances:

They carry the same phasor current.



To find the total series impedance between the terminals a-b, apply KVL for the loop:

$$\mathbf{V}_{ab} = Z_1\mathbf{I} + Z_2\mathbf{I} + \cdots + Z_n\mathbf{I}$$
$$= (Z_1 + Z_2 + \cdots + Z_n)\mathbf{I}.$$

The series impedance between a-b terminals is:

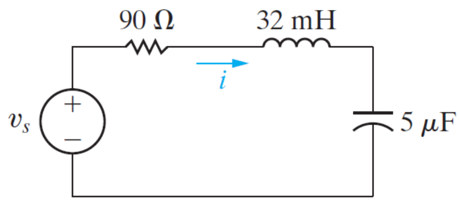
$$Z_{ab} = \frac{\mathbf{V}_{ab}}{\mathbf{I}} = Z_1 + Z_2 + \cdots + Z_n$$

Example # 3:

For the circuit shown in the figure next, the source voltage, v_s, is: $750\cos(5000t + 30^\circ)$ V

a) Construct the frequency-domain equivalent circuit.

b) Calculate the steady-state current i by the phasor method.



Solution:

a) Form the source voltage, the angular frequency is:

$$\omega = 5000 \text{ rad/s}.$$

The phasor transform of v_s is

$$\mathbf{V}_s = 750\,\underline{/30^\circ}\text{ V}.$$

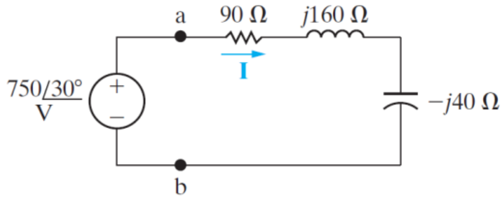
Thus, the impedance of the inductor is:

$$Z_L = j\omega L = j(5000)(32 \times 10^{-3}) = j160\ \Omega,$$

and the impedance of the capacitor is

$$Z_C = j\frac{-1}{\omega C} = -j\frac{10^6}{(5000)(5)} = -j40\ \Omega.$$

The equivalent circuit in frequency domain is shown in the figure next.



b) The equivalent impedance between the terminals a-b is:

$$\begin{aligned} Z_{ab} &= 90 + j160 - j40 \\ &= 90 + j120 = 150\,\underline{/53.13^\circ}\ \Omega. \end{aligned}$$

Thus

$$\mathbf{I} = \frac{750\,\underline{/30^\circ}}{150\,\underline{/53.13^\circ}} = 5\,\underline{/-23.13^\circ}\text{ A}.$$

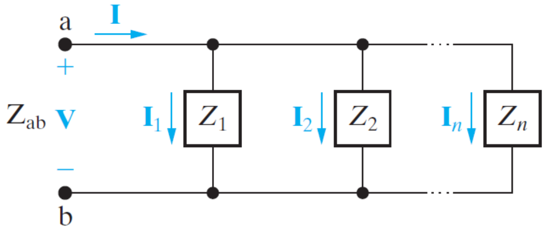
Therefore, the steady state expression for the current in time domain is:

$$i = 5\cos(5000t - 23.13^\circ)\text{ A}.$$

Parallel Impedances:

They are connected to same node-pair and they have the same voltage across them.

Applying KCL at the upper node yields:



$$\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 + \cdots + \mathbf{I}_n,$$

or

$$\frac{\mathbf{V}}{Z_{ab}} = \frac{\mathbf{V}}{Z_1} + \frac{\mathbf{V}}{Z_2} + \cdots + \frac{\mathbf{V}}{Z_n}$$

which can be reduced to:

$$\frac{1}{Z_{ab}} = \frac{1}{Z_1} + \frac{1}{Z_2} + \cdots + \frac{1}{Z_n}$$

Two impedances in parallel has the equivalent impedance as:

$$Z_{ab} = \frac{Z_1 Z_2}{Z_1 + Z_2}$$

The Admittance (Y)

- It is defined as the reciprocal of impedance;

$$Y = \frac{1}{Z} = G + jB \text{ (siemens)}.$$

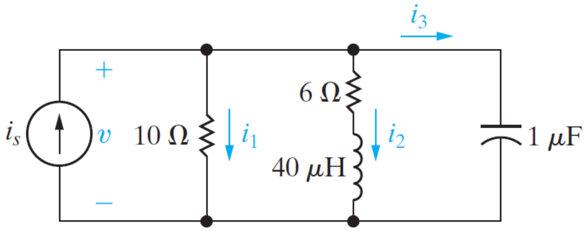
- The admittance' real part, G, is called ***conductance*** and its imaginary part, B, is called ***susceptance***.
- The admittance, conductance and susceptance are measured in siemens (S).
- For parallel admittances (impedances), the equivalent admittance is the sum of all admittances;

$$Y_{ab} = Y_1 + Y_2 + \cdots + Y_n$$

Example # 4:

The sinusoidal current source in the circuit shown in the figure produces the current:

$$i_s = 8\cos 200{,}000t \text{ A}$$



a) Construct the frequency-domain equivalent circuit.

b) Find the steady-state expressions for v, i_1, i_2, and i_3

Solution:

a) The phasor transform of the current source is $8\angle 0^\circ$; the resistors transform directly to the frequency domain as 10 and 6 Ω; the 40 μH inductor has an impedance of $j8\ \Omega$ at the given frequency of 200,000 rad/s; and at this frequency the 1 μF capacitor has an impedance of $-j5\ \Omega$. Figure 9.19 shows the frequency-domain equivalent circuit and symbols representing the phasor transforms of the unknowns.

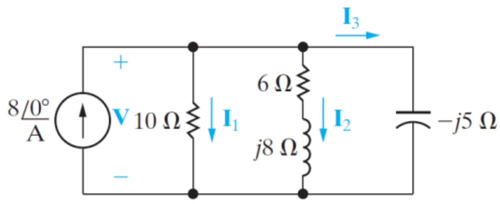


Figure 9.19 ▲ The frequency-domain equivalent circuit.

b) The admittances of the branches from left to right, respectively, are:

$$Y_1 = \frac{1}{10} = 0.1 \text{ S},$$

$$Y_2 = \frac{1}{6 + j8} = \frac{6 - j8}{100} = 0.06 - j0.08 \text{ S}$$

$$Y_3 = \frac{1}{-j5} = j0.2 \text{ S}$$

The admittance of the three branches is

$$\begin{aligned} Y &= Y_1 + Y_2 + Y_3 \\ &= 0.16 + j0.12 \\ &= 0.2\angle 36.87^\circ \text{ S}. \end{aligned}$$

The impedance at the current source is

$$Z = \frac{1}{Y} = 5\angle -36.87^\circ\ \Omega.$$

The Voltage **V** is

$$\mathbf{V} = Z\mathbf{I} = 40 \angle -36.87^\circ \text{ V}.$$

Hence

$$\mathbf{I}_1 = \frac{40 \angle -36.87^\circ}{10} = 4 \angle -36.87^\circ = 3.2 - j2.4 \text{ A},$$

$$\mathbf{I}_2 = \frac{40 \angle -36.87^\circ}{6 + j8} = 4 \angle -90^\circ = -j4 \text{ A},$$

and

$$\mathbf{I}_3 = \frac{40 \angle -36.87^\circ}{5 \angle -90^\circ} = 8 \angle 53.13^\circ = 4.8 + j6.4 \text{ A}.$$

Check the validity of KCL;

$$\mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_3 = \mathbf{I}.$$

Specifically,

$$3.2 - j2.4 - j4 + 4.8 + j6.4 = 8 + j0.$$

In time domain:

$$v = 40 \cos (200{,}000t - 36.87^\circ) \text{ V},$$

$$i_1 = 4 \cos (200{,}000t - 36.87^\circ) \text{ A},$$

$$i_2 = 4 \cos (200{,}000t - 90^\circ) \text{ A},$$

$$i_3 = 8 \cos (200{,}000t + 53.13^\circ) \text{ A}.$$

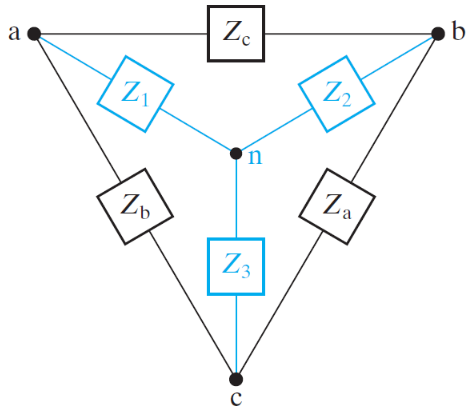
Delta-to-Wye Transformations

1) **Delta to- WYE**

$$Z_1 = \frac{Z_b Z_c}{Z_a + Z_b + Z_c},$$

$$Z_2 = \frac{Z_c Z_a}{Z_a + Z_b + Z_c},$$

$$Z_3 = \frac{Z_a Z_b}{Z_a + Z_b + Z_c}.$$



2) **WYE-to-Delta**

$$Z_a = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1}{Z_1},$$

$$Z_b = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1}{Z_2},$$

$$Z_c = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1}{Z_3}.$$

Note that, If $Z_1 = Z_2 = Z_3 \leftrightarrow Z_a = Z_b = Z_c$ then:

$\boldsymbol{Z_Y = \frac{Z_\Delta}{3}}$ and $\boldsymbol{Z_\Delta = 3Z_Y}$

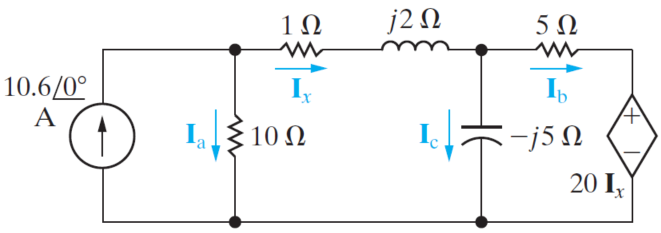
Study the example in the text book!

8.8 Techniques for Sinusoidal Steady-State Analysis

8.8.1 The Node-Voltage Method

Example # 5:

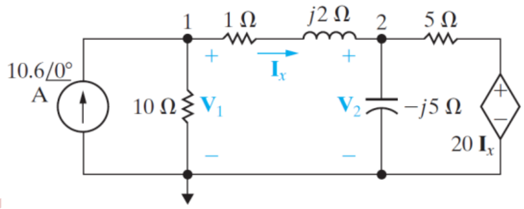
Use the node-voltage method to find the branch currents I_a, I_b and I_c in the circuit shown in the figure next.



Solution:

- The number of essential nodes, n_e, is 3, thus:

  # of equations with unknown node-voltages $= n_e - 1 = 3 - 1 = 2$.
- The circuit is already represented in frequency domain!
- Taking the bottom node as a reference node, 0V, the other two nodes voltages have to be found; V_1, & V_2.



<u>KCL at node 1:</u>

$$-10.6 + \frac{\mathbf{V}_1}{10} + \frac{\mathbf{V}_1 - \mathbf{V}_2}{1 + j2} = 0$$

Multiplying by $(1 + j2)$ and collecting coefficients yield:

$$\mathbf{V}_1(1.1 + j0.2) - \mathbf{V}_2 = 10.6 + j21.2 \quad (1)$$

<u>KCL at node 2:</u>

$$\frac{\mathbf{V}_2 - \mathbf{V}_1}{1 + j2} + \frac{\mathbf{V}_2}{-j5} + \frac{\mathbf{V}_2 - 20\mathbf{I}_x}{5} = 0 \quad (2)$$

But,

$$\mathbf{I}_x = \frac{\mathbf{V}_1 - \mathbf{V}_2}{1 + j2}$$

Substituting for I_x in eq. (2), multiplying by $(1 + j2)$, and collecting terms yield:

$$-5\mathbf{V}_1 + (4.8 + j0.6)\mathbf{V}_2 = 0. \quad (3)$$

Solving eqs. (1) and (3) yields:

$$\mathbf{V}_1 = 68.40 - j16.80 \text{ V},$$
$$\mathbf{V}_2 = 68 - j26 \text{ V}.$$

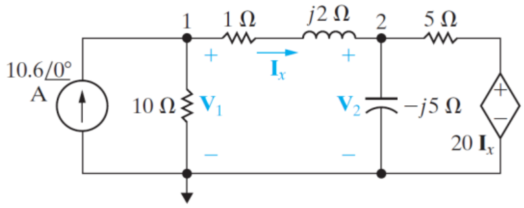
Hence the branch currents are

$$\mathbf{I}_a = \frac{\mathbf{V}_1}{10} = 6.84 - j1.68 \text{ A},$$

$$\mathbf{I}_x = \frac{\mathbf{V}_1 - \mathbf{V}_2}{1 + j2} = 3.76 + j1.68 \text{ A},$$

$$\mathbf{I}_b = \frac{\mathbf{V}_2 - 20\mathbf{I}_x}{5} = -1.44 - j11.92 \text{ A},$$

$$\mathbf{I}_c = \frac{\mathbf{V}_2}{-j5} = 5.2 + j13.6 \text{ A}.$$



To check the solution,

$$\mathbf{I}_a + \mathbf{I}_x = 6.84 - j1.68 + 3.76 + j1.68$$
$$= 10.6 \text{ A},$$

which is the same as the source current!

Also check

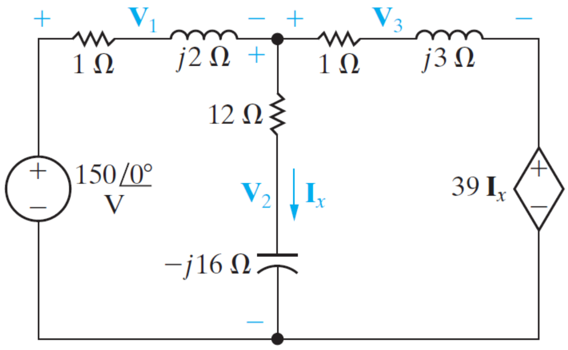
$$\mathbf{I}_x = \mathbf{I}_b + \mathbf{I}_c = -1.44 - j11.92 + 5.2 + j13.6$$
$$= 3.76 + j1.68 \text{ A}.$$

which is the same as the calculated $\boldsymbol{I_x}$ before!

8.8.2 The Mesh-Current Method

Example # 6:

Use the mesh-current method to find the branch currents $\boldsymbol{V_1}, \boldsymbol{V_2}$ and $\boldsymbol{V_3}$ in the circuit shown in the figure next.

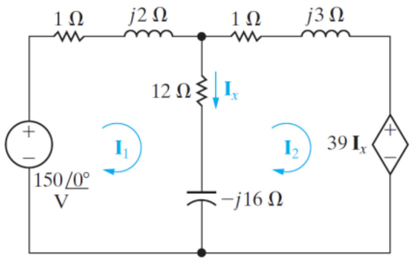


Solution:

- The number of essential nodes, n_e, is 2, and the number of essential branches, b_e, is 3, thus:

  Number of equations with unknown currents $= b_e - (n_e - 1) = 3 - (2 - 1) = 2.$
- The circuit is already represented in frequency domain!

<u>KVL for mesh 1</u>



$$150 = (1 + j2)\mathbf{I}_1 + (12 - j16)(\mathbf{I}_1 - \mathbf{I}_2)$$

or

$$150 = (13 - j14)\mathbf{I}_1 - (12 - j16)\mathbf{I}_2 \tag{1}$$

<u>KVL for mesh 2</u>

$$0 = (12 - j16)(\mathbf{I}_2 - \mathbf{I}_1) + (1 + j3)\mathbf{I}_2 + 39\mathbf{I}_x \tag{2}$$

But, the constraint equation: $\mathbf{I}_x = \mathbf{I}_1 - \mathbf{I}_2$

Substituting for $\boldsymbol{I_x}$ in eq. (2) and simplifying yield:

$$0 = (27 + j16)\mathbf{I}_1 - (26 + j13)\mathbf{I}_2 \tag{3}$$

Solving eqs. (1) and (3) for $\boldsymbol{I_1}$& $\boldsymbol{I_2}$ yields:

$$\mathbf{I}_1 = -26 - j52 \text{ A},$$

$$\mathbf{I}_2 = -24 - j58 \text{ A},$$

$$\mathbf{I}_x = -2 + j6 \text{ A}. \quad \text{; From the constraint equation!}$$

The three voltages are

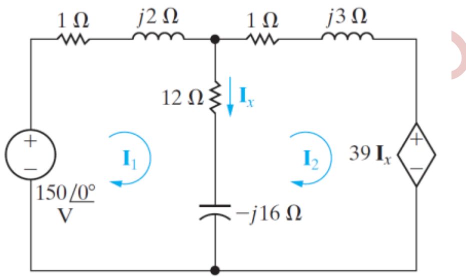
$$\mathbf{V}_1 = (1 + j2)\mathbf{I}_1 = 78 - j104 \text{ V},$$

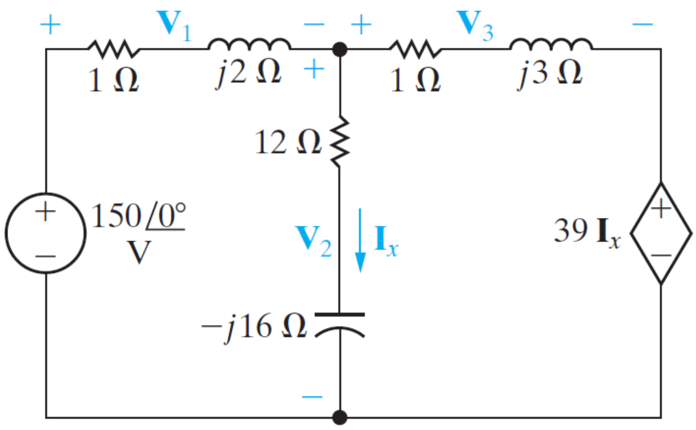
$$\mathbf{V}_2 = (12 - j16)\mathbf{I}_x = 72 + j104 \text{ V},$$

$$\mathbf{V}_3 = (1 + j3)\mathbf{I}_2 = 150 - j130 \text{ V}.$$

Also

$$39\mathbf{I}_x = -78 + j234 \text{ V}.$$





Check the answers by summing the voltages around closed paths:

Mesh 1:

$$-150 + \mathbf{V}_1 + \mathbf{V}_2 = -150 + 78 - j104 + 72 + j104 = 0,$$

Mesh 2:

$$-\mathbf{V}_2 + \mathbf{V}_3 + 39\mathbf{I}_x = -72 - j104 + 150 - j130 - 78 + j234 = 0,$$

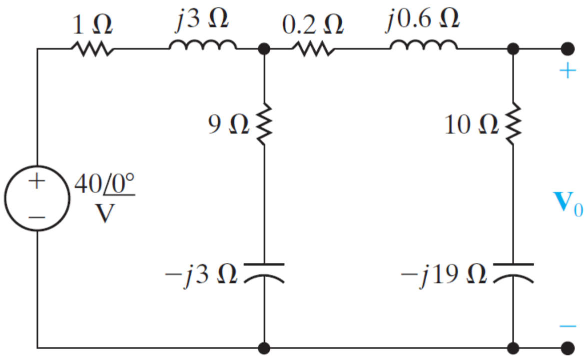
Outer mesh:

$$-150 + \mathbf{V}_1 + \mathbf{V}_3 + 39\mathbf{I}_x = -150 + 78 - j104 + 150 - j130 - 78 + j234 = 0.$$

8.8.3 Source Transformation

Example # 7:

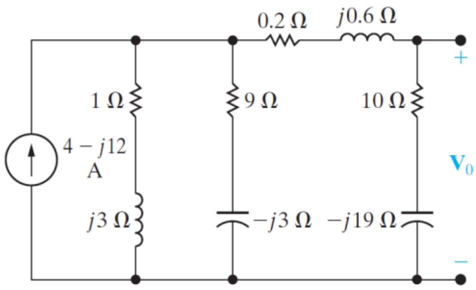
Use source transformation, in frequency domain, to find the output voltage $\boldsymbol{V_o}$ in the circuit shown in the figure next.



Solution:

1- The voltage source $(= 40\ V)$ and the series impedance $(1 + j3\ \Omega)$ can be replaced by a current source $\boldsymbol{I}$ in parallel with the same impedance. The current source equals:

$$\mathbf{I} = \frac{40}{1 + j3} = \frac{40}{10}(1 - j3) = 4 - j12\ \text{A}$$



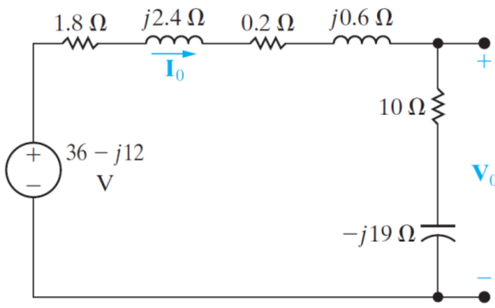
2- The parallel branches $\big((1 + j3) //\ (9 - j3)\big)$ can be replaced by an impedance, $\boldsymbol{Z}$, equal to:

$$Z = \frac{(1 + j3)(9 - j3)}{10} = 1.8 + j2.4\ \Omega$$

3- The impedance $\boldsymbol{Z}\ (= 1.8 + j2.4\ \Omega)$ and the parallel current source $\boldsymbol{I}\ (= 4 - j12\ A)$ can be replaced by a voltage source, **V,** in series with the same impedance, where $\boldsymbol{V}$ is:

$$\mathbf{V} = (4 - j12)(1.8 + j2.4) = 36 - j12\ \text{V}$$

4- The output current $\boldsymbol{I_o}$ can be obtained by dividing the voltage over the total impedance;



$$\mathbf{I}_0 = \frac{36 - j12}{12 - j16} = \frac{12(3 - j1)}{4(3 - j4)}$$

$$= \frac{39 + j27}{25} = 1.56 + j1.08\ \text{A}$$

5- The output voltage, $\boldsymbol{V_o}$, is found by applying Ohm's Law:

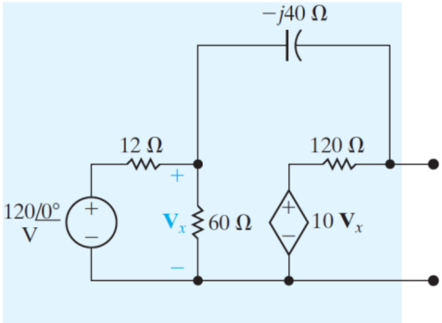
$$\mathbf{V}_0 = (1.56 + j1.08)(10 - j19) = 36.12 - j18.84\ \text{V}$$

6- *Note that, $\boldsymbol{V_o}$ can be found directly by applying voltage divider; instead of following steps 4 and 5!*

8.8.4 Thevenin's and Norton's Equivalent Circuits

Example # 8:

Find the Thévenin's equivalent circuit with respect to terminals "a-b" for the circuit shown in the figure next.



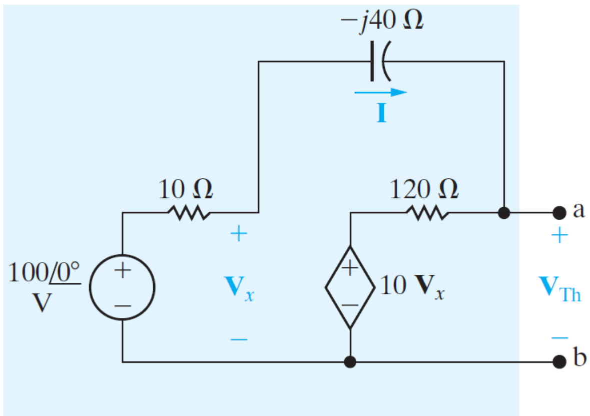
Solution:

1- $\boldsymbol{V_{Th} = V_{OC}}$

- To find the open circuit voltage, $\boldsymbol{V_{OC}}$, use source transformation to simplify the part of the circuit containing 120 V, 12 Ω and 60Ω.

  Note that, at the same time, these transformations must preserve the identity of the controlling voltage $\boldsymbol{V_x}$ *because of the dependent voltage source.*

- 120V and 12Ω resistor can be transformed to a current source of 10A in parallel with the 12Ω resistor.
- The 12Ω resistor is in parallel with 60Ω resistor, and both have the controlling voltage, $\boldsymbol{V_x}$ across them.
- The equivalent parallel resistor is 10Ω.
- Now replace the 10A source and the parallel resistor, 10Ω, with a 100V source in series with 10Ω.
- Note that, the controlling voltage, $\boldsymbol{V_x}$ is still at the terminals of the transformed part of the circuit, as shown in the figure next.



- Assume that the current $\boldsymbol{I}$ is flowing out of the transformed part of the circuit, as marked on the figure, and apply KVL for the loop yield:

$$100 = 10\mathbf{I} - j40\mathbf{I} + 120\mathbf{I} + 10\mathbf{V}_x = (130 - j40)\mathbf{I} + 10\mathbf{V}_x$$

- But,

$$\mathbf{V}_x = 100 - 10\mathbf{I}$$

- Then,

$$\mathbf{I} = \frac{-900}{30 - j40} = 18\angle{-126.87^\circ} \text{ A}$$

- Thus, the controlling voltage $\boldsymbol{V_x}$ is:

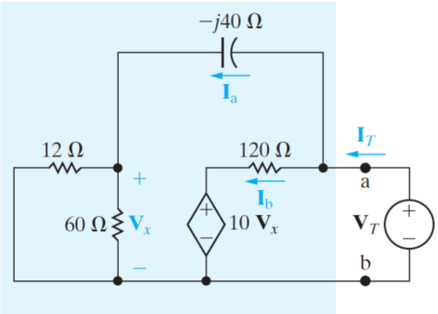
$$\mathbf{V}_x = 100 - 180\,\underline{/-126.87^\circ} = 208 + j144\text{ V}$$

- $\boldsymbol{V_{Th} = V_{OC}}$;

$$\begin{aligned}\mathbf{V}_{\text{Th}} &= 10\mathbf{V}_x + 120\mathbf{I}\\ &= 2080 + j1440 + 120(18)\,\underline{/-126.87^\circ}\\ &= 784 - j288 = 835.22\,\underline{/-20.17^\circ}\text{ V.}\end{aligned}$$

2- To find R_{Th}, use the Test Source method, as this is case 2 $\left(note\ also, R_{Th} = \frac{V_{OC}}{I_{SC}}\right)$!

- Deactivate the independent voltage source by replacing it with a short circuit (0V), and apply a Test voltage, $\boldsymbol{V_T}$, with a current $\boldsymbol{I_T}$, as shown in the figure next.
- Assume that, the branch currents $\boldsymbol{I_a}$ & $\boldsymbol{I_b}$, are as assigned in the figure. These currents are (noting that, 12Ω//60Ω=10Ω):



$$\mathbf{I}_a = \frac{\mathbf{V}_T}{10 - j40}, \quad \mathbf{V}_x = 10\mathbf{I}_a,$$

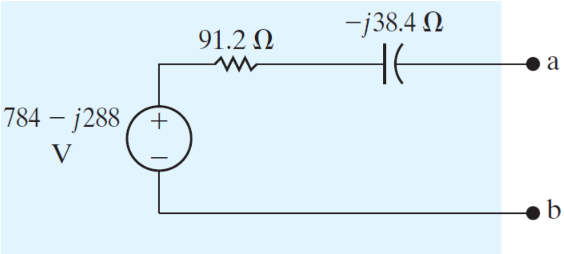
$$\mathbf{I}_b = \frac{\mathbf{V}_T - 10\mathbf{V}_x}{120} = \frac{-\mathbf{V}_T(9 + j4)}{120(1 - j4)},$$

- Apply KCL at the output node:

$$\mathbf{I}_T = \mathbf{I}_a + \mathbf{I}_b = \frac{\mathbf{V}_T}{10 - j40}\left(1 - \frac{9 + j4}{12}\right) = \frac{\mathbf{V}_T(3 - j4)}{12(10 - j40)},$$

$$Z_{\text{Th}} = \frac{\mathbf{V}_T}{\mathbf{I}_T} = 91.2 - j38.4\ \Omega.$$

- Therefore, the equivalent Thevenin's circuit is depicted in the figure next.

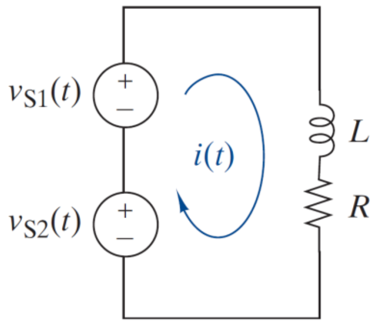


8.8.5 Superposition

- When the sources have different frequencies, this problem can be solved only by superposition.
- Draw the equivalent circuit when each independent source is acting alone, with the impedances calculated at the particular frequency of the active independent source, then find the particular response due to the active source; in frequency and time domains.
- Repeat the procedure for other source.
- When deactivating an independent source, the voltage source is replaced by a short circuit, and the current source is replaced by an open circuit.
- Dependent sources stay in the circuit, no matter which independent source is active.
- Superimpose all the responses ***in time domain*** (if the sources have different frequencies, do not superimpose responses in the frequency domain).

Example # 9:

Use superposition to find the steady-state current $i(t)$ in the figure next for $R = 10k\Omega$, $L = 200mH, v_{s1}(t) = 24\,cos20{,}000t\ V$, and $v_{s2}(t) = 8\cos(60{,}000t + 30)\ V$.



Solution:

Source 1 acting alone (the other voltage source is replaced by a short circuit):

$$\mathbf{V}_{S1} = 24\angle 0^\circ \text{ V at a frequency of } \omega = 20 \text{ krad/s.}$$

The impedance is:

$$Z_{EQ1} = R + j\omega L = (10 + j4)\text{ k}\Omega$$

The phasor current due to source 1 is:

$$\mathbf{I}_1 = \frac{\mathbf{V}_{S1}}{Z_{EQ1}} = \frac{24\angle 0^\circ}{10{,}000 + j4000} = 2.23\angle{-21.8^\circ}\text{ mA}$$

Source 2 acting alone (the other voltage source is replaced by a short circuit):

$$\mathbf{V}_{S2} = 8\angle 30^\circ \text{ V at a frequency of } \omega = 60 \text{ krad/s}$$

The impedance is:

$$Z_{EQ2} = R + j\omega L = (10 + j12) \text{ k}\Omega$$

The phasor current due to source 2 is

$$\mathbf{I}_2 = \frac{\mathbf{V}_{S2}}{Z_{EQ2}} = \frac{8\angle 30^\circ}{10{,}000 + j12000} = 0.512\angle -20.2^\circ \text{ mA}$$

Since the two sources have different frequencies, the phasor currents cannot be added, but the time domain responses can be added:

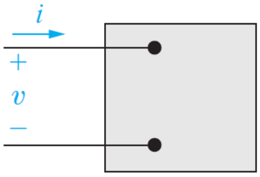
$$i(t) = 2.23\cos(20{,}000t - 21.8^\circ) + 0.512\cos(60{,}000t - 20.2^\circ) \text{ mA}$$

Chapter 9

Sinusoidal Steady-State Power Calculations

9.1 Instantaneous Power

- The instantaneous power consumed, at any time, by a network is:



$$p = vi$$

where v & i are steady-state sinusoidal signals of the voltage (measured in Volts) and current (measured in Amperes), respectively.

- If the current is reversed, then p is negative; and the power is delivered or supplied by the network.
- If the voltage and current, respectively, are:

$$v = V_m \cos(\omega t + \theta_v),$$

$$i = I_m \cos(\omega t + \theta_i),$$

where θ_v is the voltage phase angle, and θ_i is the current phase angle.

- Choosing a reference of time that corresponds to the time when the current has its maximum value, then:

$$v = V_m \cos(\omega t + \theta_v - \theta_i),$$

$$i = I_m \cos \omega t.$$

Thus, instantaneous power becomes:

$$p = V_m I_m \cos(\omega t + \theta_v - \theta_i) \cos \omega t$$

- Using trigonometric identity:

$$\cos\alpha \cos\beta = \frac{1}{2}\cos(\alpha - \beta) + \frac{1}{2}\cos(\alpha + \beta)$$

letting $\alpha = \omega t + \theta_v - \theta_i$ and $\beta = \omega t$ gives

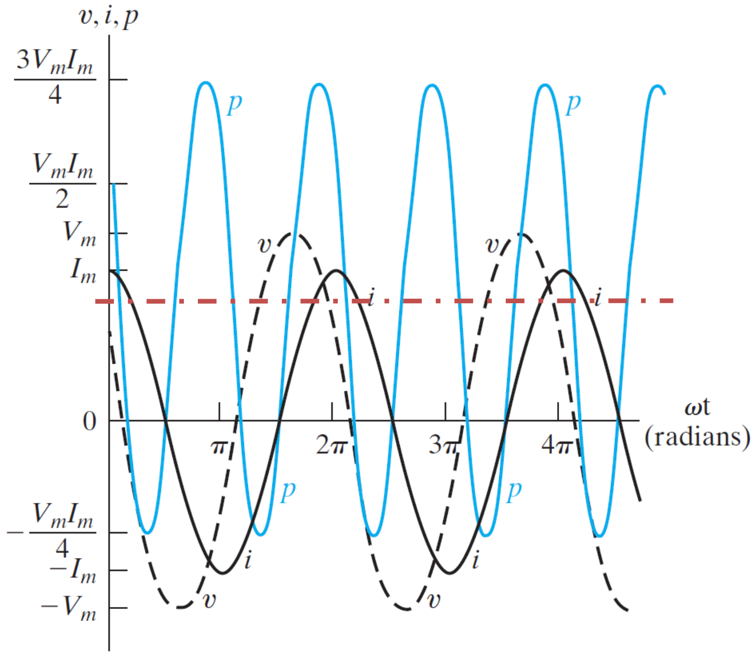
$$p = \frac{V_m I_m}{2}\cos(\theta_v - \theta_i) + \frac{V_m I_m}{2}\cos(2\omega t + \theta_v - \theta_i)$$

- Using trigonometric identity:

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$p = \frac{V_m I_m}{2}\cos(\theta_v - \theta_i) + \frac{V_m I_m}{2}\cos(\theta_v - \theta_i)\cos 2\omega t - \frac{V_m I_m}{2}\sin(\theta_v - \theta_i)\sin 2\omega t.$$

A plot of the instantaneous power for $\theta_v = 60^o$ and $\theta_i = 0^o$ is shown in the figure below:



Notes:

- The instantaneous power is not constant and varies with time, causing **vibrations** to motors.
- The frequency of the instantaneous power is **twice the frequency** of the voltage or current.
- The dashed line represents the **average power**.
- The negative power implies that **energy stored** in the inductors or capacitors is now being **extracted**.

9.2 Average and Reactive Power

Recall the instantaneous power equation,

$$p = \frac{V_m I_m}{2}\cos(\theta_v - \theta_i) + \frac{V_m I_m}{2}\cos(\theta_v - \theta_i)\cos 2\omega t$$

$$- \frac{V_m I_m}{2}\sin(\theta_v - \theta_i)\sin 2\omega t.$$

which can be rewritten in the form:

$$p = P + P\cos 2\omega t - Q\sin 2\omega t$$

where

P is the **Average (Real, or Active) power,** it represents the power converted from electric to nonelectric power, measured in Watts, and is:

$$P = \frac{V_m I_m}{2}\cos(\theta_v - \theta_i)$$

Also, ***P*** can be found by integrating the instantaneous power as:

$$P = \frac{1}{T}\int_{t_0}^{t_0+T} p\,dt$$

Note that, integrating the instantaneous power equation over one period will yield the first term of p only, as the integration of the second and the third terms over one complete cycle is zero.

and ***Q*** the **Reactive power,** measured in Volt-Ampere Reactive **(VAR)** and is:

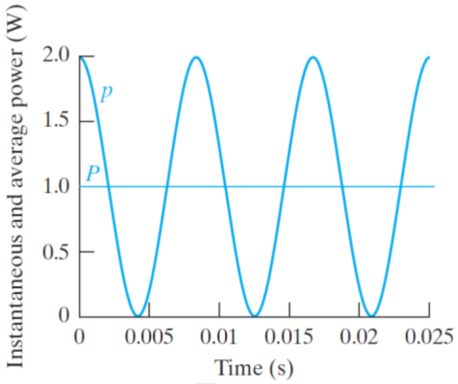
$$Q = \frac{V_m I_m}{2}\sin(\theta_v - \theta_i)$$

9.2.1 Power for Purely Resistive Circuits

For a purely resistive circuit, the voltage and current are in phase, $\theta_v = \theta_i$, thus the instantaneous power is:

$$p = P + P\cos 2\omega t$$

$$\boldsymbol{P} = \frac{\boldsymbol{V_m I_m}}{\boldsymbol{2}} \boldsymbol{cos}\,(\theta_v - \theta_i) = \frac{\boldsymbol{V_m I_m}}{\boldsymbol{2}}$$



Note that, the instantaneous power cannot be negative, and no power can be extracted from resistors, as resistors cannot store energy; all *electric energy is dissipated as thermal energy*.

9.2.2 Power for Purely Inductive Circuits

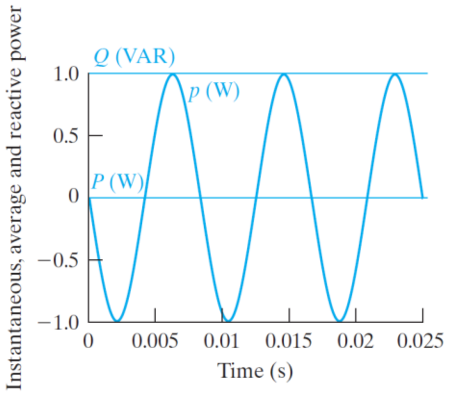
- For a purely inductive circuit, the current lags the voltage by 90^o; $\theta_i = \theta_v - 90^o\ or\ \theta_v - \theta_i = 90^o$. Thus, the instantaneous power is:

  $$p = -Q\sin 2\omega t$$

  because the first two terms are zero; $\cos(\theta_v - \theta_i) = \cos 90 = 0$.
- ***Q is positive for inductors (inductors absorb magnetizing VARs)***;

$$\boldsymbol{Q} = \frac{\boldsymbol{V_m I_m}}{\boldsymbol{2}} \boldsymbol{sin}\,(\theta_v - \theta_i) = \frac{\boldsymbol{V_m I_m}}{\boldsymbol{2}} \boldsymbol{sin}\,(90) = \frac{\boldsymbol{V_m I_m}}{\boldsymbol{2}}$$

- *Q is the reactive power* (VAR), because the inductor is a reactive element; its impedance is purely reactive.
- The *average power for a purely inductive circuit is zero*, and no power is dissipated in the circuit.
- A plot of the instantaneous power is shown in the figure next.
- When $p(t)$ is positive, the energy is stored in the magnetic field of the inductor, and when it is negative it is extracted from the magnetic field.
- The instantaneous power at the terminals in a purely inductive circuit is continually *exchanged between the circuit and the source* driving the circuit, at a frequency of 2ω.



9.2.3 Power for Purely Capacitive Circuits

- For a purely capacitive circuit, the current leads the voltage by 90^o; $\theta_i = \theta_v + 90^o \; or \; \theta_v - \theta_i = -90^o$. Thus, the instantaneous power is:

$$p = -Q \sin 2\omega t$$

because the first two terms are zero; $\cos(\theta_v - \theta_i) = \cos -90 = 0$.

- The *average power for a purely capacitive circuit is zero*, and no power is dissipated in the circuit.
- ***Q is negative for capacitors (capacitors supply magnetizing VARs);***

$$\boldsymbol{Q} = \frac{\boldsymbol{V_m I_m}}{\boldsymbol{2}} \boldsymbol{sin}\,(\theta_v - \theta_i)$$

$$\boldsymbol{Q} = \frac{\boldsymbol{V_m I_m}}{\boldsymbol{2}} (-90)$$

$$\boldsymbol{Q} = -\frac{\boldsymbol{V_m I_m}}{\boldsymbol{2}}$$

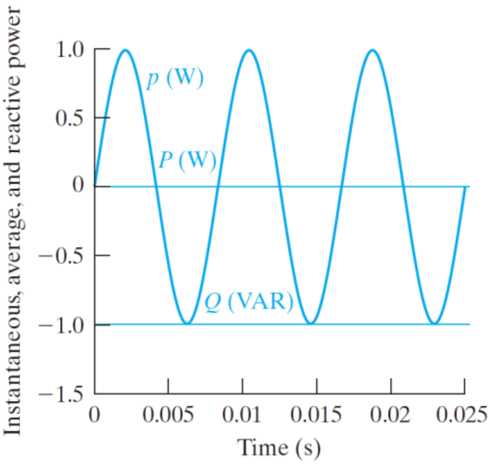
- *Q is the reactive power* (VAR), because the capacitor is a reactive element; its impedance is purely reactive.
- The instantaneous power can be expressed as:

$$\boldsymbol{p} = -\frac{\boldsymbol{V_m I_m}}{\boldsymbol{2}} \boldsymbol{sin}\,(\theta_v - \theta_i)\boldsymbol{sin\,2\omega t}$$

$$\boldsymbol{p} = -\frac{\boldsymbol{V_m I_m}}{\boldsymbol{2}} \boldsymbol{sin}\,(-90)\boldsymbol{sin\,2\omega t}$$

$$\boldsymbol{p} = -(-\frac{\boldsymbol{V_m I_m}}{\boldsymbol{2}})\boldsymbol{sin\,2\omega t}$$

- A plot of the instantaneous power is shown in the figure next.
- When $p(t)$ is positive, the energy is stored in the electric field of the capacitor, and when it is negative energy is extracted from the electric field.
- The instantaneous power at the terminals of a purely capacitive circuit is continually *exchanged between the circuit and the source* driving the circuit, at a frequency of 2ω.



9.3 Power Factor

- The angle is $\boldsymbol{\theta_v - \theta_i}$ plays an important role in power calculations, and is called ***power factor angle.***

- *The power factor angle is the same as the load impedance angle;* $\boldsymbol{\theta_{ZL} = \theta_v - \theta_i}$

- ***Power Factor (pf)*** *is defined as the cosine of the power factor angle*

$$\boldsymbol{pf = cos(\theta_{ZL}) = cos(\theta_v - \theta_i)}$$

- Because $\boldsymbol{cos(\theta_v - \theta_i) = cos(\theta_i - \theta_v)}$**,** is always positive, specify:
  1. ***Lagging power factor*** is for inductive load, where $\boldsymbol{\theta_v - \theta_i}$ **is positive;**
     The current lags the voltage: $0 < (\theta_v - \theta_i) < 90^o$
  2. ***Leading power factor*** is for capacitive load, where $\boldsymbol{\theta_v - \theta_i}$ **is negative;**
     The current leads the voltage: $-90^o < (\theta_v - \theta_i) < 0^o$
  3. ***Unity power factor*** is for purely resistive load, where $\theta_v = \theta_i$, and

$$\boldsymbol{pf = cos\,(\theta_v - \theta_i) = 1}$$

- ***The reactive factor (rf)*** is defined as:

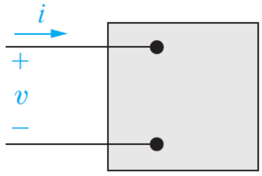
$$\boldsymbol{rf = sin(\theta_v - \theta_i)}$$

Example # 1:

e) Calculate the average power and the reactive power at the terminals of the network shown in the figure next if:

$$v = 100\, cos(\omega t + 15)\ V$$

$$i = 4\, sin(\omega t - 15)\ A$$



f) State whether the network inside the box is absorbing or delivering average power.

g) State whether the network inside the box is absorbing or supplying magnetizing vars.

Solution:

a) Because i is expressed in terms of the sine function, the first step in the calculation for P and Q is to rewrite i as a cosine function:

$$i = 4\cos(\omega t - 105^\circ)\ \mathrm{A}$$

Thus,

$$P = \frac{V_m I_m}{2} cos\,(\theta_v - \theta_i)$$

$$P = \frac{100(4)}{2} cos\,(15 - -105)$$

$$P = 200\, cos\,(120) = -100W$$

And $$Q = \frac{V_m I_m}{2}\ sin\,(\theta_v - \theta_i)$$

$$Q = \frac{100(4)}{2}\ sin\,(15 - -105)$$

$$Q = 200\, sin\,(120) = 173.21\ VAR$$

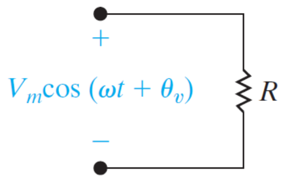
b) Since P is negative $(= -100W)$, the network is delivering (or supplying) power to its terminals.

c) Since Q is positive $(= +173.21\ VAR)$, the network is consuming magnetizing VARs at its terminals.

9.4 The rms Value and Power Calculations

Assuming that, the voltage across the resistor is:

$$v = V_m\, cos(\omega t + \theta_v)\ V$$



Then, the instantaneous power of the resistor is:

$$p = \frac{v(t)^2}{R}$$

$$p = \frac{\left(V_m\, cos(\omega t+\theta_v)\right)^2}{R}$$

Thus, the average power delivered to the resistor is:

$$P = \frac{1}{T}\int_{t_0}^{t_0+T} \frac{V_m^2 \cos^2(\omega t + \theta_v)}{R} dt$$

$$= \frac{1}{R}\underbrace{\left[\frac{1}{T}\int_{t_0}^{t_0+T} V_m^2 \cos^2(\omega t + \theta_v) dt\right]}_{V_{rms}^2}$$

Therefore,

$$P = \frac{V_{\text{rms}}^2}{R}$$

If the resistor current is:

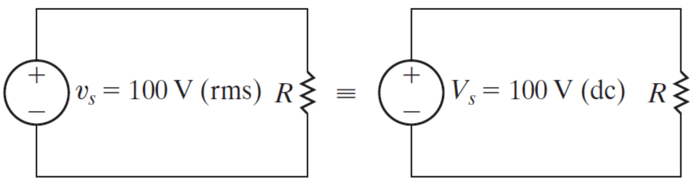
$$i = I_m\, cos(\omega t + \theta_i)\ A$$

Then, the average power is

$$P = I_{\text{rms}}^2 R$$

The rms value is called the ***effective value***!

Note that, the rms value of a sinusoidal source delivers the same energy to a resistor as a DC source does of the same value during the same period.



The average power can be expressed in terms of the rms value as:

$$\begin{aligned} P &= \frac{V_m I_m}{2} \cos(\theta_v - \theta_i) \\ &= \frac{V_m}{\sqrt{2}} \frac{I_m}{\sqrt{2}} \cos(\theta_v - \theta_i) \\ &= V_{\text{eff}} I_{\text{eff}} \cos(\theta_v - \theta_i); \end{aligned}$$

or $$\boldsymbol{P = V_{rms}\, I_{rms} \cos(\theta_v - \theta_i)}$$

$$\boldsymbol{P = V_{rms}\, I_{rms}\, pf}$$

Similarly,

$$Q = V_{\text{eff}} I_{\text{eff}} \sin(\theta_v - \theta_i)$$

or $$\boldsymbol{Q = V_{rms}\, I_{rms} \sin(\theta_v - \theta_i)}$$

where $I_{\text{eff}} = I_{rms}$ and $V_{\text{eff}} = V_{rms}$!

Example # 2:

a) A sinusoidal voltage having a maximum amplitude of 625 V is applied to the terminals of a 50Ω resistor. Find the average power delivered to the resistor.

b) Repeat (a) by first finding the current in the resistor.

Solution:

a) The rms value of the applied voltage is:

$$V_{rms} = \frac{V_m}{\sqrt{2}} = \frac{625}{\sqrt{2}} = 441.94V$$

The average (real) power dissipated in the resistor is:

$$P = \frac{V_{\text{rms}}^2}{R} \Rightarrow P = \frac{(441.94)^2}{50} = 3906.25 \text{ W}$$

b) The maximum current in the resistor is:

$$I_m = \frac{V_m}{R} = \frac{625}{50} = 12.5A \Rightarrow I_{rms} = \frac{I_m}{\sqrt{2}} = \frac{12.5}{\sqrt{2}} = 8.84A$$

or the rms value of the current can found directly from the rms value of the voltage as:

$$I_{rms} = \frac{V_{rms}}{R} = \frac{441.94}{50} = 8.84A!$$

The average (real) power dissipated in the resistor is:

$$P = \frac{(441.94)^2}{50} = 3906.25\text{ W}$$

$$P = (8.84)^2 50 = 3906.25\text{ W}$$

Note that, the reactive power, Q, is zero because the current and the voltage of a resistor are in phase!

9.5 The Complex Power

- **Complex power** is the complex sum of the average (real) power and the reactive power, or

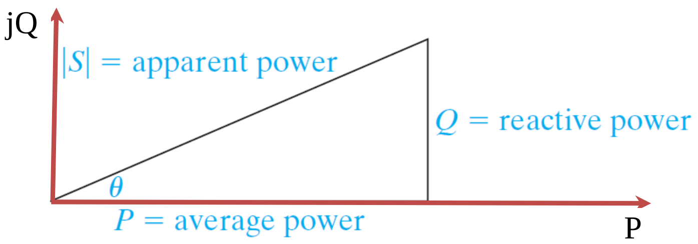
$$S = P + jQ.$$

- The complex power is measured in Volt-Ampere (VA).
- The magnitude of the complex power called the ***apparent power***;

$$|\boldsymbol{S}| = \sqrt{P^2 + Q^2}$$

- The apparent power represents the volt-amp capacity required to supply the average power.
- The ***power triangle*** is a plot of the complex power in the complex plane.

The power triangle for an inductive load.



Note that, the power triangle for a capacitive load lies in the 4th quadrant of the complex plane.

- The angle of the complex power is the same as the power factor angle;

$$\boldsymbol{\theta} = \boldsymbol{\theta_{ZL}} = \boldsymbol{\theta_v} - \boldsymbol{\theta_i}$$

$$\tan\theta = \frac{Q}{P}$$

But, $$\frac{Q}{P} = \frac{(V_m I_m/2)\sin(\theta_v - \theta_i)}{(V_m I_m/2)\cos(\theta_v - \theta_i)}$$

$$= \tan(\theta_v - \theta_i).$$

- The Average (real) power is real part of the complex power;

$$P = \Re\{S\}$$

$$\boldsymbol{P = |S| \cos\theta}$$

$$\boldsymbol{P = |S|\, pf}$$

- And the reactive power is imaginary part of the complex power;

$$Q = \Im\{S\}$$

$$\boldsymbol{Q = |S| \sin\theta}$$

$$\boldsymbol{Q = |S|\, rf}$$

Example # 3:

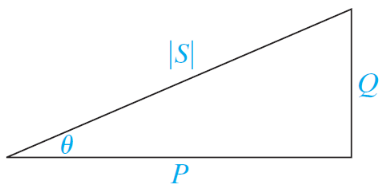
An electrical load operates at 240 V rms. The load absorbs an average power of 8 kW at a lagging power factor of 0.8.

a) Calculate the complex power of the load.

b) Calculate the impedance of the load.

Solution:

a) From the power triangle shown



$$P = |S| \cos\theta,$$

$$Q = |S| \sin\theta.$$

Now, because $\cos\theta = 0.8$, $\sin\theta = 0.6$. Therefore

$$|S| = \frac{P}{\cos\theta} = \frac{8\text{ kW}}{0.8} = 10\text{ kVA},$$

$$Q = 10\sin\theta = 6\text{ kVAR},$$

and

$$S = 8 + j6\text{ kVA}.$$

b) To calculate the load impedance,

$$|Z| = \frac{|V_{\text{eff}}|}{|I_{\text{eff}}|}$$

To find the current,

$$P = V_{\text{eff}} I_{\text{eff}} \cos(\theta_v - \theta_i)$$

$$8000 = (240) I_{\text{eff}} (0.8)$$

$$I_{\text{eff}} = 41.67A$$

The power angle is the same as the impedance angle;

$$\theta = \cos^{-1}(0.8) = 36.87^\circ.$$

Therefore, the load impedance is:

$$|Z| = \frac{|V_{\text{eff}}|}{|I_{\text{eff}}|} = \frac{240}{41.67} = 5.76.$$

Hence,

$$Z = 5.76 \underline{/36.87^\circ}\ \Omega = 4.608 + j3.456\ \Omega.$$

9.6 Power Calculations

Recall,

$$S = P + jQ.$$

But, $P = \frac{V_m I_m}{2} \cos(\theta_v - \theta_i)$ and $Q = \frac{V_m I_m}{2} \sin(\theta_v - \theta_i)$

Thus, $\boldsymbol{S} = \frac{V_m I_m}{2} \cos(\theta_v - \theta_i) + j\frac{V_m I_m}{2} \sin(\theta_v - \theta_i)$

$$= \frac{V_m I_m}{2}[\cos(\theta_v - \theta_i) + j\sin(\theta_v - \theta_i)]$$

$$= \frac{V_m I_m}{2} e^{j(\theta_v - \theta_i)} = \frac{1}{2} V_m I_m \underline{/(\theta_v - \theta_i)}.$$

$$S = V_{\text{eff}} I_{\text{eff}} \underline{/(\theta_v - \theta_i)}$$

$$= V_{\text{eff}} I_{\text{eff}} e^{j(\theta_v - \theta_i)}$$

$$= V_{\text{eff}} e^{j\theta_v} I_{\text{eff}} e^{-j\theta_i}$$

Note that,

$$I_{\text{eff}}e^{-j\theta_i} = I_{\text{eff}}\cos(-\theta_i) + jI_{\text{eff}}\sin(-\theta_i)$$
$$= I_{\text{eff}}\cos(\theta_i) - jI_{\text{eff}}\sin(\theta_i)$$
$$= \mathbf{I}^*_{\text{eff}}.$$

where $\mathbf{I}_{\mathbf{eff}}{}^*$ is the complex conjugate of $\mathbf{I}_{\mathbf{eff}}$

Therefore,

$$S = V_{\text{eff}}I_{\text{eff}}{}^* \quad \Rightarrow \quad S = V_{\text{rms}}I_{rms}{}^*$$

where $\mathbf{I}_{\mathbf{rms}}{}^*$ is the complex conjugate of $\mathbf{I}_{\mathbf{rms}}$

If the maximum amplitude of the voltage and current are known, then:

$$S = \frac{1}{2}\mathbf{VI}^*$$

Example # 4:

Find the complex power if

$$\mathbf{V} = 100\angle 15^\circ \text{ V},$$
$$\mathbf{I} = 4\angle -105^\circ \text{ A}.$$

Solution:

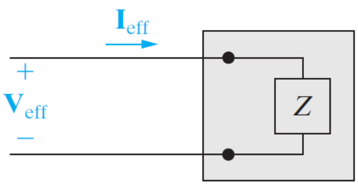
$$S = \frac{1}{2}(100\angle 15^\circ)(4\angle +105^\circ) = 200\angle 120^\circ$$
$$= -100 + j173.21 \text{ VA}.$$

Such that,

$$P = -100 \text{ W},$$
$$Q = 173.21 \text{ VAR}.$$

Alternate Forms for Complex Power

If the equivalent impedance of the circuit is Z, then the voltage across the circuit is:



$$\mathbf{V}_{\text{eff}} = Z\mathbf{I}_{\text{eff}}.$$

Recall, the complex power is:

$$S = \mathbf{V}_{\text{eff}}\mathbf{I}_{\text{eff}}^{*}$$

$$S = Z\mathbf{I}_{\text{eff}}\mathbf{I}_{\text{eff}}^{*}$$

$$= |\mathbf{I}_{\text{eff}}|^2 Z$$

$$= |\mathbf{I}_{\text{eff}}|^2 (R + jX)$$

$$= |\mathbf{I}_{\text{eff}}|^2 R + j|\mathbf{I}_{\text{eff}}|^2 X = P + jQ,$$

from which

$$P = |\mathbf{I}_{\text{eff}}|^2 R = \frac{1}{2}I_m^2 R,$$

$$Q = |\mathbf{I}_{\text{eff}}|^2 X = \frac{1}{2}I_m^2 X.$$

Also,

$$S = \mathbf{V}_{\text{eff}}\left(\frac{\mathbf{V}_{\text{eff}}}{Z}\right)^{*} = \frac{|\mathbf{V}_{\text{eff}}|^2}{Z^{*}} = P + jQ$$

Note that if Z is a pure resistive element,

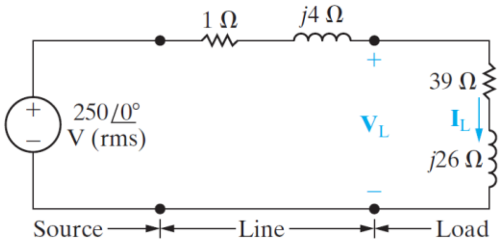
$$P = \frac{|\mathbf{V}_{\text{eff}}|^2}{R}$$

and if Z is a pure reactive element,

$$Q = \frac{|\mathbf{V}_{\text{eff}}|^2}{X}$$

Example # 5:

In the circuit shown in the figure, a load having an impedance of $39 + j26\ \Omega$ is fed from a voltage source through a line having an impedance of $1 + j4\ \Omega$. The effective, or rms, value of the source voltage is 250 V.



a) Calculate the load current $\mathbf{I}_L$ and voltage $\mathbf{V}_L$.

b) Calculate the average and reactive power delivered to the load.

c) Calculate the average and reactive power delivered to the line.

d) Calculate the average and reactive power supplied by the source.

Solution:

a) The load current is:

$$\mathbf{I}_L = \frac{250\ \angle 0^\circ}{40 + j30} = 4 - j3 = 5\ \angle -36.87^\circ \text{ A (rms)}$$

The load voltage is:

$$\mathbf{V}_L = (39 + j26)\mathbf{I}_L = 234 - j13$$

$$= 234.36\ \angle -3.18^\circ \text{ V (rms)}.$$

b) The load complex power is:

$$S = \mathbf{V}_L\mathbf{I}_L^* = (234 - j13)(4 + j3)$$

$$= 975 + j650 \text{ VA}.$$

Since, $S = P + jQ$, therefore, $P = 975W$ and $Q = 650VAR$

c) The line power is:

$$P = |\mathbf{I}_{\text{eff}}|^2 R \Rightarrow P = (5)^2(1) = 25 \text{ W},$$

$$Q = |\mathbf{I}_{\text{eff}}|^2 X \Rightarrow Q = (5)^2(4) = 100 \text{ VAR}.$$

d) The complex power delivered by the source can be obtained either by adding the complex power of the line and the load

$$S = 25 + j100 + 975 + j650$$

$$= 1000 + j750 \text{ VA}.$$

or

$$S_s = -250\mathbf{I}_L^*.$$

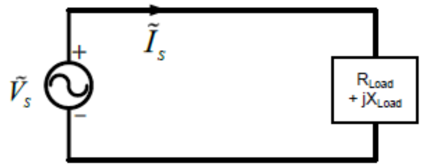
$$S_s = -250(4 + j3) = -(1000 + j750) \text{ VA}.$$

9.7 Power Factor Correction (Improvement)

- Most load are inductive load and have a lagging power factor.
- The original complex load power is:

$$\mathbf{S}_{\mathbf{old}} = P_{old} + jQ_{old}$$

$$\mathbf{S}_{\mathbf{old}} = |\mathbf{S}_{\mathbf{old}}| \angle \theta_{old}$$



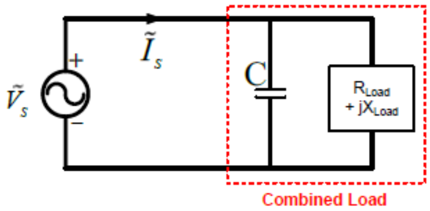
- The new complex power that results from *adding a capacitor in parallel with the load* is:

$$\mathbf{S}_{\mathbf{new}} = P_{old} + jQ_{new}$$

$$\mathbf{S}_{\mathbf{new}} = |\mathbf{S}_{\boldsymbol{new}}| \angle \theta_{new}$$

Because the capacitor does not have real power.

θ_{new} is specified by the (new) required power factor!

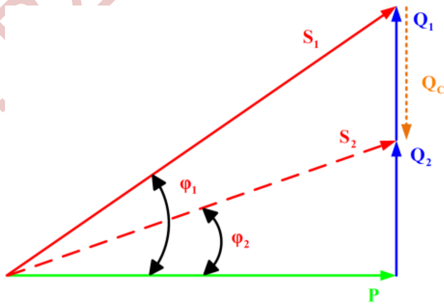


- Since $\mathbf{S}_{\mathbf{new}} = \mathbf{S}_{\mathbf{old}} + \mathbf{S}_{\mathbf{cap}}$, the added capacitor complex power is:

$$\mathbf{S}_{\mathbf{cap}} = \mathbf{S}_{\mathbf{new}} - \mathbf{S}_{\mathbf{old}}$$

- But, the capacitor is purely reactive;

$$\mathbf{S}_{\mathbf{cap}} = 0 + jQ_{cap}$$



and,

$$\mathbf{S}_{\mathbf{cap}} = \boldsymbol{V}_{\boldsymbol{rms}}\boldsymbol{I}^*_{\boldsymbol{rms}}$$

$$\mathbf{S}_{\mathbf{cap}} = (V_{rms} \angle \theta_v)\boldsymbol{I}^*_{\boldsymbol{rms}}$$

Noting that, $\boldsymbol{I}_{\boldsymbol{rms}} = \dfrac{V_{rms} \angle \theta_v}{1/j\omega C}$

$$\boldsymbol{I}_{\boldsymbol{rms}} = \omega C\, V_{rms} \angle(\theta_v + 90)$$

Therefore, $\boldsymbol{I}^*_{\boldsymbol{rms}} = \omega C\, V_{rms} \angle(-\theta_v - 90)$

Substituting $\boldsymbol{I}^*_{\boldsymbol{rms}}$ in the capacitor complex power yields:

$$\therefore \mathbf{S}_{\mathbf{cap}} = (V_{rms} \angle \theta_v)(\omega C\, V_{rms} \angle(-\theta_v - 90))$$

$$\mathbf{S}_{\mathbf{cap}} = \omega C\, V^2_{rms} \angle -90$$

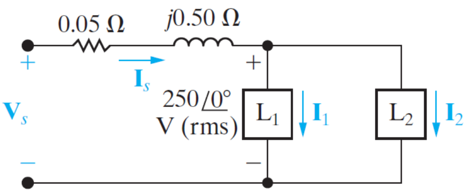
$$\mathbf{S}_{\mathbf{cap}} = -\boldsymbol{j}\omega C\, V^2_{rms}$$

Also, $Q_{cap} = Q_{new} - Q_{old} = -\boldsymbol{j}\omega C\, V^2_{rms}$

Note that, most electricity distribution companies require the load power factor to be greater than 0.92 lag!

Example # 6:

The two loads in the circuit shown in the figure can be described as follows: Load 1 absorbs an average power of 8 kW at a leading power factor of 0.8. Load 2 absorbs 20 kVA at a lagging power factor of 0.6.



a) Determine the power factor of the two loads in parallel.

b) Determine the apparent power required to supply the loads, the magnitude of the current, $\mathbf{I_s}$, and the average power loss in the transmission line.

c) Given that the frequency of the source is 60 Hz, compute the value of the capacitor that would correct the power factor to 1 if placed in parallel with the two loads. Recompute the values in (b) for the load with the corrected power factor.

Solution:

a) The voltage is given as rms value. The source current is:

$$\mathbf{I}_s = \mathbf{I}_1 + \mathbf{I}_2$$

The complex power of the two loads:

$$S = (250)\mathbf{I}_s^*$$

$$= (250)(\mathbf{I}_1 + \mathbf{I}_2)^*$$

$$= (250)\mathbf{I}_1^* + (250)\mathbf{I}_2^*$$

$$= S_1 + S_2.$$

But,

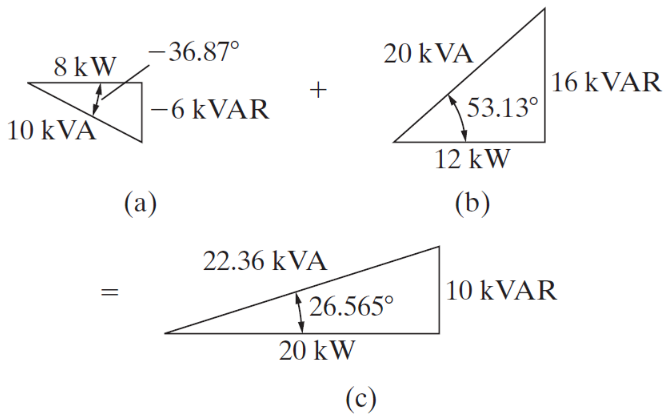
$$S_1 = 8000 - j\frac{8000(.6)}{(.8)}$$

$$= 8000 - j6000 \text{ VA},$$

And

$$S_2 = 20{,}000(.6) + j20{,}000(.8)$$

$$= 12{,}000 + j16{,}000 \text{ VA}.$$



It follows that

$$S = 20{,}000 + j10{,}000 \text{ VA},$$

and

$$\mathbf{I}_s^* = \frac{20{,}000 + j10{,}000}{250} = 80 + j40 \text{ A}.$$

Therefore

$$\mathbf{I}_s = 80 - j40 = 89.44 \angle -26.57^\circ \text{ A}.$$

The power factor of the combined load is:

$$pf = \mathbf{cos}\ (\boldsymbol{\theta_v - \theta_i})$$

$$pf = \cos\ (0 - -26.57) = 0.8944 \text{ lagging}$$

Note that, the total reactive power (Q) of the load is positive

b) The apparent power which must be supplied to these loads is:

$$|S| = |20 + j10| = 22.36 \text{ kVA}$$

The magnitude of the current that supplies this apparent power is:

$$|\mathbf{I}_s| = |80 - j40| = 89.44 \text{ A}$$

The average power lost in the line is:

$$P_{\text{line}} = |\mathbf{I}_s|^2 R = (89.44)^2(0.05) = 400\ W$$

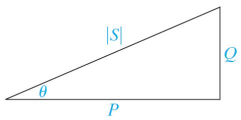
The average power supplied by the source is:

$$P_s = P_L + P_{Line} = 20{,}000 + 400 = 20{,}400W$$

c) The new power factor is unity; which means that the reactive power of the loads and the capacitor is zero;

$$\theta_{new} = \cos^{-1} 1 = 0$$

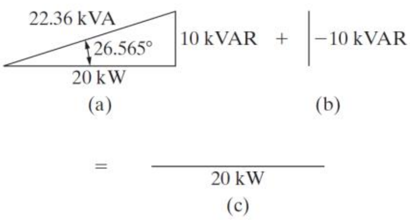
$$\text{S}_{\text{new}} = P_{old} + jQ_{new}$$



But,

$$Q_{new} = P_{old} \tan \theta_{new} = 20{,}000 \tan 0 = 0$$

Therefore, $\text{S}_{\text{new}} = 20{,}000 + j0$



$$\mathbf{S_{cap} = S_{new} - S_{old}}$$

$$\mathbf{S_{cap}} = 20{,}000 - (20{,}000 + j10{,}000)$$

$$\mathbf{S_{cap}} = -j10{,}000 = -j\omega C\ V_{rms}^2$$

$$C = \frac{10{,}000}{\omega V_{rms}^2}$$

where $\omega = 2\pi f = 2\pi(60) = 376.99\ rad/s$

$$C = \frac{10{,}000}{376.99\ (250^2)} = 424.4\ \mu F$$

Alternative way to calculate C:

Since the new power factor is unity,

$$Q_{cap} = Q_{old} = 10{,}000VAR$$

And the reactance of the capacitor is:

$$X = \frac{|V_{eff}|^2}{Q} = \frac{(250)^2}{-10{,}000} = -6.25\ \Omega.$$

$$C = \frac{-1}{\omega X} = \frac{-1}{(376.99)(-6.25)} = 424.4\ \mu\text{F}.$$

Since the new power factor is unity,

$$|S| = P = 20\ \text{kVA}.$$

The new current from the supply is:

$$|\mathbf{I_s}| = \frac{P_L}{V_{rms}} = \frac{20{,}000}{250} = 80A$$

The current is reduced compared to the case without the capacitor!

The average power lost in the line is reduced to:

$$P_{\text{line}} = |\mathbf{I}_s|^2 R = (80)^2(0.05) = 320\ W.$$

The average power supplied by the source is:

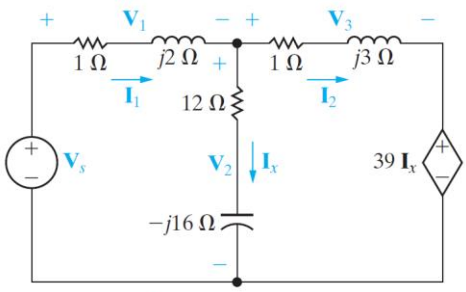
$$P_s = P_L + P_{Line} = 20{,}000 + 320 = 20{,}320W$$

Example # 7:

a) Calculate the total average and reactive power delivered to each impedance in the circuit shown in Fig. 10.17.

b) Calculate the average and reactive powers associated with each source in the circuit.

c) Verify that the average power delivered equals the average power absorbed, and that the magnetizing reactive power delivered equals the magnetizing reactive power absorbed.



$\mathbf{V}_s = 150\angle 0^\circ$ V

$\mathbf{V}_1 = (78 - j104)$ V $\mathbf{I}_1 = (-26 - j52)$ A

$\mathbf{V}_2 = (72 + j104)$ V $\mathbf{I}_x = (-2 + j6)$ A

$\mathbf{V}_3 = (150 - j130)$ V $\mathbf{I}_2 = (-24 - j58)$ A

Figure 10.17 ▲ The circuit, with solution, for Example 10.7.

Solution:

Note that, the source voltage is given as a maximum amplitude not as an rms value!

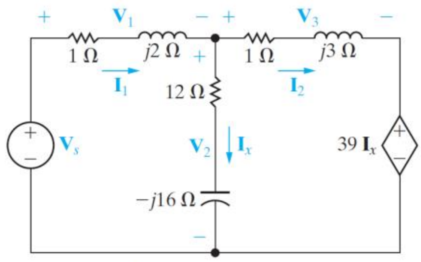
a) The complex power delivered to the $1 + j2\Omega$ impedance is:

$$S_1 = \frac{1}{2}\mathbf{V}_1\mathbf{I}_1^* = P_1 + jQ_1$$

$$= \frac{1}{2}(78 - j104)(-26 + j52)$$

$$= \frac{1}{2}(3380 + j6760)$$

$$= 1690 + j3380 \text{ VA.}$$



$\mathbf{V}_s = 150\angle 0^\circ$ V

$\mathbf{V}_1 = (78 - j104)$ V $\mathbf{I}_1 = (-26 - j52)$ A

$\mathbf{V}_2 = (72 + j104)$ V $\mathbf{I}_x = (-2 + j6)$ A

$\mathbf{V}_3 = (150 - j130)$ V $\mathbf{I}_2 = (-24 - j58)$ A

The complex power delivered to the $12 - j16\Omega$ impedance is:

$$S_2 = \frac{1}{2}\mathbf{V}_2\mathbf{I}_x^* = P_2 + jQ_2$$

$$= \frac{1}{2}(72 + j104)(-2 - j6)$$

$$= 240 - j320 \text{ VA.}$$

The complex power delivered to the $1 + j3\Omega$ impedance is:

$$S_3 = \frac{1}{2}\mathbf{V}_3\mathbf{I}_2^* = P_3 + jQ_3$$

$$= \frac{1}{2}(150 - j130)(-24 + j58)$$

$$= 1970 + j5910 \text{ VA.}$$

b) The complex power associated with the independent source is:

$$S_s = -\frac{1}{2}\mathbf{V}_s\mathbf{I}_1^* = P_s + jQ_s$$

$$= -\frac{1}{2}(150)(-26 + j52)$$

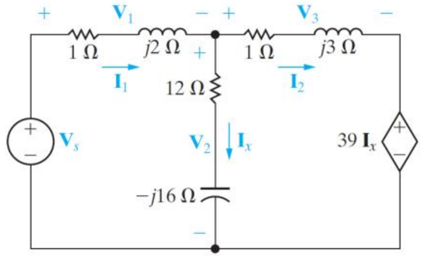
$$= 1950 - j3900 \text{ VA.}$$

The complex power associated with the current-controlled voltage source is:

$$S_x = \frac{1}{2}(39\mathbf{I}_x)(\mathbf{I}_2^*) = P_x + jQ_x$$

$$= \frac{1}{2}(-78 + j234)(-24 + j58)$$

$$= -5850 - j5070 \text{ VA.}$$



$\mathbf{V}_s = 150\angle 0^\circ$ V

$\mathbf{V}_1 = (78 - j104)$ V $\mathbf{I}_1 = (-26 - j52)$ A

$\mathbf{V}_2 = (72 + j104)$ V $\mathbf{I}_x = (-2 + j6)$ A

$\mathbf{V}_3 = (150 - j130)$ V $\mathbf{I}_2 = (-24 - j58)$ A

c) The total power absorbed by the passive impedances and the independent voltage source is:

$$P_{\text{absorbed}} = P_1 + P_2 + P_3 + P_s = 5850 \text{ W.}$$

The dependent voltage source is the only circuit element delivering average power. Thus:

$$P_{\text{delivered}} = 5850 \text{ W.}$$

Magnetizing reactive power is being absorbed by the two horizontal branches. Thus:

$$Q_{\text{absorbed}} = Q_1 + Q_3 = 9290 \text{ VAR.}$$

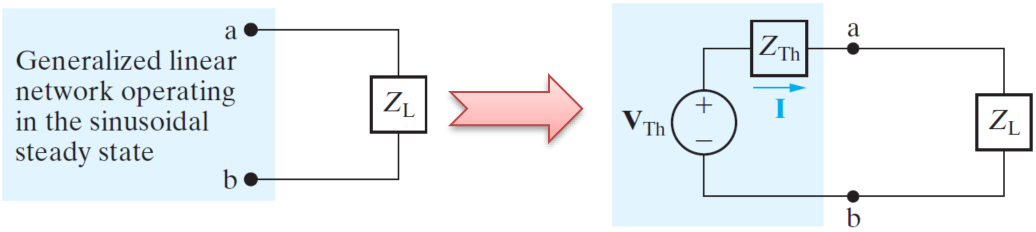
Magnetizing reactive power is being delivered by the independent voltage source, the capacitor in the vertical impedance branch, and the dependent voltage source. Therefore:

$$Q_{\text{delivered}} = 9290 \text{ VAR.}$$

The absorbed and delivered average and reactive powers are balanced!

9.8 Maximum Power Transfer

- For any network, to find the load impedance Z_L which draws the maximum power, the network is represented by its Thevenin's or Norton's equivalent circuit.



- Assuming that, the Thevenin's impedance (Z_{Th}) and the load impedance (Z_L), respectively are:

$$Z_{Th} = R_{Th} + jX_{Th},$$

$$Z_L = R_L + jX_L.$$

- Assuming that, the Thevenin's voltage ($\boldsymbol{V_{Th}}$) is given as an "rms" value, the rms value of the current is found by KVL:

$$\mathbf{I} = \frac{\mathbf{V}_{Th}}{(R_{Th} + R_L) + j(X_{Th} + X_L)}.$$

- The average power delivered to the load is:

$$P = |\mathbf{I}|^2 R_L$$

- Therefore, the load power is

$$P = \frac{|\mathbf{V}_{Th}|^2 R_L}{(R_{Th} + R_L)^2 + (X_{Th} + X_L)^2}$$

- For any network, $\boldsymbol{V_{Th}}, R_{Th}, and\ X_{Th}$ are known, whilst $R_L, and\ X_L$ are unknown and can be adjusted for maximum power transferred to the load.
- To maximize the load power P, $R_L, and\ X_L$ must be found such that;

$$\frac{\partial P}{\partial X_L} = 0 \quad \text{and} \quad \frac{\partial P}{\partial R_L} = 0$$

Thus,

$$\frac{\partial P}{\partial X_L} = \frac{-|\mathbf{V}_{Th}|^2 2R_L(X_L + X_{Th})}{[(R_L + R_{Th})^2 + (X_L + X_{Th})^2]^2},$$

Equating the derivative with zero yields:

$$\boxed{X_L = -X_{Th}} \qquad (1)$$

Also,

$$\frac{\partial P}{\partial R_L} = \frac{|\mathbf{V}_{Th}|^2[(R_L + R_{Th})^2 + (X_L + X_{Th})^2 - 2R_L(R_L + R_{Th})]}{[(R_L + R_{Th})^2 + (X_L + X_{Th})^2]^2}$$

Again, equating the derivative with zero yields:

$$\boxed{R_L = \sqrt{R_{Th}^2 + (X_L + X_{Th})^2}} \qquad (2)$$

- Combining the conditions in eqs. (1) and (2) for maximum power transferred to the load yields:

$$R_L = \sqrt{R_{Th}^2 + (-X_{Th} + X_{Th})^2}$$

$$R_L = \sqrt{R_{Th}^2} \qquad R_L = R_{Th}$$

$$\boldsymbol{Z_L = R_{Th} - jX_{Th}}$$

$$\boxed{Z_L = Z_{Th}^*}$$

- The current under the above condition is:

$$\boldsymbol{I} = \frac{\boldsymbol{V_{Th_{rms}}}}{R_{Th} + jX_{Th} + R_L + jX_L}$$

$$\boldsymbol{I} = \frac{\boldsymbol{V_{Th_{rms}}}}{R_{Th} + jX_{Th} + R_{Th} - jX_{Th}}$$

$$\boldsymbol{I} = \frac{\boldsymbol{V_{Th_{rms}}}}{R_{Th} + R_{Th}}$$

- The maximum average power transferred to the load is:

$$P_{max} = |I|^2 R_{Th}$$

$$P_{max} = \frac{|V_{Th_{rms}}|^2}{(R_{Th} + R_{Th})^2} R_{Th}$$

$$P_{max} = \frac{|V_{Th_{rms}}|^2}{(2R_{Th})^2} R_{Th}$$

$$P_{max} = \frac{|V_{Th_{rms}}|^2}{4R_{Th}} \quad \text{or} \quad P_{max} = \frac{|V_{Th_{rms}}|^2}{4R_L}$$

For Norton's equivalent:

$$P_{max} = \frac{|\boldsymbol{I_{N_{rms}}}|^2}{\boldsymbol{4}} R_N \quad \text{or} \quad P_{max} = \frac{|\boldsymbol{I_{N_{rms}}}|^2}{\boldsymbol{4}} R_L$$

- In terms of the maximum amplitude of the voltage (or current), the maximum power is:

$$P_{max} = \frac{|V_{Th_m}|^2}{8R_{Th}} \quad \text{or} \quad P_{max} = \frac{|I_{N_m}|^2}{8} R_N$$

For a Restricted Z_L:

1. If the values of R_L and $X_L$are restricted to a limited range of values, but the phase can be changed, then for maximum power transfer:
   - ➢ Adjust X_L as close to $-X_{Th}$ as possible
   - ➢ Choose R_L as close $\sqrt{R_{\text{Th}}^2 + (X_{\text{L}} + X_{\text{Th}})^2}$ to as possible

2. If the magnitude of Z_L can be varied, but not the phase, then set:

$$|Z_{\text{L}}| = |Z_{\text{Th}}|$$

Example # 8:

a) For the circuit shown in Fig. 10.20, determine the impedance Z_{L} that results in maximum average power transferred to Z_{L}.

b) What is the maximum average power transferred to the load impedance determined in (a)?

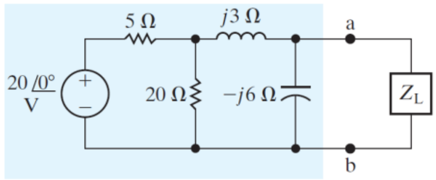


Figure 10.20 ▲ The circuit for Example 10.8.

Solution:

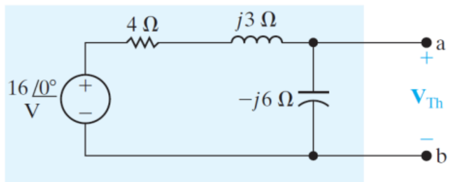
a) Using source transformation, 20V in series with 5Ω can be transformed to a current source in parallel with the same resistor;

$$I_1 = \frac{20}{5} = 4A$$

The resistors 5Ω and 20Ω are in parallel and have an equivalent resistor of:

$$R_{eq1} = \frac{5(20)}{25} = 4\Omega$$

The current source I_1 in parallel with 4Ω can be transformed to a voltage source in series with the same resistor;



$$V_1 = I_1(R_{eq1}) = 4\,(4) = 16V$$

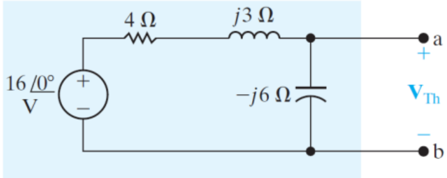
Now, to find $\boldsymbol{V_{Th}}$, use potential divider;

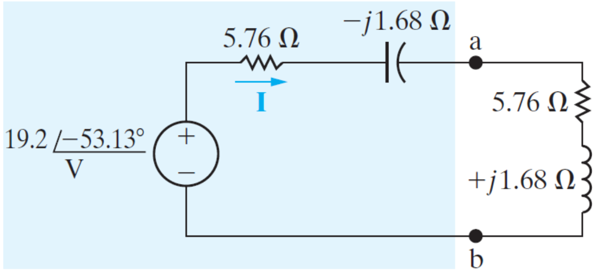
$$\mathbf{V}_{\text{Th}} = \frac{16\,\angle 0^\circ}{4 + j3 - j6}(-j6)$$

$$= 19.2\,\angle -53.13^\circ = 11.52 - j15.36\text{ V}$$

To find $\boldsymbol{Z_{Th}}$, this is case 1, deactivate the independent source and look into the terminal a-b:

$$Z_{Th} = \frac{(-j6)(4 + j3)}{4 + j3 - j6} = 5.76 - j1.68\ \Omega$$





For maximum power transfer,

$$Z_L = Z_{Th}^*$$

$$Z_L = 5.76 + j1.68\ \Omega$$

b) To find the maximum power transferred to the load, calculate the effective current first as,

$$I_{eff} = \frac{19.2/\sqrt{2}}{2(5.76)} = 1.1785\ \text{A}$$

then $$P = I_{eff}^2(5.76) = 8\ \text{W}$$

or directly,

$$P_{max} = \frac{|V_{Thm}|^2}{8R_{Th}} = \frac{|19.2|^2}{8(5.76)} = 8W$$

Chapter 10

Introduction to Laplace Transform

- Laplace Transform is a powerful analytical technique that is widely used to study the behavior of linear, lumped-parameter circuits.
- Besides, it is used to analyze circuits described by sets of linear differential equations.
- Moreover, it relates, in a systematic fashion, the time domain behavior of a circuit to its frequency-domain behavior.

10.1 Definition of Laplace Transform

- The Laplace transform of a function is given by the expression:

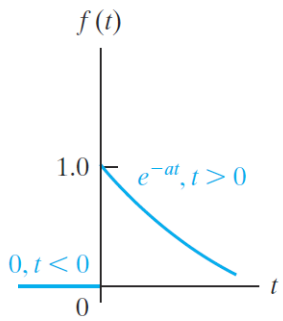
$$F(s) = \mathscr{L}\{f(t)\} = \int_0^{\infty} f(t)e^{-st}\,dt$$

which is one sided or unilateral Laplace Transform since its lower limit is zero!

$'s'$ represents the frequency, the complex frequency; $s = \sigma + j\omega$

- The Laplace transform transforms the problem from the time domain to the frequency domain to make manipulation easier; manipulating a set of algebraic equations is easier than manipulating a set of integro-differential equations!
- After obtaining the frequency-domain expression for the unknown, inverse-transform is used to get back the unknown in time domain.
- Though the upper limit of the integral is ∞ (improper integral), here in linear circuits, we are interested in functions that have Laplace Transforms, and hence the integral converges.
- If the function is discontinuous at 0, then use 0^- as the lower limit of the integral!

The initial conditions account for $t < 0^-$!



- **Two Types of Laplace Transform**:
  1- **A Functional Transform:** is the Laplace Transform of a specific function
  2- **An Operational Transform:** defines a general mathematical property of the Laplace Transform; e.g., the derivative of $f(t)$

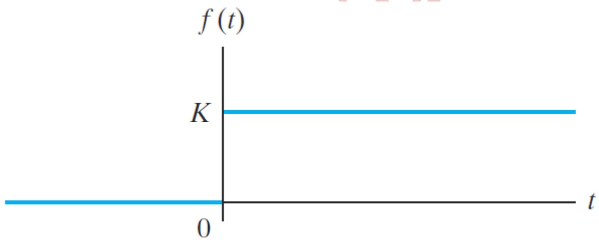
10.2 The Step Function

- It has a discontinuity of a jump at a specific time!
- The **Unit Step Function** is zero for $t < 0$, and has a value of 1 for $t \geq 0$;

$$u(t) = 0 \quad \text{for} \quad t < 0$$
$$u(t) = 1 \quad \text{for} \quad t > 0$$

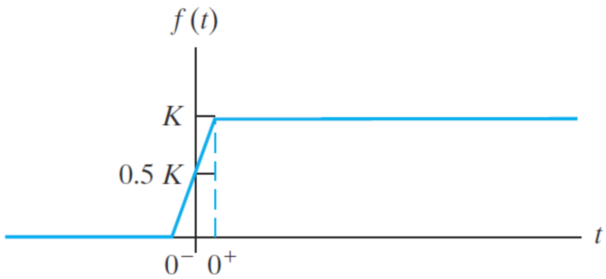
- The **Step Function** is:

$$Ku(t) = 0 \quad \text{for} \quad t < 0$$
$$Ku(t) = K \quad \text{for} \quad t > 0$$



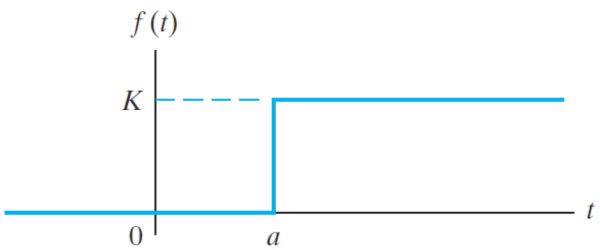
- $Ku(t)$ is not defined at $t = 0$, but it may be assumed that:

$$Ku(0) = 0.5K$$



- The **Shifted Step Function**
  A step that occurs at $t = a$ is:

$$Ku(t-a) = 0 \quad \text{for} \quad t < a$$
$$Ku(t-a) = K \quad \text{for} \quad t > a$$



The plot next is for $Ku(t-a)$ for $a > 0$!

If $'a'$ is positive, the step occurs to the right of the origin, and if $'a'$ is negative the step occurs to the left of the origin.

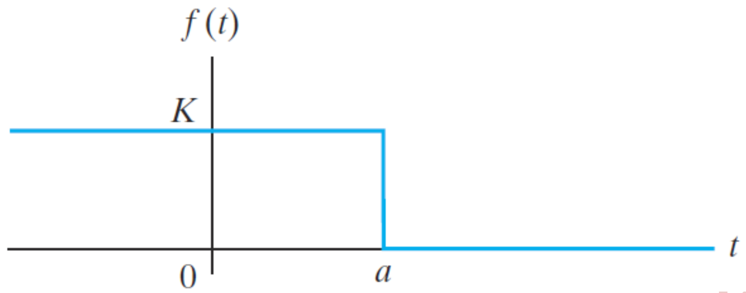
The Step Function is zero when $(t-a)$ is negative, and is K when $(t-a)$ is positive!

➢ The Step Function can be also expressed as:

$$Ku(a - t) = K \quad \text{for} \quad t < a$$

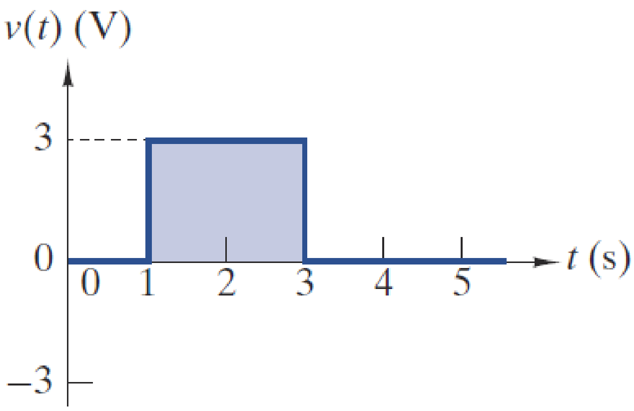
$$Ku(a - t) = 0 \quad \text{for} \quad t > a$$

The plot below is for $Ku(a - t)$ for $a > 0$



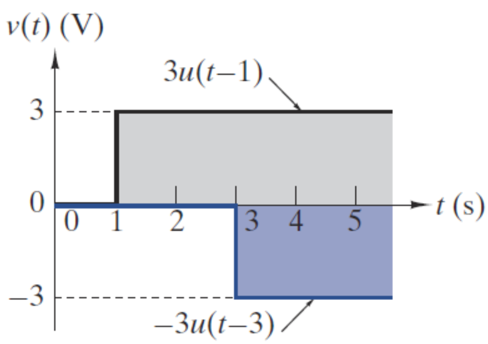
Example # 1:

Express the waveform, the Rectangular Pulse Function, shown in the Figure below in terms of step functions!



Solution:

✓ The amplitude of the pulse jumps to a value of 3V at $t = 1s$; therefore, $3u(t - 1)$ is part of the equation for the waveform.



✓ The pulse returns to zero at $t = 3s$, so an equal and opposite step must occur at $t = 3s$; therefore, $-3u(t - 3)$ is part of the equation of the waveform.

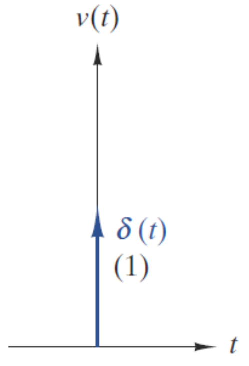
✓ Putting these observations together, the rectangular pulse can be expressed as:

$$v(t) = 3u(t - 1) - 3u(t - 3) \; V$$

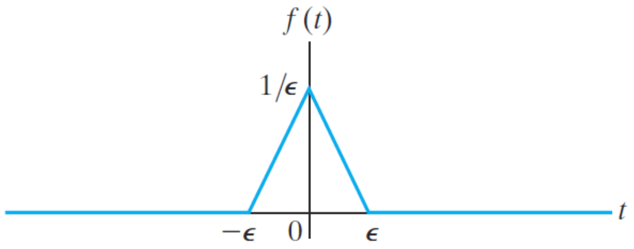
10.3 The Impulse Function (Dirac (Delta) Function)

- It is a signal of infinite amplitude and zero duration.
- Such signals don't exist in nature, but some circuit signals come very close to approximating this definition.
- Impulse function enables defining the derivative of a function at the point of discontinuity.
- The Unit Impulse Function is defined as:

$$\delta(t) = \infty \qquad , t = 0$$
$$\delta(t) = 0 \qquad , t \neq 0$$



- As $\epsilon \to 0$, the function $f(t)$ approaches a unit Impulse Function



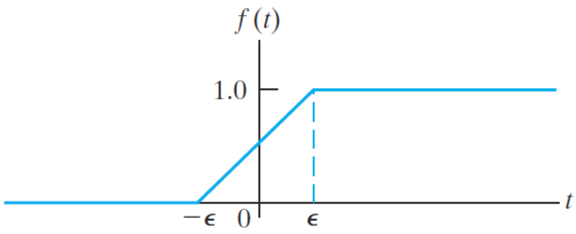
- The area under a delta function is unity!

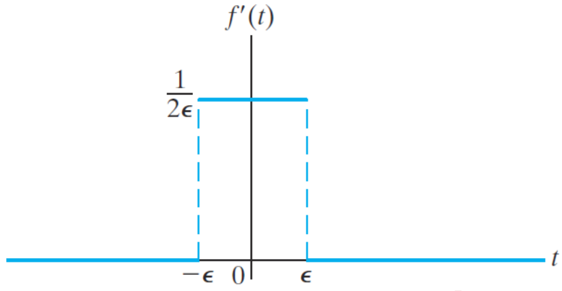
$$\int_{-\infty}^{\infty} \delta(t)dt = \int_{-\epsilon}^{\epsilon} \delta(t)dt = 1$$

- If $K\delta(t)$, then K is the strength of the Impulse Function (also the area of the impulse function).

$$\int_{-\infty}^{\infty} K\delta(t)dt = K$$

❖ Definition: $\delta(t) = \frac{du(t)}{dt}$

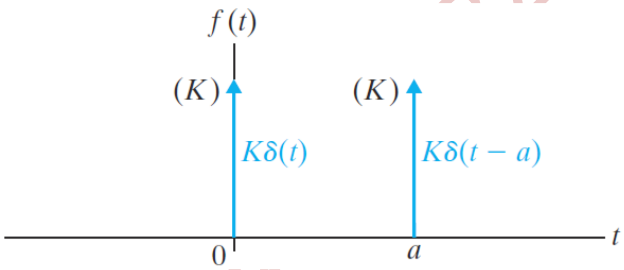




Again as $\epsilon \to 0$, the function $f(t)$ a Unit Step Function, and as $\epsilon \to 0$, the function $f'(t)$ approaches a Unit Impulse Function! As $\epsilon \to 0$, the amplitude $\to \infty$ and the duration $\to 0$!

❖ If the impulse occurs at $t = a$, then $K\delta(t - a)$ is the impulse function.

The graphical symbol of the Impulse function is shown in the Figure below!



10.4 Sifting Property

Assuming that, $f(t)$ is a continuous function at $t = a$ then,

$$\int_{-\infty}^{\infty} f(t)\delta(t-a)dt = \int_{a-\epsilon}^{a+\epsilon} f(t)\delta(t-a)dt, \qquad \text{because } \delta(t-a) = 0 \text{ at } t \neq a$$

$$= \int_{a-\epsilon}^{a+\epsilon} f(a)\delta(t-a)dt, \qquad \text{because } f(t) \text{ is continuous at } t = a$$

$$= f(a)\int_{a-\epsilon}^{a+\epsilon} \delta(t-a)dt$$

$$= f(a)$$

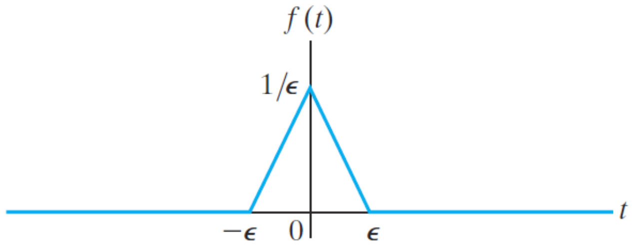
Therefore, $\int_{-\infty}^{\infty} f(t)\delta(t-a)dt = f(a)$

Using the Sifting Property,

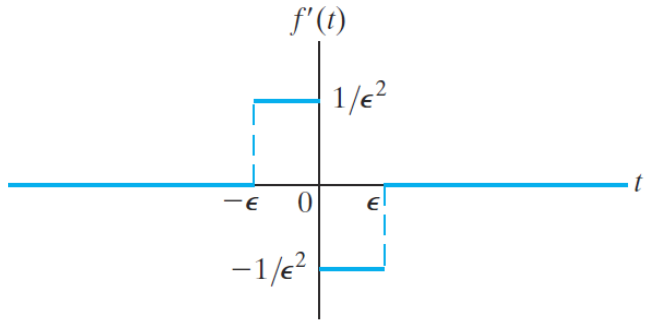
$$\mathcal{L}\{\delta(t)\} = \int_{0^-}^{\infty} \delta(t)e^{-st}\,dt = \int_{0^-}^{\infty} \delta(t)dt = 1$$

That is because, $a = 0$, with $f(a) = f(0) = e^{-s(0)} = 1$!

Again reconsider the function shown in the Figure next, as $\epsilon \to 0$, the function generates an Impulse Function, $\delta(t)$.



The derivative of $f(t)$ is $\delta'(t)$.



The derivative of the Impulse Function is referred to a moment function or unit doublet!

To find the Laplace Transform of $\delta'(t)$,

$$L\{\delta'(t)\} = \lim_{\epsilon \to 0}\left[\int_{-\epsilon}^{0^-} \frac{1}{\epsilon^2} e^{-st}\,dt + \int_{0^+}^{\epsilon}\left(-\frac{1}{\epsilon^2}\right) e^{-st}\,dt\right]$$

$$= \lim_{\epsilon \to 0} \frac{e^{s\epsilon} + e^{-s\epsilon} - 2}{s\epsilon^2}, \qquad \frac{\mathbf{1+1-2}}{\mathbf{0}} = \frac{\mathbf{0}}{\mathbf{0}}$$, use L'Hopital's rule

$$= \lim_{\epsilon \to 0} \frac{se^{s\epsilon} - se^{-s\epsilon}}{2\epsilon s}$$, again use L'Hopital's rule

$$= \lim_{\epsilon \to 0} \frac{s^2 e^{s\epsilon} + s^2 e^{-s\epsilon}}{2s}$$

$$\boxed{L\{\delta'(t)\} = s}$$

In a similar manner, the Laplace Transform of the nth derivative of the Impulse Function is:

$$\boxed{\mathcal{L}\{\delta^{(n)}(t)\} = s^n}$$

10.5 Functional Laplace Transforms

They are the Laplace Transform of specific functions of t.

10.5.1 Derivation of Laplace Transform of Some Functions

❖ **Step Function**

$$\mathcal{L}\{u(t)\} = \int_{0^-}^{\infty} f(t)e^{-st}\,dt = \int_{0^+}^{\infty} 1e^{-st}dt$$

$$= \frac{e^{-st}}{-s}\bigg|_{0^+}^{\infty} = \frac{1}{s}.$$

$$\boxed{\mathcal{L}\{u(t)\} = \frac{1}{s}}$$

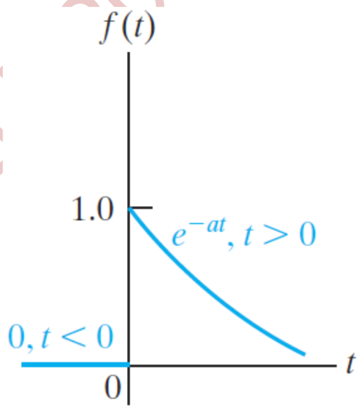
Note that, the function is defined to be zero for $t < 0^-$.

❖ **Decaying Exponential**

$$\mathcal{L}\{e^{-at}\} = \int_{0^+}^{\infty} e^{-at}\,e^{-st}\,dt$$

$$= \int_{0^+}^{\infty} e^{-(a+s)t}\,dt = \frac{1}{s+a}$$

$$\boxed{\mathcal{L}\{e^{-at}\} = \frac{1}{s+a}}$$



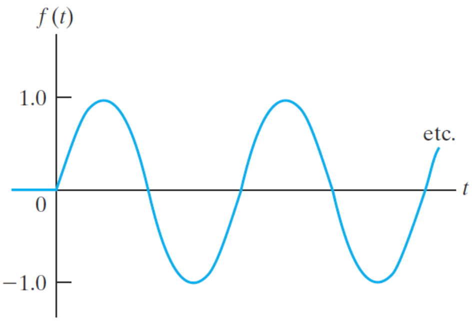
❖ **Sinusoidal Function**

$$\mathcal{L}\{\sin\omega t\} = \int_{0^-}^{\infty} (\sin\omega t)e^{-st}\,dt$$

$$= \int_{0^-}^{\infty} \left(\frac{e^{j\omega t} - e^{-j\omega t}}{2j}\right)e^{-st}\,dt$$

$$= \int_{0^-}^{\infty} \frac{e^{-(s-j\omega)t} - e^{-(s+j\omega)t}}{2j}\,dt$$

$$= \frac{1}{2j}\left(\frac{1}{s-j\omega} - \frac{1}{s+j\omega}\right)$$



$$\boxed{\mathcal{L}\{\sin\omega t\} = \frac{\omega}{s^2+\omega^2}}$$

10.5.2 Laplace Transform Pairs

All functions are defined to be zero for $t < 0^-$!

Type	$f(t)$ $(t > 0-)$	$F(s)$
(impulse)	$\delta(t)$	1
(step)	$u(t)$	$\frac{1}{s}$
(ramp)	t	$\frac{1}{s^2}$
(exponential)	e^{-at}	$\frac{1}{s+a}$
(sine)	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
(cosine)	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
(damped ramp)	te^{-at}	$\frac{1}{(s+a)^2}$
(damped sine)	$e^{-at}\sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
(damped cosine)	$e^{-at}\cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$

10.6 Operational Laplace Transforms

10.6.1 Multiplication by a Constant, *K*

If $\mathscr{L}\{f(t)\} = F(s)$, then $\mathscr{L}\{Kf(t)\} = KF(s)$.

10.6.2 Addition and/or Subtraction

If $\mathscr{L}\{f_1(t)\} = F_1(s)$, $\mathscr{L}\{f_2(t)\} = F_2(s)$, $\mathscr{L}\{f_3(t)\} = F_3(s)$,

Then $\mathscr{L}\{f_1(t) + f_2(t) - f_3(t)\} = F_1(s) + F_2(s) - F_3(s)$,

10.6.3 Differentiation

$$\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = sF(s) - f(0^-)$$, Subtracting the initial value of $f(t)$

Also for the n^{th} derivative,

$$\mathcal{L}\left\{\frac{d^n f(t)}{dt^n}\right\} = s^n F(s) - s^{n-1}f(0^-) - s^{n-2}\frac{df(0^-)}{dt} - s^{n-3}\frac{d^2 f(0^-)}{dt^2} - \cdots - \frac{d^{n-1}f(0^-)}{dt^{n-1}}$$

10.6.4 Integration

$$\mathcal{L}\left\{\int_{0^-}^{t} f(x)\,dx\right\} = \frac{F(s)}{s}$$

10.6.5 Translation in the Time Domain

Translating $f(t)u(t)$ in time by a constant "a" yields $f(t-a)u(t-a)$, then

$$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s), \quad a > 0$$

For example, knowing that,

$$\mathcal{L}\{tu(t)\} = \frac{1}{s^2}$$

Then,

$$\mathcal{L}\{(t-a)u(t-a)\} = \frac{e^{-as}}{s^2}$$

Note that, the translation in the time domain corresponds to multiplication by an exponential in the frequency domain!

10.6.6 Translation in the Frequency Domain

$$\mathcal{L}\{e^{-at}f(t)\} = F(s+a)$$

Note that, the translation in the frequency domain corresponds to multiplication by an exponential in the time domain!

For example,

If

$$\mathcal{L}\{\cos \omega t\} = \frac{s}{s^2 + \omega^2}$$

Then, it can be deduced that,

$$\mathcal{L}\{e^{-at} \cos\omega t\} = \frac{s + a}{(s + a)^2 + \omega^2}$$

10.6.7 Scale Changing

$$\mathcal{L}\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right), \quad a > 0,$$

For example,

If

$$\mathcal{L}\{\cos t\} = \frac{s}{s^2 + 1}$$

Then,

$$\mathcal{L}\{\cos\omega t\} = \frac{1}{\omega}\frac{s/\omega}{(s/\omega)^2 + 1} = \frac{s}{s^2 + \omega^2}$$

10.6.8 First Derivative

$$\mathcal{L}\ \{tf(t)\} = -\frac{dF(s)}{ds}$$

10.6.9 n^th Derivatives

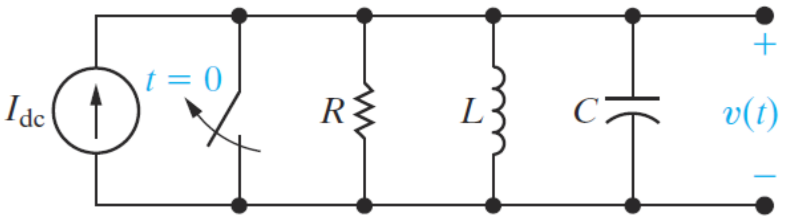
$$\mathcal{L}\ \{t^n f(t)\} = (-1)^n \frac{d^n F(s)}{ds^n}$$

10.6.10 s Integral

$$\mathcal{L}\ \{\frac{f(t)}{t}\} = \int_s^{\infty} F(u)du$$

10.7 Applying the Laplace Transform

Example # 2: Use Laplace Transform to solve the ordinary integro-differential equation that describes the circuit shown in the Figure below, assume that the initial energy stored is zero before opening the switch at $t = 0s$!, and find $v(t)$ for $t \geq 0s$!!



Solution:

Applying KCL to the upper node,

$$\frac{v(t)}{R} + \frac{1}{L}\int_0^t v(x)\,dx + C\frac{dv(t)}{dt} = I_{dc}u(t)$$

Transforming the equation to s-domain yields,

$$\frac{V(s)}{R} + \frac{1}{L}\frac{V(s)}{s} + C[sV(s) - v(0^-)] = I_{dc}\left(\frac{1}{s}\right)$$

Collecting terms of $V(s)$ yields,

$$V(s)\left(\frac{1}{R} + \frac{1}{sL} + sC\right) = \frac{I_{dc}}{s} \quad , \quad \text{since } V(0^-) = 0V$$

Thus,

$$V(s) = \frac{I_{dc}/C}{s^2 + (1/RC)s + (1/LC)}$$

But, $v(t)$ is the inverse Laplace Transform of $V(s)$;

$$v(t) = \mathscr{L}^{-1}\{V(s)\}$$

10.8 Inverse Laplace Transform

- The corresponding s-domain variables (upper case) and the time domain variables are illustrated as:

$$\mathcal{L}\{v\} = V \quad \text{or} \quad v = \mathcal{L}^{-1}\{V\},$$

$$\mathcal{L}\{i\} = I \quad \text{or} \quad i = \mathcal{L}^{-1}\{I\},$$

$$\mathcal{L}\{f\} = F \quad \text{or} \quad f = \mathcal{L}^{-1}\{F\},$$

- In general, for linear, lumped-parameter circuits whose component values are constant, the s-domain expressions for the unknown voltages and currents are always rational functions of s;

$$F(s) = \frac{N(s)}{D(s)} = \frac{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}$$

where, a and b are real constants, and m and n are positive integers!

Notes,

1- The ratio $\frac{N(s)}{D(s)}$ is called a proper rational function if $m > n$

2- The ratio $\frac{N(s)}{D(s)}$ is called an improper rational function if $m \leq n$

3- Only a proper rational function can be expanded as a sum of partial fractions.

10.8.1 Partial Fraction Expansion

- Proper rational functions can be expanded into a sum of partial fractions by writing a term or a series of terms for each root of $D(s)$; poles of $F(s)$!
- For each distinct root of $D(s)$, a single term appears in the sum of partial fractions.
- For each multiple root of $D(s)$ of multiplicity "r", the expansion contains "r" terms.

For example, the rational function

$$\frac{s+6}{s(s+3)(s+1)^2}$$

can be expanded as,

$$\frac{s+6}{s(s+3)(s+1)^2} \equiv \frac{K_1}{s} + \frac{K_2}{s+3} + \frac{K_3}{(s+1)^2} + \frac{K_4}{s+1}$$

And

$$\mathcal{L}^{-1}\left\{\frac{s+6}{s(s+3)(s+1)^2}\right\} = (K_1 + K_2e^{-3t} + K_3te^{-t} + K_4e^{-t})u(t)$$

The roots of $D(s)$ can be either,

1- real and distinct

2- complex and distinct

3- real and repeated

4- complex and repeated

10.8.1.1 Partial Fraction Expansion - Distinct Real Roots of $D(s)$

If

$$F(s) = \frac{K_1}{s-P_1} + \frac{K_2}{s-P_2} + \cdots + \frac{K_m}{s-P_m}$$

Then

$$K_1 = F(s) * (s - P_1)|_{s=P_1}$$

$$\vdots$$

$$K_m = F(s) * (s - P_m)|_{s=P_m}$$

Example # 3: Use Partial Fraction to find the coefficients; K's, and $f(t)$, given that,

$$F(s) = \frac{96(s+5)(s+12)}{s(s+8)(s+6)} \equiv \frac{K_1}{s} + \frac{K_2}{s+8} + \frac{K_3}{s+6}$$

Solution:

To find K_1, multiply both sides by "s" and evaluate both sides at $s = 0$;

$$\left.\frac{96(s+5)(s+12)}{(s+8)(s+6)}\right|_{s=0} \equiv K_1 + \left.\frac{K_2 s}{s+8}\right|_{s=0} + \left.\frac{K_3 s}{s+6}\right|_{s=0}$$

$$\frac{96(5)(12)}{8(6)} \equiv K_1 = 120$$

To find K_2, multiply both sides by "$(s+8)$" and evaluate both sides at $s = -8$;

$$\left.\frac{96(s+5)(s+12)}{s(s+6)}\right|_{s=-8} \equiv \left.\frac{K_1(s+8)}{s}\right|_{s=-8} + K_2 + \left.\frac{K_3(s+8)}{(s+6)}\right|_{s=-8}$$

or

$$\frac{96(-3)(4)}{(-8)(-2)} = K_2 = -72$$

To find K_3, multiply both sides by "$(s+6)$" and evaluate both sides at $s = -6$;

$$\left.\frac{96(s+5)(s+12)}{s(s+8)}\right|_{s=-6} = K_3 = 48$$

Therefore,

$$F(s) = \frac{96(s+5)(s+12)}{s(s+8)(s+6)} \equiv \frac{120}{s} + \frac{48}{s+6} - \frac{72}{s+8}$$

Check, $F(s) = 0$, at $s = -5$ or $s = -12$; choosing $s = -5$, the right hand side yields:

$$F(s) = \frac{120}{-5} + \frac{48}{1} - \frac{72}{3} = -24 + 48 - 24 = 0$$

Thus,

$$f(t) = \mathscr{L}^{-1}\left\{\frac{96(s+5)(s+12)}{s(s+8)(s+6)}\right\} = (120 + 48e^{-6t} - 72e^{-8t})u(t)$$

10.8.1.2 Partial Fraction Expansion - Distinct Complex Roots of $D(s)$

Example # 4: Use Partial Fraction to find the coefficients; K's, and $f(t)$, given that,

$$F(s) = \frac{100(s+3)}{(s+6)(s^2+6s+25)}$$

Solution:

$F(s)$ is a proper rational function.

The roots of the quadratic equation,

$$s^2 + 6s + 25 = (s + 3 - j4)(s + 3 + j4)$$

Thus, with the denominator in a factored form,

$$F(s) = \frac{K_1}{s+6} + \frac{K_2}{s+3-j4} + \frac{K_3}{s+3+j4}$$

To find K_1, multiply both sides by "$s+6$" and evaluate both sides at $s=-6$;

$$K_1 = \left.\frac{100(s+3)}{s^2+6s+25}\right|_{s=-6} = \frac{100(-3)}{25} = -12$$

To find K_2, multiply both sides by "$s+3-j4$" and evaluate both sides at $s=-3+j4$;

$$K_2 = \left.\frac{100(s+3)}{(s+6)(s+3+j4)}\right|_{s=-3+j4} = \frac{100(j4)}{(3+j4)(j8)}$$

$$\therefore \quad K_2 = 6 - j8 = 10e^{-j53.13^\circ}$$

To find K_3, multiply both sides by "$s+3+j4$" and evaluate both sides at $s=-3-j4$;

$$K_3 = \left.\frac{100(s+3)}{(s+6)(s+3-j4)}\right|_{s=-3-j4} = \frac{100(-j4)}{(3-j4)(-j8)}$$

$$\therefore \quad K_3 = 6 + j8 = 10e^{j53.13^\circ}$$

Note that, complex roots coefficients are also conjugates!

Therefore,

$$F(s) = \frac{100(s+3)}{(s+6)(s^2+6s+25)} = \frac{-12}{s+6} + \frac{10\angle{-53.13^\circ}}{s+3-j4} + \frac{10\angle{53.13^\circ}}{s+3+j4}$$

Testing at $s = -3$ is attractive because the left-hand side reduces to zero at this value;

$$F(s) = \frac{-12}{3} + \frac{10\,\angle{-53.13^\circ}}{-j4} + \frac{10\,\angle{53.13^\circ}}{j4}$$
$$= -4 + 2.5\,\angle{36.87^\circ} + 2.5\,\angle{-36.87^\circ}$$
$$= -4 + 2.0 + j1.5 + 2.0 - j1.5 = 0$$

Taking the inverse Laplace Transform of $F(s)$ yields,

$$\mathcal{L}^{-1}\left\{\frac{100(s+3)}{(s+6)(s^2+6s+25)}\right\} = (-12e^{-6t} + 10e^{-j53.13^\circ}e^{-(3-j4)t} + 10e^{j53.13^\circ}e^{-(3+j4)t})u(t)$$

$$= (-12e^{-6t} + 10e^{-3t}(e^{j(4t-53.13^\circ)} + e^{-j(4t-53.13^\circ)}))u(t)$$

$$= [-12e^{-6t} + 20e^{-3t}\cos(4t - 53.13^\circ)]u(t)$$

Note that, $\cos\theta = \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right)$

In general, for the complex roots, the coefficients are also complex conjugates; if the roots are:

$$(s + \alpha - j\beta)(s + \alpha + j\beta)$$

Then, the partial fraction yields,

$$\frac{K}{s + \alpha - j\beta} + \frac{K^*}{s + \alpha + j\beta}$$

where $K = |K|e^{j\theta} = |K|\angle{\theta^\circ}$ and $K^* = |K|e^{-j\theta} = |K|\angle{-\theta^\circ}$

Hence, the complex conjugate pair inverse transform is:

$$\mathcal{L}^{-1}\left\{\frac{K}{s + \alpha - j\beta} + \frac{K^*}{s + \alpha + j\beta}\right\} = 2|K|e^{-\alpha t}\cos(\beta t + \theta)$$

Note that, ***K*** is associated with the root $s + \alpha - j\beta$

10.8.1.3 Partial Fraction Expansion - Repeated Real Roots of $D(s)$

Example # 5: Use Partial Fraction to find the coefficients; K's, and $f(t)$, given that,

$$F(s) = \frac{100(s+25)}{s(s+5)^3} = \frac{K_1}{s} + \frac{K_2}{(s+5)^3} + \frac{K_3}{(s+5)^2} + \frac{K_4}{s+5}$$

Solution:

To find K_1, multiply both sides by "s" and evaluate both sides at $s = 0$;

$$K_1 = \left.\frac{100(s+25)}{(s+5)^3}\right|_{s=0} = \frac{100(25)}{125} = 20$$

To find K_2, multiply both sides by "$(s+5)^3$" and evaluate both sides at $s = -5$;

$$\left.\frac{100(s+25)}{s}\right|_{s=-5} = \left.\frac{K_1(s+5)^3}{s}\right|_{s=-5} + K_2 + K_3(s+5)|_{s=-5} + \left.K_4(s+5)^2\right|_{s=-5}$$

$$\Rightarrow \quad \frac{100(20)}{(-5)} = K_1 \times 0 + K_2 + K_3 \times 0 + K_4 \times 0$$

$$K_2 = -400$$

To find K_3, multiply both sides by "$(s+5)^3$". **Next, differentiate both sides once with respect to "s"** and evaluate both sides at $s = -5$;

$$\frac{d}{ds}\left[\frac{100(s+25)}{s}\right]_{s=-5} = \frac{d}{ds}\left[\frac{K_1(s+5)^3}{s}\right]_{s=-5} + \frac{d}{ds}[K_2]_{s=-5}$$

$$+ \frac{d}{ds}[K_3(s+5)]_{s=-5} + \frac{d}{ds}[K_4(s+5)^2]_{s=-5}$$

$$\therefore \quad 100\left[\frac{s-(s+25)}{s^2}\right]_{s=-5} = K_3 = -100$$

To find K_4, multiply both sides by "$(s+5)^3$". **Next, differentiate both sides TWICE with respect to "*s*"** and evaluate both sides at $s = -5$;

After simplifying the first derivative, the second derivative becomes:

$$100\frac{d}{ds}\left[-\frac{25}{s^2}\right]_{s=-5} = K_1\frac{d}{ds}\left[\frac{(s+5)^2(2s-5)}{s^2}\right]_{s=-5}$$

$$+ 0 + \frac{d}{ds}[K_3]_{s=-5} + \frac{d}{ds}[2K_4(s+5)]_{s=-5}$$

which yields, $\quad -40 = 2K_4 \quad$ or $\quad K_4 = -20$

Therefore,

$$\frac{100(s+25)}{s(s+5)^3} = \frac{20}{s} - \frac{400}{(s+5)^3} - \frac{100}{(s+5)^2} - \frac{20}{s+5}$$

Check both sides at $s = -25$, both sides must be equal to zero!

The inverse Laplace Transform of $F(s)$ yields,

$$\mathscr{L}^{-1}\left\{\frac{100(s+25)}{s(s+5)^3}\right\} = [20 - 200t^2e^{-5t} - 100te^{-5t} - 20e^{-5t}]u(t)$$

Note that, if $F(s) = \frac{1}{s+5}$ then $\frac{dF(s)}{ds} = \frac{-1}{(s+5)^2}$ and $\frac{d^2F(s)}{ds^2} = \frac{2}{(s+5)^3}$

$$\therefore \frac{400}{(s+5)^3} = (-1)^2(200)\frac{d^2F(s)}{ds^2}$$

Thus, $\mathcal{L}^{-1}\left\{\frac{400}{(s+5)^3}\right\} = 200\,\mathcal{L}^{-1}\left\{(-1)^2\frac{2}{(s+5)^2}\right\} = 200\,\mathcal{L}^{-1}\left\{(-1)^2\frac{d^2F(s)}{ds^2}\right\}$

$$= 200t^2f(t)$$

But, $f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s+5}\right\} \Rightarrow f(t) = e^{-5t}$

$$\therefore \mathcal{L}^{-1}\left\{\frac{400}{(s+5)^3}\right\} = 200t^2e^{-5t}$$

Besides, $\frac{-100}{(s+5)^2} = -100(-1)\frac{-1}{(s+5)^2} = -100(-1)\frac{dF(s)}{ds}$

Therefore, $\mathcal{L}^{-1}\left\{\frac{-100}{(s+5)^2}\right\} = -100\mathcal{L}^{-1}\left\{(-1)\frac{dF(s)}{ds}\right\} = -100te^{-5t}$

10.8.1.4 Partial Fraction Expansion - Repeated Complex Roots of $D(s)$

Repeated complex roots are treated in a similar manner to that was done to repeated real roots; the only difference is that the algebra involves complex numbers.

Example # 6: Find $f(t)$, given that,

$$F(s) = \frac{768}{(s^2 + 6s + 25)^2}$$

Solution:

The first step is factoring the denominator polynomial;

$$F(s) = \frac{768}{(s + 3 - j4)^2(s + 3 + j4)^2}$$

$$= \frac{K_1}{(s + 3 - j4)^2} + \frac{K_2}{s + 3 - j4} + \frac{K_1^*}{(s + 3 + j4)^2} + \frac{K_2^*}{s + 3 + j4}$$

To find K_1, multiply $F(s)$ by $(s + 3 - j4)^2$ and evaluate at $s = -3 + j4$;

$$K_1 = \left.\frac{768}{(s + 3 + j4)^2}\right|_{s=-3+j4}$$

$$= \frac{768}{(j8)^2} = -12$$

To find K_2, multiply $F(s)$ by $(s + 3 - j4)^2$. **Next, differentiate once with respect to "s"** and evaluate both sides at $s = -3 + j4$;

$$K_2 = \frac{d}{ds}\left[\frac{768}{(s + 3 + j4)^2}\right]_{s=-3+j4}$$

$$= -\left.\frac{2(768)}{(s + 3 + j4)^3}\right|_{s=-3+j4}$$

$$= -\frac{2(768)}{(j8)^3}$$

$$= -j3 = 3\angle -90^\circ$$

Thus,

$$K_1^* = -12$$

and

$$K_2^* = j3 = 3 \angle 90°$$

Now, grouping the partial fraction expansion by conjugate terms to obtain:

$$\Rightarrow \quad F(s) = \left[\frac{-12}{(s+3-j4)^2} + \frac{-12}{(s+3+j4)^2}\right] + \left(\frac{3\angle -90°}{s+3-j4} + \frac{3\angle 90°}{s+3+j4}\right)$$

θ

α β

Taking the inverse Laplace transform of $F(s)$ yields:

$$f(t) = [-24te^{-3t}\cos 4t + 6e^{-3t}\cos(4t - 90°)]u(t)$$

$2|K|$

Notes,

- If $F(s)$ has a real root a of multiplicity *"r"* in its denominator, the term in a partial fraction expansion is of the form:

$$\frac{K}{(s+a)^r}$$

The inverse Transform of this term is

$$\mathscr{L}^{-1}\left\{\frac{K}{(s+a)^r}\right\} = \frac{Kt^{r-1}e^{-at}}{(r-1)!}u(t)$$

- If $F(s)$ has a complex root of $\alpha + j\beta$ of multiplicity *"r"* in its denominator, the term in partial fraction expansion is the conjugate pair:

$$\frac{K}{(s+\alpha-j\beta)^r} + \frac{K^*}{(s+\alpha+j\beta)^r}$$

Then the inverse Laplace Transform of this pair is:

$$\mathcal{L}^{-1}\left\{\frac{K}{(s+\alpha-j\beta)^r} + \frac{K^*}{(s+\alpha+j\beta)^r}\right\}$$

$$= \left[\frac{2|K|t^{r-1}}{(r-1)!} e^{-\alpha t} \cos(\beta t + \theta)\right] u(t)$$

Summary:

Nature of Roots	$F(s)$	$f(t)$
Distinct real	$\frac{K}{s+a}$	$Ke^{-at}u(t)$
Repeated real	$\frac{K}{(s+a)^2}$	$Kte^{-at}u(t)$
Distinct complex	$\frac{K}{s+\alpha-j\beta} + \frac{K^*}{s+\alpha+j\beta}$	$2\lvert K\rvert e^{-\alpha t}\cos(\beta t+\theta)u(t)$
Repeated complex	$\frac{K}{(s+\alpha-j\beta)^2} + \frac{K^*}{(s+\alpha+j\beta)^2}$	$2t\lvert K\rvert e^{-\alpha t}\cos(\beta t+\theta)u(t)$

10.8.2 Partial Fraction Expansion: Improper Rational Functions

- An improper rational function can always be expanded into a polynomial plus a proper rational function.
- The polynomial is then inverse-transformed into impulse functions and derivatives of impulse functions.
- The proper rational function is inverse-transformed by the techniques outlined.

Example # 7: For the function $F(s)$ find $f(t)$!

$$F(s) = \frac{s^4 + 13s^3 + 66s^2 + 200s + 300}{s^2 + 9s + 20}$$

Solution:

1- $F(s)$ is an improper rational function!

2- Divide the numerator by the denominator (Algebraic Long Division) until the remainder is a proper rational function, which yields,

$$F(s) = s^2 + 4s + 10 + \frac{30s + 100}{s^2 + 9s + 20}$$

3- The remainder, $\frac{30s+100}{s^2+9s+20}$, is a proper rational function that can be expanded by partial fractions as:

$$\frac{30s + 100}{s^2 + 9s + 20} = \frac{30s + 100}{(s + 4)(s + 5)} = \frac{-20}{s + 4} + \frac{50}{s + 5}$$

4- Therefore,

$$F(s) = s^2 + 4s + 10 - \frac{20}{s + 4} + \frac{50}{s + 5}$$

5- The inverse Transform yields,

$$f(t) = \frac{d^2\delta(t)}{dt^2} + 4\frac{d\delta(t)}{dt} + 10\delta(t) - (20e^{-4t} - 50e^{-5t})u(t)$$

10.9 Poles and Zeros of *F(s)*

The function $F(s)$ can be expressed as,

$$F(s) = \frac{K(s + z_1)(s + z_2)\cdots(s + z_n)}{(s + p_1)(s + p_2)\cdots(s + p_m)}$$

- The roots of the denominator polynomial are called the poles of $F(s), \{-p_1, -p_2, \dots, -p_m\}$; they are the values of "s" at which $F(s)$ goes to infinity.
- Whilst, the roots of the numerator polynomial are called the zeros of $F(s)$, $\{-z_1, -z_2, \dots, -z_n\}$; ; they are the values of "s" at which $F(s)$ becomes zero.
- The poles and zeros may be represented as points in the complex s-plane!

Example # 8: For the function $F(s)$ find the poles and zeros!

$$F(s) = \frac{8s^2 + 120s + 400}{2s^4 + 20s^3 + 70s^2 + 100s + 48}$$

Solution:

The function can be expressed as:

$$F(s) = \frac{8(s^2 + 15s + 50)}{2(s^4 + 10s^3 + 35s^2 + 50s + 24)}$$

The numerator and denominator polynomials can be can be factored as:

$$F(s) = \frac{4(s + 5)(s + 10)}{(s + 1)(s + 2)(s + 3)(s + 4)}$$

$\therefore$ The poles of $F(s)$ are: $-1, -2, -3$, and -4

And the zeros of $F(s)$ are: -5, and -10

Example # 9: Plot the poles and zeros of $F(s)$ in the s-plane

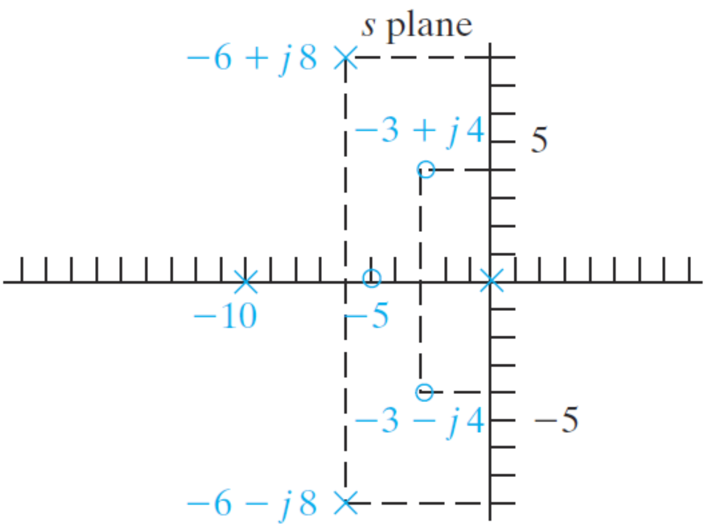
$$F(s) = \frac{10(s + 5)(s + 3 - j4)(s + 3 + j4)}{s(s + 10)(s + 6 - j8)(s + 6 + j8)}$$

Solution:

The poles of $F(s)$ are: $0, -10, -6 + j8$, and $-6 - j8$

The zeros of $F(s)$ are: $-5, -3 + j4$, and $-3 - j4$

The poles and zeros are assigned in the s-plane as shown in the Figure next.



10.10 Initial- and Final-Value Theorems

- They enable determining from $F(s)$ the behavior of $f(t)$ at 0 and ∞.
- Hence, they enable checking whether $F(s)$ is correct and predicts the initial and final values of $f(t)$; to see if they conform with known circuit behavior, before actually finding the inverse transform of $F(s)$.
- The Initial-value theorem states that:

$$\lim_{t \to 0^+} f(t) = \lim_{s \to \infty} sF(s)$$

- The Final-value theorem states that:

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$$

Notes:

1- The Initial-value theorem is based on the assumption that $f(t)$ contains no impulse functions.

2- The Final-value theorem is valid only if the **poles** of F(s), except for a first-order pole at the origin, lie in the left half of the s-plane.

Chapter 11

The Laplace Transform in Circuit Analysis

- The Laplace Transform is an attractive tool in circuit analysis because:
  1- It transforms a set of linear constant coefficient differential equations into a set of linear polynomial equations, which are easier to manipulate.
  2- It automatically introduces into the polynomial equations the initial values of the current and voltage.
- The Laplace Transform tool permits extending the sinusoidal steady state phasor analysis methods to a much wider setting, where transient and steady state analysis are both possible for a broad range of input excitations not amenable to phasor analysis.
- Recall that, transient is not possible with phasors.

11.1 Circuit Elements in the s-Domain

11.1.1 A Resistor in the s-Domain

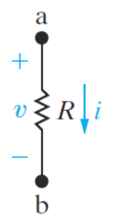
- Ohm's Law in time domain is:

$$v = Ri$$

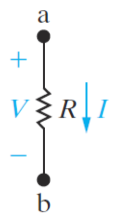
The Laplace Transform is:

$$V = RI,$$

where, $V = \mathscr{L}\{v\}$ and $I = \mathscr{L}\{i\}$

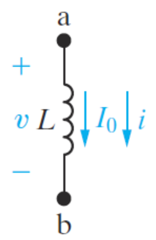


- In the s-domain, The resistor is measured in Ohms
  The current, I, is measured in Amperes-seconds
  The voltage, V is measured in Volts- seconds



- The Impedance, $Z(s) = \frac{V(s)}{I(s)} = R$ $(Ohms,\ \Omega)$
- The admittance, $Y(s) = \frac{I(s)}{V(s)} = \frac{1}{R}$ (Siemens, S)

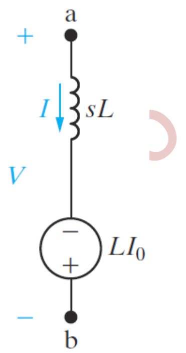
11.1.2 An Inductor in s-Domain



- In time domain,

$$v = L\frac{di}{dt}$$

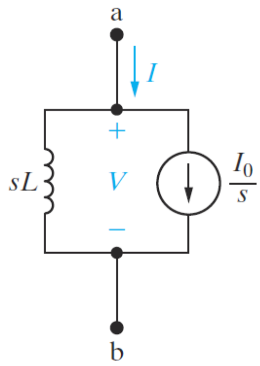
where, I_0 is the initial current



- The Laplace Transform gives:

$$V = L[sI - i(0^{-})] = sLI - LI_0$$

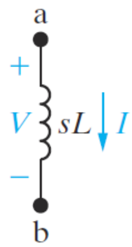
- The equivalent circuit is shown in the Figure next.



- Arranging for the current, I, yields:

$$I = \frac{V}{sL} + \frac{I_0}{s}$$

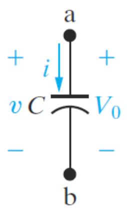
which can be represented by an inductor in parallel with a current source, as shown in the Figure next.



- If the initial current is zero, then the equivalent circuit is represented by an inductor only, as shown in the Figure next, whose impedance is sL, such that:

$$Z_L = sL = \frac{V}{I}$$

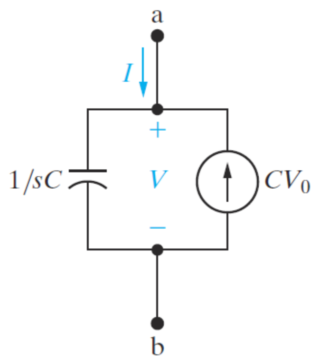
11.1.3 A Capacitor in s-Domain



- In time domain,

$$i = C\frac{dv}{dt}$$

where, C_0 is the initial voltage at the capacitor



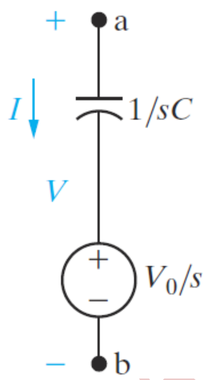
- The Laplace Transform gives:

$$I = C[sV - v(0^{-})] = sCV - CV_0$$

- The equivalent circuit is shown in the Figure next.

- Arranging for the voltage, V, yields:

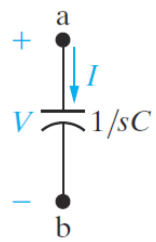
$$V = \frac{I}{sC} + \frac{V_0}{s}$$



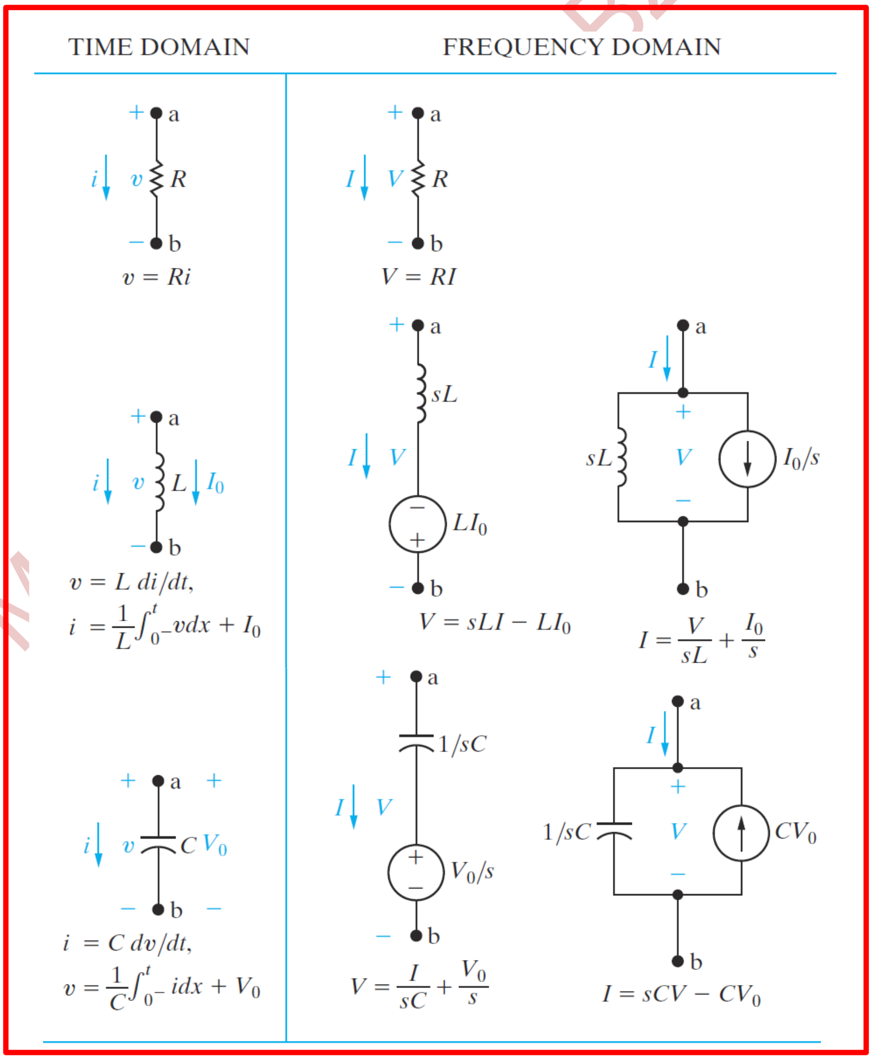
which can be represented by a capacitor in series with a voltage source, as shown in the Figure next.

- If the initial voltage is zero, then the equivalent ciruit is represented by a capacitor only, as shown in the Figure next, whose impedance is $\frac{1}{sC}$, such that:

$$Z_c = \frac{1}{sC} = \frac{V}{I}$$



Summary:



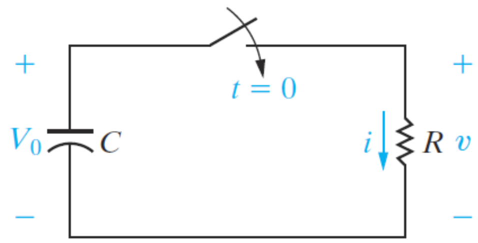
11.2 Circuit Analysis in s-Domain

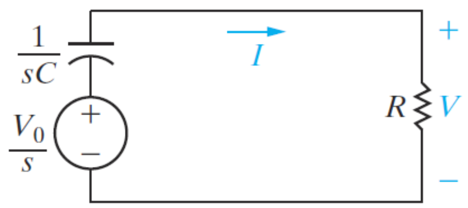
- The first step for circuit analysis is to transfer the time domain circuit into its s-domain equivalent circuit
- Each element is represented by its s-domain equivalent circuit. The stored energy in inductors or capacitors is represented by an independent source.
- The rules for combining impedances and admittances in the s-domain are the same as those for frequency-domain circuits.
- Thus, series-parallel simplifications and Δ-to-Y conversions also are applicable to s-domain analysis.
- In addition, Ohm's law, Kirchhoff's laws apply to s-domain for currents and voltages.
- Node voltages, mesh currents, source transformations, and Thévenin-Norton equivalents are all valid techniques, even when energy is stored initially in the inductors and capacitors.
- The response in s-domain can be inverse-transformed to find the time domain.

11.3 Applications

11.3.1 The Natural Response of an RC Circuit

Example # 1: If the capacitor in the circuit is initially charged to V_0, then find $i(t)$ and $v(t)$.





Solution:

- The first step is to construct the s-domain equivalent circuit
- To find the current, use the series representation of the capacitor as it yields one mesh equation.
- Applying KVL yields:

$$\frac{V_0}{s} = \frac{1}{sC}I + RI$$

- Solving for I yields:

$$I = \frac{CV_0}{RCs + 1} = \frac{V_0/R}{s + (1/RC)}$$

which has a proper rational function of s.

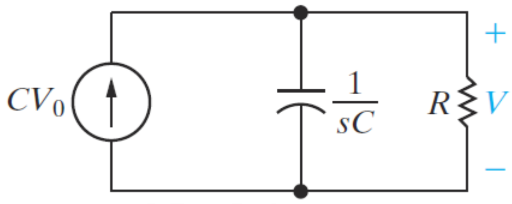
- The inverse transform of I yields the time domain function of current:

$$i = \frac{V_0}{R} e^{-t/RC} u(t)$$

and $$v = Ri = V_0 e^{-t/RC} u(t)$$

- Otherwise, the parallel representation of a capacitor may also be used to find V first, by applying KCL to the upper node as:

$$\frac{V}{R} + sCV = CV_0$$



- Arranging for V yields:

$$V = \frac{V_0}{s + (1/RC)}$$

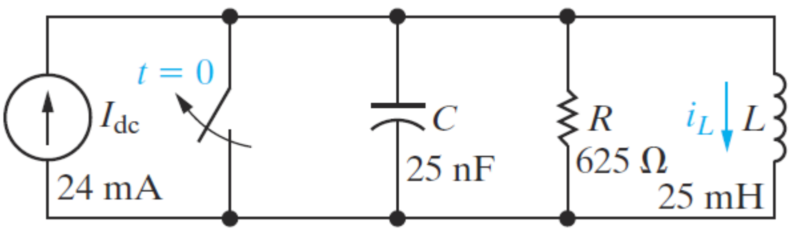
- The inverse Laplace transform yields:

$$v = V_0 e^{-t/RC} = V_0 e^{-t/\tau} u(t)$$

where, the time constant, $\tau = RC$

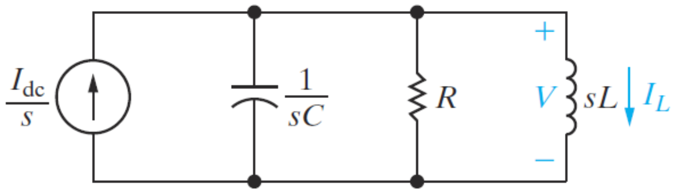
11.3.2 The Step Response of a Parallel Circuit

Example # 2: For the circuit shown below, find the current, $i_L(t)$ after the constant current source is switched across the parallel elements. The initial energy stored in the circuit is zero.



Solution:

- The first step is to construct the s-domain equivalent circuit as shown in the Figure below.



- The Laplace transform of the switched current source is:

$$\mathscr{L}\{I_{dc}u(t)\} = I_{dc}/s$$

- Applying KCL at the upper node yields:

$$sCV + \frac{V}{R} + \frac{V}{sL} = \frac{I_{dc}}{s}$$

Arranging for V yields:

$$V = \frac{I_{dc}/C}{s^2 + (1/RC)s + (1/LC)}$$

But, $$I_L = \frac{V}{sL}$$

Thus, $$I_L = \frac{I_{dc}/LC}{s[s^2 + (1/RC)s + (1/LC)]}$$

- Substituting numerical values of $R, L, and\ C$, yields:

$$I_L = \frac{384 \times 10^5}{s(s^2 + 64{,}000s + 16 \times 10^8)}$$

- Factoring the quadratic term in the denominator yields:

$$I_L = \frac{384 \times 10^5}{s(s + 32{,}000 - j24{,}000)(s + 32{,}000 + j24{,}000)}$$

- Check the latter expression of I_L using the final value theorem at $t = \infty$, all poles of I_L, except the first order pole at the origin, lie in the left half of the s-plane, so the theorem is applicable.

As $t \to \infty$, the inductor becomes a short circuit, and, in time domain, $I_L(\infty) = 24mA,$

- Check, for s-domain,

  Since currents in the s-domain carry the dimension of ampere-seconds, so the dimension of sI_L will be Amperes; $sI_L(0)$ =??

$$\lim_{s\to 0} sI_L = \frac{384 \times 10^5}{16 \times 10^8} = 24 \text{ mA}$$

  Therefore, the expression is correct!

- Expanding the last equation into sum of partial fractions:

$$I_L = \frac{K_1}{s} + \frac{K_2}{s + 32{,}000 - j24{,}000} + \frac{K_2^*}{s + 32{,}000 + j24{,}000}$$

- The partial fraction coefficients are:

$$K_1 = \frac{384 \times 10^5}{16 \times 10^8} = 24 \times 10^{-3}$$

$$K_2 = \frac{384 \times 10^5}{(-32{,}000 + j24{,}000)(j48{,}000)}$$

$$= 20 \times 10^{-3} \angle 126.87^\circ.$$

and $K_2^* = 20 \times 10^{-3} \angle - 126.87^o$

- Note that, K_2 corresponds to the pole at $-\alpha + j\beta$
- Note also that, the inverse Laplace transform of the second and the third terms is:

$$2|K_2|e^{-\alpha t}\cos(\beta t + \theta);$$

  where, $|K_2| = 20X10^{-3}, \alpha = -32{,}000, \beta = 24{,}000, and\ \theta = 126.87$

- Substituting the numerical values of $K_1, K_2, and K_2^*$ in the current equation and inverse transforming the resulting expression of I_L yields:

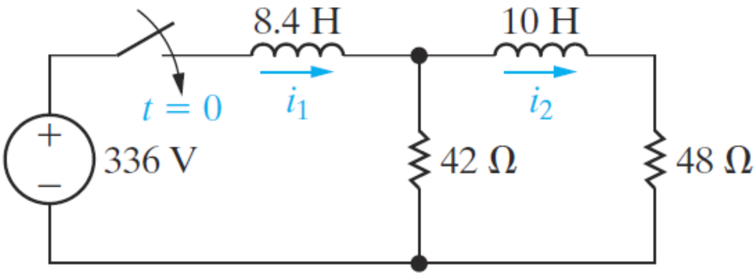
$$i_L = [24 + 40e^{-32{,}000t}\cos(24{,}000t + 126.87^\circ)]u(t)\text{mA}$$

Test the latter equation to make sure that, $I_L(0)$ satisfied the given initial conditions and $I_L(\infty)$ satisfies the known behaviour of the circuit!

11.3.3 The Step Response of a Multiple Mesh Circuit

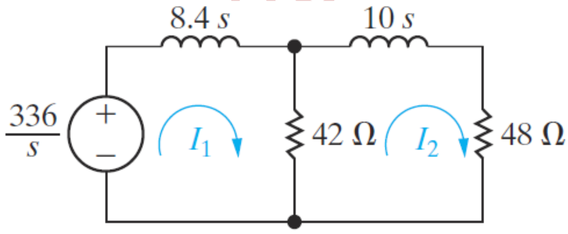
The Laplace techniques enable solving simultaneous differential equations.

Example # 3: For the circuit shown below, find the current, $i_1(t)$ and $i_2(t)$



Solution:

The first step is to find the s-domain equivalent circuit as shown in the Figure below. The two mesh-current equations are:



$$\frac{336}{s} = (42 + 8.4s)I_1 - 42I_2$$

$$0 = -42I_1 + (90 + 10s)I_2$$

Using Cramer's method to solve for I_1 and I_2, yields:

$$\Delta = \begin{vmatrix} 42 + 8.4s & -42 \\ -42 & 90 + 10s \end{vmatrix}$$

$$= 84(s^2 + 14s + 24)$$

$$= 84(s + 2)(s + 12)$$

$$N_1 = \begin{vmatrix} 336/s & -42 \\ 0 & 90 + 10s \end{vmatrix}$$

$$= \frac{3360(s + 9)}{s}$$

$$N_2 = \begin{vmatrix} 42 + 8.4s & 336/s \\ -42 & 0 \end{vmatrix}$$

$$= \frac{14{,}112}{s}$$

Therefore,

$$I_1 = \frac{N_1}{\Delta} = \frac{40(s + 9)}{s(s + 2)(s + 12)}$$

$$I_2 = \frac{N_2}{\Delta} = \frac{168}{s(s + 2)(s + 12)}$$

Expanding I_1 and I_2 into a sum of partial fractions yields,

$$I_1 = \frac{15}{s} - \frac{14}{s + 2} - \frac{1}{s + 12}$$

$$I_2 = \frac{7}{s} - \frac{8.4}{s + 2} + \frac{1.4}{s + 12}$$

Inverse transforming the latter equations yields expressions for $i_1(t)$ and $i_2(t)$;

$$i_1 = (15 - 14e^{-2t} - e^{-12t})u(t) \text{ A}$$

$$i_2 = (7 - 8.4e^{-2t} + 1.4e^{-12t})u(t) \text{ A}$$

- Test the equations for $i_1(t = 0) = 0$ and $i_2(t = 0) = 0$, which is true as no energy was stored in the inductors!
- Check $i_1(t = \infty) = ?$ and $i_2(t = \infty) = ?$ (Note that, the inductors at steady state are short circuits);

$$i_1(\infty) = \frac{336(90)}{42(48)} = 15 \text{ A}$$

$$i_2(\infty) = \frac{15(42)}{90} = 7 \text{ A}$$

These values are the values obtained if the inductors were replaced by short circuits.

- The voltage across the 42Ω is:

$$v = 42(i_1 - i_2)$$

⋮

$$v = (336 - 235.2e^{-2t} - 100.80e^{-12t})u(t) \text{ V}$$

Note that,

For a step response, $\mathcal{L}\{u(t)\} = \frac{1}{s}$

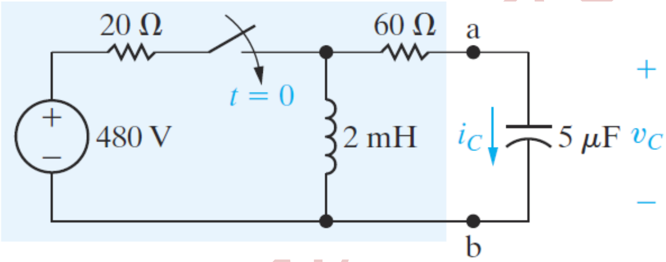
If switching occurs at t = a, $\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}$

$$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)$$

$$\mathcal{L}\{(t-a)u(t-a)\} = \frac{e^{-as}}{s^2}$$

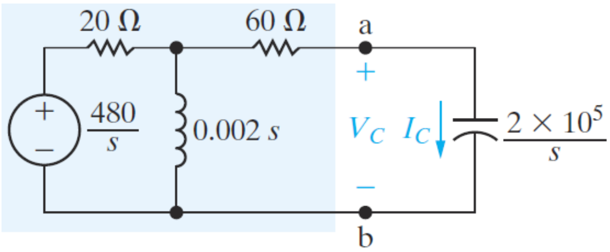
11.3.4 The Use of Thévenin's Equivalent Circuit

Example # 4: For the circuit shown in the Figure below, find the current, $i_C(t)$ using Thévenin's Theorem, if the energy stored in the circuit prior to closing the switch is zero.



Solution: Construct the s-domain equivalent circuit, as shown in the Figure below, and then find the Thévenin's equivalence of this circuit with respect to the terminals of the capacitor.

When opening the terminals a-b, the voltage across the 60Ω-resistor is zero; it's floating.



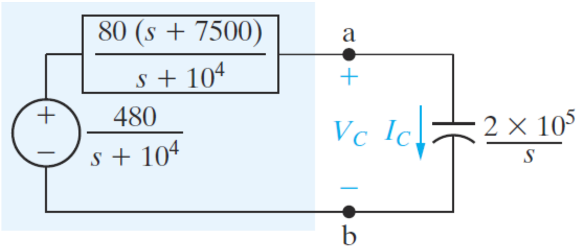
The open circuit voltage can be found by potential divider;

$$V_{Th} = \frac{(480/s)(0.002s)}{20 + 0.002s} = \frac{480}{s + 10^4}$$

This is Case 1; looking into the terminal a-b, after replacing the independent voltage source by a short circuit, the 60Ω-resistor is in series with the parallel combination of the 20Ω-resistor and the 2 mH inductor. Thus,

$$Z_{Th} = 60 + \frac{0.002s(20)}{20 + 0.002s} = \frac{80(s + 7500)}{s + 10^4}$$

The Thévenin's equivalent circuit is shown in the Figure next.



Thus, the capacitor current I_C equals the Thévenin voltage divided by the total series impedance:

$$I_C = \frac{480/(s+10^4)}{[80(s+7500)/(s+10^4)] + [(2\times 10^5)/s]}$$

Simplifying I_C yields:

$$I_C = \frac{6s}{s^2 + 10{,}000s + 25\times 10^6} = \frac{6s}{(s+5000)^2}$$

A partial fraction expansion generates:

$$I_C = \frac{-30{,}000}{(s+5000)^2} + \frac{6}{s+5000}$$

The inverse Laplace transform yields:

$$i_C = (-30{,}000te^{-5000t} + 6e^{-5000t})u(t)\text{ A}$$

Note that, the inverse Laplace Transform of $\frac{K}{s+a}$ is $Ke^{-at}u(t)$

To test the equation, from the circuit, $i_C(t=0) = \frac{480}{80} = 6A$, and this agrees with the result obtained from the equation.

Also, $i_C(t=\infty) = 0$, as expected!

To find $v_C(t)$, in s-domain,

$$V_C = \frac{1}{sC}I_C = \frac{2\times 10^5}{s}\frac{6s}{(s+5000)^2}$$

$$= \frac{12\times 10^5}{(s+5000)^2}$$

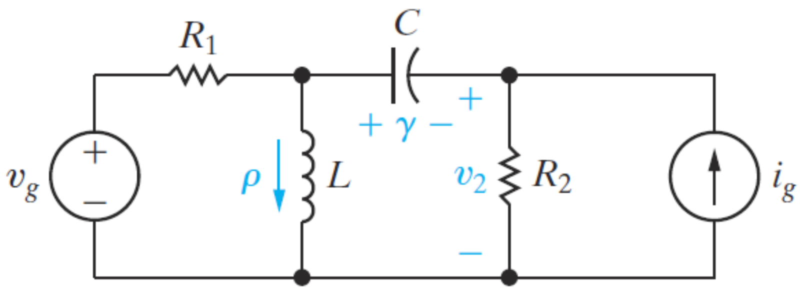
$$\therefore \qquad v_C = 12\times 10^5 te^{-5000t}u(t)$$

Alternatively, in time domain,

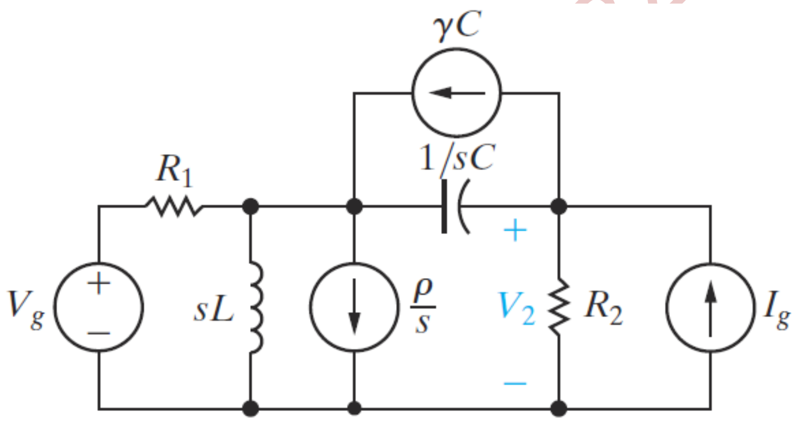
$$v_C = 2\times 10^5\int_{0^-}^{t}(6 - 30{,}000x)e^{-5000x}\,dx$$

11.3.5 The Use of Superposition

Example # 5: For the circuit shown in the Figure below, let $v_C(0^-) = V_o = \gamma$, and $i_L(0^-) = I_o = \rho$, find the voltage, $v_2(t)$ using Superposition Theorem.



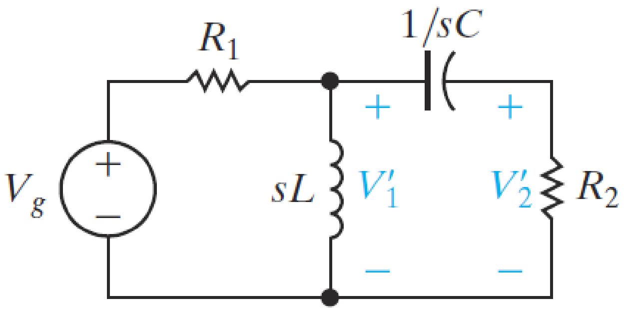
The first step is to find the s-domain equivalent circuit as shown in the Figure below; use the parallel equivalence of L and C for the initial conditions, as nodal Analysis will be used.



To find V_2 by superposition, calculate the component of V_2 resulting from each source acting alone, and then sum the components.

V_g acting alone:

The equivalent circuit is as shown next.



Applying KCL at node 1 yields:

$$\left(\frac{1}{R_1} + \frac{1}{sL} + sC\right)V_1' - sCV_2' = \frac{V_g}{R_1}$$

Applying KCL at node 2 yields:

$$-sCV_1' + \left(\frac{1}{R_2} + sC\right)V_2' = 0$$

Let,

$$Y_{11} = \frac{1}{R_1} + \frac{1}{sL} + sC$$

$$Y_{12} = -sC$$

$$Y_{22} = \frac{1}{R_2} + sC$$

Substituting in the node equations yields:

$$Y_{11}V_1' + Y_{12}V_2' = V_g/R_1$$

and

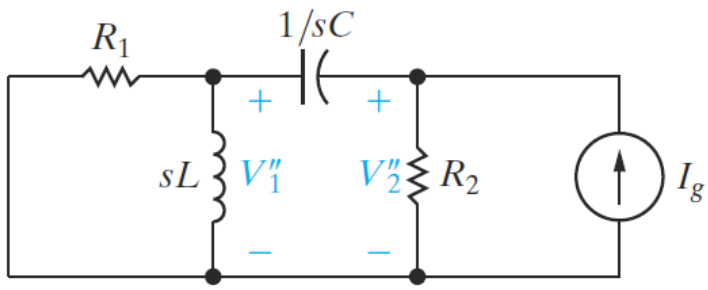
$$Y_{12}V_1' + Y_{22}V_2' = 0$$

Solving the latter equations yields:

$$V_2' = \frac{-Y_{12}/R_1}{Y_{11}Y_{22} - Y_{12}^2}V_g$$

I_g acting alone:

The new equivalent circuit is as shown next.



Applying KCL at node 1 and rearranging yield:

$$Y_{11}V_1'' + Y_{12}V_2'' = 0$$

Applying KCL at node 2 and rearranging yield:

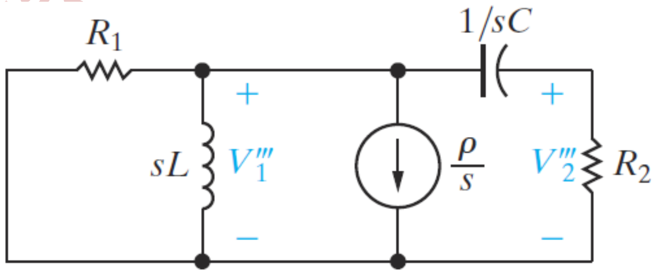
$$Y_{12}V_1'' + Y_{22}V_2'' = I_g$$

Solving the latter node equations yields:

$$V_2'' = \frac{Y_{11}}{Y_{11}Y_{22} - Y_{12}^2} I_g$$

The source due to the stored inductor's energy, $\frac{\rho}{s}$, acting alone:

The new equivalent circuit is as shown in the Figure next.



Applying KCL at node 1 and rearranging yield:

$$Y_{11}V_1''' + Y_{12}V_2''' = -\rho/s$$

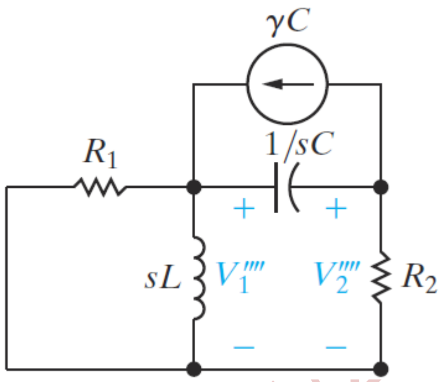
Applying KCL at node 2 and rearranging yield:

$$Y_{12}V_1''' + Y_{22}V_2''' = 0$$

Solving the latter node equations yields:

$$V_2''' = \frac{Y_{12}/s}{Y_{11}Y_{22} - Y_{12}^2}\rho$$

The source due to the stored capacitor's energy, γC, acting alone:



The new equivalent circuit is as shown in the Figure next.

Applying KCL at node 1 and rearranging yield:

$$Y_{11}V_1'''' + Y_{12}V_2'''' = \gamma C$$

Applying KCL at node 2 and rearranging yield:

$$Y_{12}V_1'''' + Y_{22}V_2'''' = -\gamma C$$

Solving the latter node equations yields:

$$V_2'''' = \frac{-(Y_{11} + Y_{12})C}{Y_{11}Y_{22} - Y_{12}^2}\gamma$$

Thus, the expression for V_2 is:

$$V_2 = V_2' + V_2'' + V_2''' + V_2''''$$

Substituting,

$$V_2 = \frac{-(Y_{12}/R_1)}{Y_{11}Y_{22} - Y_{12}^2}V_g + \frac{Y_{11}}{Y_{11}Y_{22} - Y_{12}^2}I_g + \frac{Y_{12}/s}{Y_{11}Y_{22} - Y_{12}^2}\rho + \frac{-C(Y_{11} + Y_{12})}{Y_{11}Y_{22} - Y_{12}^2}\gamma$$

Note that, the problem can be solved directly by applying nodal analysis to the original circuit in s-domain;

$$Y_{11}V_1 + Y_{12}V_2 = \frac{V_g}{R_1} + \gamma C - \frac{\rho}{s}$$

$$Y_{12}V_1 + Y_{22}V_2 = I_g - \gamma C$$

Solving the latter equations must give the same result for V_2, check!

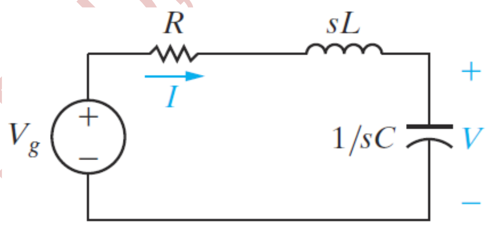
11.4 The Transfer Function

- The **transfer function,** $H(s)$, is defined as the s-domain ratio of the Laplace transform of the output (response), $Y(s)$, to the Laplace transform of the input (source), $X(s)$; with all initial conditions are set to zero.

$$H(s) = \frac{Y(s)}{X(s)}$$

- If there is multiple independent sources (inputs), the transfer function of each source is found, then Superposition is used to find the response to all sources.
- The transfer function depends on the output signal!
- For the circuit shown next, if the current is defined as the output signal, then the transfer function is:

$$H(s) = \frac{I}{V_g} = \frac{1}{R + sL + 1/sC} = \frac{sC}{s^2LC + RCs + 1}$$



- For the same circuit, if the voltage across "C" is defined as the output signal, then the transfer function is:

$$H(s) = \frac{V}{V_g} = \frac{1/sC}{R + sL + 1/sC} = \frac{1}{s^2LC + RCs + 1}$$

The Location of Poles and Zeros of *H*(*s*):

- For linear lumped-parameter circuits, H(s) is always a rational function of s.
- Complex poles and zeros always appear in conjugate pairs.
- The poles of H(s) must lie in the left half of the s-plane if the response to a bounded source is to be bounded.
- The zeros of H(s) may lie in either the right half or the left half of the s-plane.

11.5 The Transfer Function in Partial Fraction Expansions

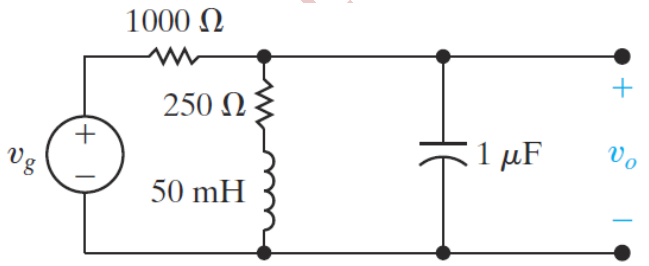
- The output in s-domain is the product of the transfer function and the driving function:

$$Y(s) = H(s)X(s)$$

- Expanding the right-hand side into a sum of partial fractions produces a term for each pole of H(s) and X(s).
- The terms generated by the poles of H(s) correspond to the transient component of the total response.
- The terms generated by the poles of X(s) corresponds to the steady state component of the response.

Example # 6: Deriving the Transfer Function of a Circuit

The voltage source, v_g, drives the circuit shown in the Figure next. The response signal is the voltage across the capacitor, v_o,

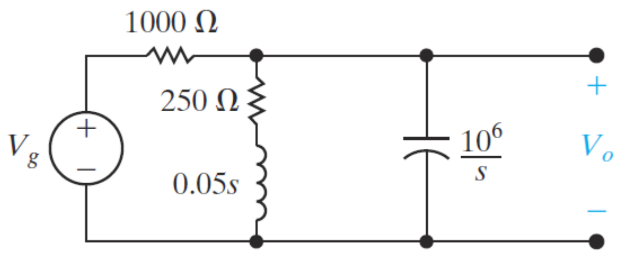


a) Calculate the numerical expression for the transfer function.

b) Calculate the numerical values for the poles and zeros of the transfer function.

Solution

a) The first step in finding the transfer function is to construct the s-domain equivalent circuit, as shown in Figure next.



KCL at the upper node yields,

$$\frac{V_o - V_g}{1000} + \frac{V_o}{250 + 0.05s} + \frac{V_o s}{10^6} = 0$$

Solving for V_o yields

$$V_o = \frac{1000(s + 5000)V_g}{s^2 + 6000s + 25 \times 10^6}$$

Hence the transfer function is

$$H(s) = \frac{V_o}{V_g}$$

$$= \frac{1000(s + 5000)}{s^2 + 6000s + 25 \times 10^6}$$

b) The poles of $H(s)$ are the roots of the denominator polynomial;

$$p_1 = -3000 - j4000$$

$$p_2 = -3000 + j4000$$

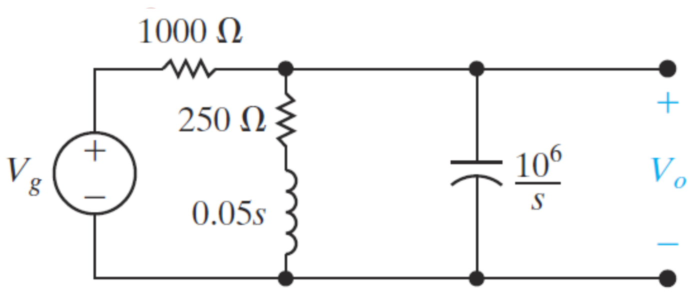
The zeros of $H(s)$ are the roots of the numerator;

$$z_1 = -5000$$

Example # 7: Analyzing the Transfer Function of a Circuit

The circuit in Example # 6 is driven by a voltage source whose voltage increases linearly with time, namely,

$$v_g = 50tu(t)$$



a) Use the transfer function to find v_o

b) Identify the transient component of the response

c) Identify the steady-state component of the response

d) Sketch v_o versus t for $0 \leq t \leq 1.5$ ms

Solution

a) From Example # 6, the transfer function is:

$$H(s) = \frac{1000(s + 5000)}{s^2 + 6000s + 25 \times 10^6}$$

The Laplace transform of the driving voltage $(v_g = 50tu(t))$, is $\left(V_g = \frac{50}{s^2}\right)$

The s-domain expression for the output voltage is:

$$V_o = \frac{1000(s + 5000)}{(s^2 + 6000s + 25 \times 10^6)} \frac{50}{s^2}$$

The partial fraction expansion of V_o is:

$$V_o = \frac{K_1}{s + 3000 - j4000} + \frac{K_1^*}{s + 3000 + j4000} + \frac{K_2}{s^2} + \frac{K_3}{s}$$

Evaluating the coefficients yields:

$$K_1 = 5\sqrt{5} \times 10^{-4} \angle 79.70^\circ;$$

$$K_1^* = 5\sqrt{5} \times 10^{-4} \angle -79.70^\circ,$$

$$K_2 = 10,$$

$$K_3 = -4 \times 10^{-4}.$$

Therefore, the s-domain expression for the output voltage, v_o, is:

$$v_o = [10\sqrt{5} \times 10^{-4} e^{-3000t} \cos(4000t + 79.70^\circ) + 10t - 4 \times 10^{-4}]u(t) \text{ V}$$

b) The transient component of v_o is:

$$10\sqrt{5} \times 10^{-4} e^{-3000t} \cos(4000t + 79.70^\circ)$$

which is generated by the poles of the transfer function;

$$(-3000 + j4000) \text{ and } (-3000 - j4000)$$

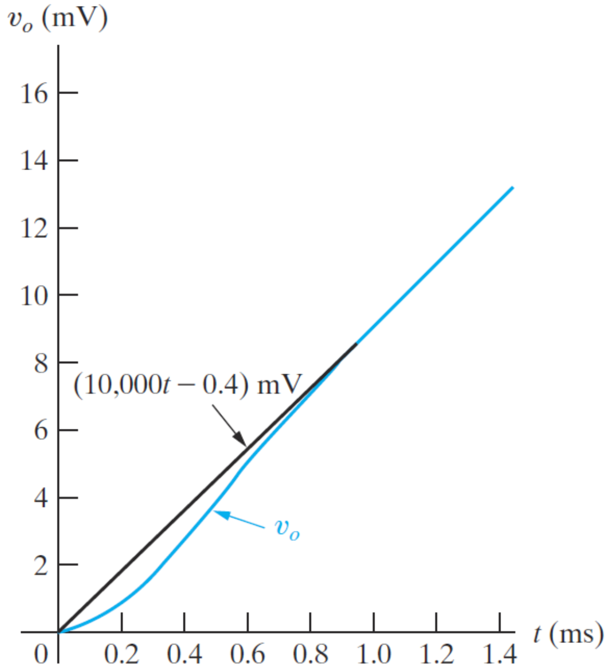
c) The steady-state component of the response is:

$$(10t - 4 \times 10^{-4})u(t)$$

These two terms are generated by the second order pole $\left(\frac{K}{s^2}\right)$ of the driving voltage.

d) A sketch of v_o versus t is shown in the Figure below.

Note that, after $1ms$, $v_o = 10{,}000t - 0.4\ mV$, which is the steady state component only!



Notes,

1- Recall,

$$Y(s) = H(s)X(s)$$

If the input is delayed by "*a*" seconds, then,

$$\mathcal{L}\{x(t-a)u(t-a)\} = e^{-as}X(s)$$

and the response becomes,

$$Y(s) = H(s)X(s)e^{-as}$$

If

$$y(t) = \mathcal{L}^{-1}\{H(s)X(s)\}$$

Then

$$y(t-a)u(t-a) = \mathcal{L}^{-1}\{H(s)X(s)e^{-as}\}$$

i.e., delaying the input by "a" seconds, delays the response by "a" seconds! The circuit which exhibits this characteristics, is called **time invariant**!

2- (Introduction to Convolution)

If a unit impulse source drives the circuit, the response of the circuit equals the inverse transform of the transfer function. Thus if,

$$x(t) = \delta(t), \quad \text{then } X(s) = 1$$

Then

$$Y(s) = H(s)$$

Hence,

$$y(t) = h(t)$$

i.e., The inverse transform of the transfer function equals the unit impulse response of the circuit. Note that this is also the natural response of the circuit.

In fact, the unit impulse response of a circuit, $h(t)$, contains enough information to compute the response to any source that drives the circuit, (convolution integral).

Chapter 12

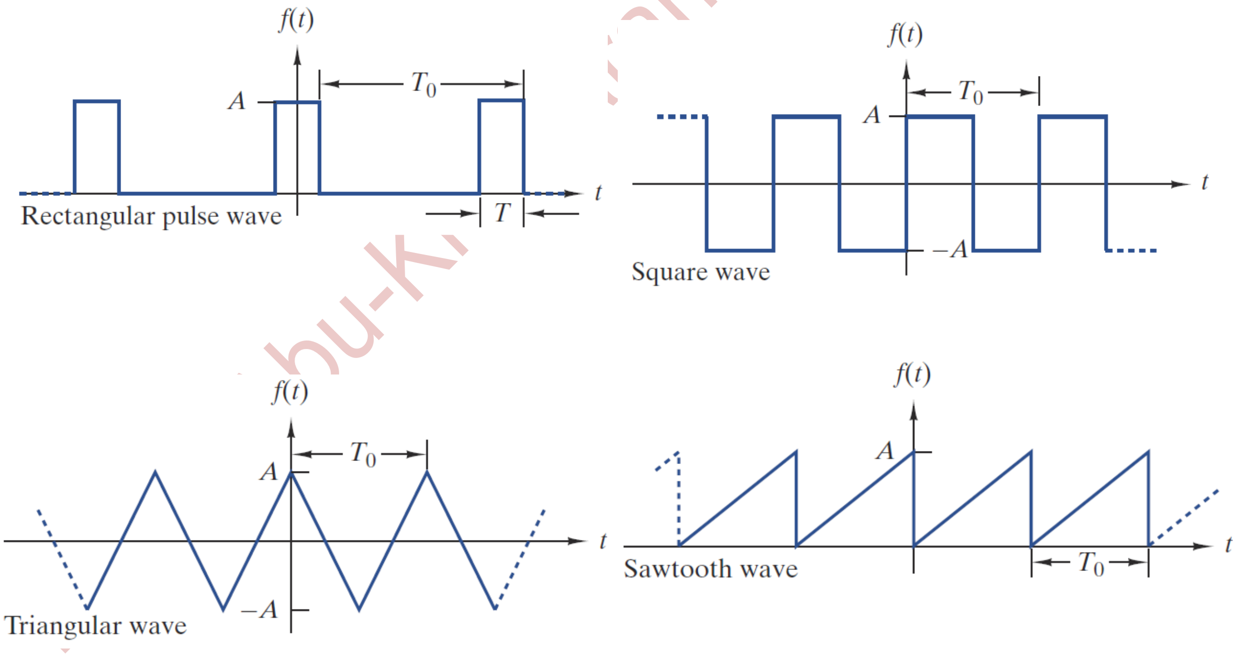
Introduction to Frequency Selective Circuits (Filters)

12.1 Overview of Fourier Analysis

If a signal $f(t)$ is periodic with period T_0 and is reasonably well behaved, then $f(t)$ can be expressed as a Fourier Series of the form:

$$f(t) = a_0 + a_1\cos(2\pi f_0 t) + a_2\cos(2\pi 2 f_0 t) + \cdots + a_n\cos(2\pi n f_0 t) + \cdots$$
$$+ b_1\sin(2\pi f_0 t) + b_2\sin(2\pi 2 f_0 t) + \cdots + b_n\sin(2\pi n f_0 t) + \cdots$$

Examples of periodic signals:



The Fourier Series can be expressed as:

$$f(t) = \underbrace{a_0 +}_{\text{dc}} \underbrace{\sum_{n=1}^{\infty}\left[a_n\cos(2\pi n f_0 t) + b_n\sin(2\pi n f_0 t)\right]}_{\text{ac}}$$

The Fourier Coefficients are:

$$a_0 = \frac{1}{T_0}\int_{-T_0/2}^{+T_0/2} f(t)dt$$

$$a_n = \frac{2}{T_0}\int_{-T_0/2}^{+T_0/2} f(t)\cos(2\pi nt/T_0)dt$$

$$b_n = \frac{2}{T_0}\int_{-T_0/2}^{+T_0/2} f(t)\sin(2\pi nt/T_0)dt$$

where $n = 1, 2, 3, ..$

Alternative form of Fourier Series Expansion (FSE) is:

$$f(t) = A_0 + A_1\cos(2\pi f_0 t + \phi_1) + A_2\cos(2\pi 2 f_0 t + \phi_2) + \cdots + A_n\cos(2\pi n f_0 t + \phi_n) + \cdots$$

where

$$A_n = \sqrt{a_n^2 + b_n^2} \quad \text{and} \quad \phi_n = \tan^{-1}\frac{-b_n}{a_n}$$

Note that:

- A function is even if: $\boldsymbol{f(t) = f(-t)}$**;** symmetry around y-axis
- A function is odd if: $\boldsymbol{f(t) = -f(-t)}$**;** symmetry around origin
- The product of two even functions is even.
- The product of two odd functions is even.
- The product of an even function and an odd function is odd.
- FSE of an even periodic function (with period 2π) does not have terms with sines;

  $$\boldsymbol{f(t) = F_{dc} + \sum_{n=1,2,\ldots}^{\infty} a_n \cos n\omega t;}$$

  where, $\boldsymbol{b_n}$ is zero for all n
- FSE of an odd periodic function (with period 2π) has sine terms only;

  $$\boldsymbol{f(t) = \sum_{n=1,2,\ldots}^{\infty} b_n \sin n\omega t;}$$

  where, the average value and $\boldsymbol{a_n}$ are zero for all n

Example # 1:

Find the Fourier Series Expansion of the square wave!

Solution:

An expression for a square wave on the interval $0 < t < T_0$ is

$$f(t) = \begin{cases} A & 0 < t < T_0/2 \\ -A & T_0/2 < t < T_0 \end{cases}$$

The coefficients for the square wave,

The DC value is:

$$\begin{aligned} a_0 &= \frac{1}{T_0}\int_0^{T_0/2} A\,dt + \frac{1}{T_0}\int_{T_0/2}^{T_0}(-A)dt \\ &= \frac{A}{T_0}\left[\frac{T_0}{2} - 0 - T_0 + \frac{T_0}{2}\right] = 0 \end{aligned}$$

The coefficients of the cosine function,

$$\begin{aligned} a_n &= \frac{2}{T_0}\int_0^{T_0/2} A\cos(2\pi nt/T_0)dt + \frac{2}{T_0}\int_{T_0/2}^{T_0}(-A)\cos(2\pi nt/T_0)dt \\ &= \frac{2A}{T_0}\left[\frac{\sin(2\pi nt/T_0)}{2\pi n/T_0}\right]_0^{T_0/2} - \frac{2A}{T_0}\left[\frac{\sin(2\pi nt/T_0)}{2\pi n/T_0}\right]_{T_0/2}^{T_0} \\ &= \frac{A}{n\pi}[\sin(n\pi) - \sin(0) - \sin(2n\pi) + \sin(n\pi)] = 0 \end{aligned}$$

No cosine terms, as the square function is an odd function!

The coefficients of the sine function,

$$\begin{aligned} b_n &= \frac{2}{T_0}\int_0^{T_0/2} A\sin(2\pi nt/T_0)dt + \frac{2}{T_0}\int_{T_0/2}^{T_0}(-A)\sin(2\pi nt/T_0)dt \\ &= \frac{2A}{T_0}\left[-\frac{\cos(2\pi nt/T_0)}{2\pi n/T_0}\right]_0^{T_0/2} - \frac{2A}{T_0}\left[-\frac{\cos(2\pi nt/T_0)}{2\pi n/T_0}\right]_{T_0/2}^{T_0} \\ &= \frac{A}{n\pi}[-\cos(n\pi) + \cos(0) + \cos(2n\pi) - \cos(n\pi)] \\ &= \frac{2A}{n\pi}[1 - \cos(n\pi)] \end{aligned}$$

The term $[1 - \cos(n\pi)] = 2$ if n is odd and zero if n is even. Hence b_n can be written as

$$b_n = \begin{cases} \dfrac{4A}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

The first three nonzero terms in the Fourier series of the square wave are

$$f(t) = \frac{4A}{\pi}\left[\sin 2\pi f_0 t + \frac{1}{3}\sin 2\pi 3 f_0 t + \frac{1}{5}\sin 2\pi 5 f_0 t + \cdots\right]$$

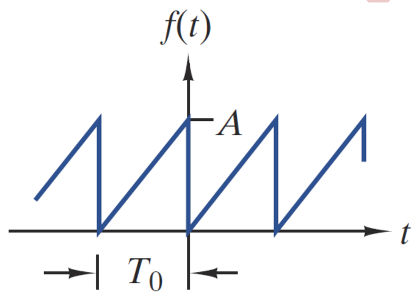
Note that this series contains only odd harmonic terms.

Amplitude (Frequency) Spectrum

The Amplitude (Frequency) Spectrum plots the amplitude of each term versus frequency;

Example # 2:

Plot the Amplitude (Frequency) Spectrum for the sawtooth signal shown (with A= 5, and T_0 = 4ms)

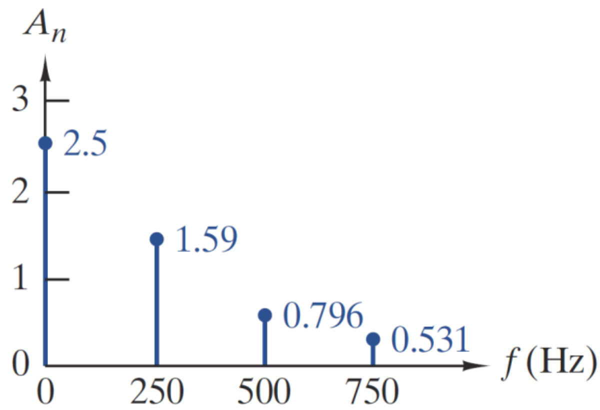


Solution

The Fourier series expansion can be found as:

$$f(t) = 2.5 + 1.59\cos(2\pi 250t + 90^\circ) + 0.796\cos(2\pi 500t + 90^\circ) + 0.531\cos(2\pi 750t + 90^\circ)$$

The Amplitude spectrum is:



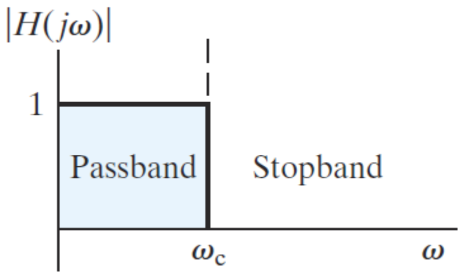
12.2 Passive Filters

- Varying the frequency of a sinusoidal voltage/current source changes impedance of capacitors and inductors.
- Therefore, a careful choice of circuit elements, their values, and their connections to other elements enables constructing circuits that pass to the output only those input signals that reside in a desired range of frequencies.
- Such circuits are called frequency-selective circuits (Filters); employed in Telephones, radios, TVs, Computers, Power Electronic circuits…

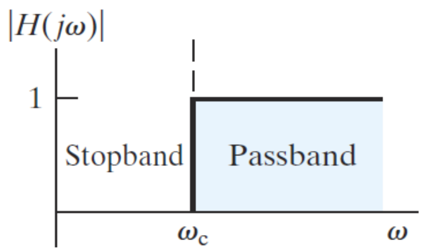
- The frequency-selective circuits (Filters) are classified according to frequency response plot of the magnitude of the transfer function $|(H(j\omega))|$, as a frequency function, into four types:

1. **Low Pass Filter**

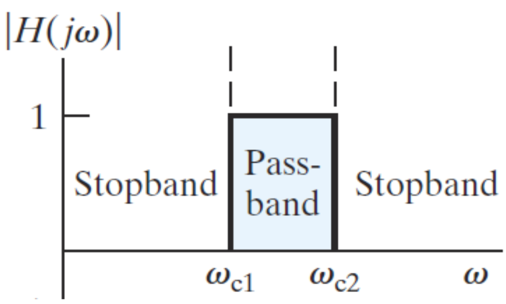
   The magnitude plot is shown next, where ω_c is the cut-off frequency



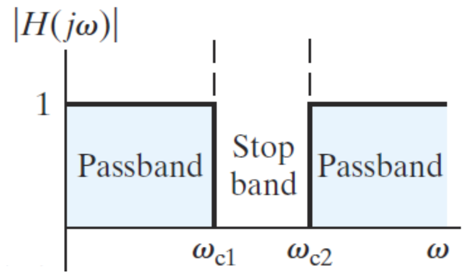
2. **High Pass Filter**



3. **Band Pass Filter**



4. Band Reject Filter



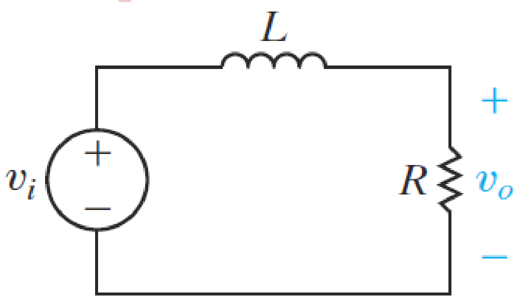
12.3 Types of Passive Filter

- They are constructed using passive components; resistors, capacitors, and inductors.
- The largest output amplitude that can be achieved is "1", except if the filter is a series RLC resonant circuit.
- They may cause loading effect of the input.
- On the other hand, **Active Filters**, that can amplify the inputs are constructed using op-amps and passive components; they have large input impedance and low output impedance; no loading effect!

12.3.1 Low-Pass Filters (LPF)

12.3.1.1 Series RL Circuits LPF

Assuming zero initial conditions, the voltage transfer function of the circuit shown next is:



$$H(s) = \frac{R/L}{s + R/L}$$

To study the frequency response, substitute for $s = j\omega$,

$$H(j\omega) = \frac{R/L}{j\omega + R/L}$$

The magnitude of the transfer function is:

$$|H(j\omega)| = \frac{R/L}{\sqrt{\omega^2 + (R/L)^2}}$$

The phase angle of the transfer function is:

$$\theta(j\omega) = -\tan^{-1}\left(\frac{\omega L}{R}\right)$$

To plot the magnitude of the Transfer function,

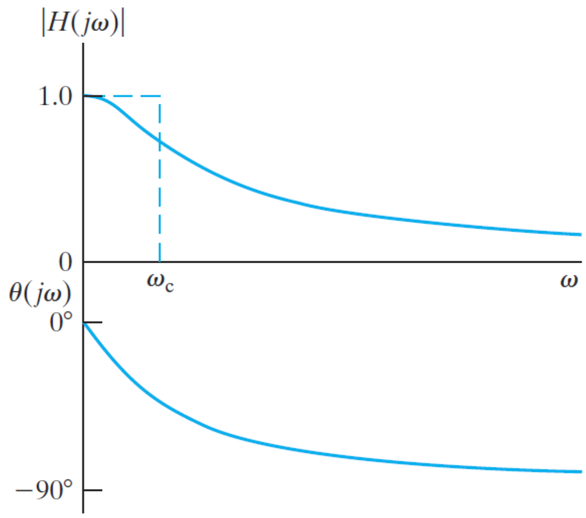
as $\omega \to 0, \omega L \to 0$, and $|H(j\omega)| \to 1$

as $\omega \to \infty, \omega L \to \infty$, and $|H(j\omega)| \to 0$

To plot the phase angle of the Transfer function,

as $\omega \to 0, \omega L \to 0$, and $\theta(j\omega) \to 0$

as $\omega \to \infty, \omega L \to \infty$, and $\theta(j\omega) \to -90^o$



Define:

The Cut-off Frequency

It is the frequency for which the transfer function magnitude is decreased by a factor of $\frac{1}{\sqrt{2}}$ from its maximum value,

$$|H(j\omega_c)| = \frac{1}{\sqrt{2}} H_{\max}$$

But,

$$\begin{aligned} |V_L(j\omega_c)| &= |H(j\omega_c)||V_i| \\ &= \frac{1}{\sqrt{2}} H_{\max}|V_i| \\ &= \frac{1}{\sqrt{2}} V_{L\max}. \\ &= 0.707 V_{Lmax} \end{aligned}$$

The average power delivered to the load, R, using the formula for peak voltage (not rms value) is:

$$\begin{aligned} P(j\omega_c) &= \frac{1}{2} \frac{|V_L^2(j\omega_c)|}{R} \\ &= \frac{1}{2} \frac{\left(\frac{1}{\sqrt{2}} V_{L\max}\right)^2}{R} \\ &= \frac{1}{2} \frac{V_{L\max}^2/2}{R} \\ &= \frac{P_{\max}}{2}. \end{aligned}$$

Note that,

- $P_{max} = \frac{V_{Lmax}^2}{2R}$
- At the cut-off frequency, the average power delivered to the load is one half the maximum average power. Thus, ω_c is also called the **half-power frequency**.

At the cut-off frequency,

$$|H(j\omega_c)| = \frac{1}{\sqrt{2}} H_{\max}$$

But, for a low pass filter, $H_{max} = |H(j0)| = 1$

Therefore, the cut-off frequency can be found as,

$$|H(j\omega_c)| = \frac{1}{\sqrt{2}}|1| = \frac{R/L}{\sqrt{\omega_c^2 + (R/L)^2}}$$

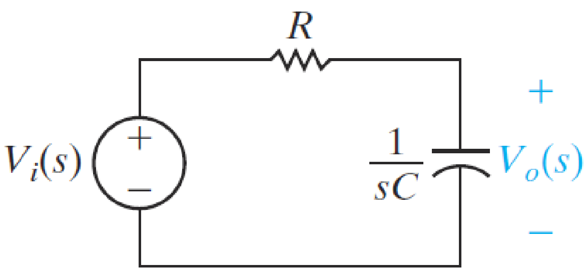
Solving for ω_c yields,

$$\omega_c = \frac{R}{L}$$

ω_c is measured in rad/s, $\left(\omega_c = \frac{1}{\tau}\right)$; thus, $f_c = \frac{\omega_c}{2\pi}$ and is measured in Hz

12.3.1.2 Series RC Circuits LPF

Assuming zero initial conditions, the voltage transfer function of the circuit shown in the Figure next is obtained by voltage divider as:



$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{\frac{1}{sC}}{R + \frac{1}{sC}}$$

Simplifying yields,

$$H(s) = \frac{\frac{1}{RC}}{s + \frac{1}{RC}}$$

Thus,

$$|H(j\omega)| = \frac{\frac{1}{RC}}{\sqrt{\omega^2 + \left(\frac{1}{RC}\right)^2}}$$

At the cut-off frequency,

$$|H(j\omega_c)| = \frac{1}{\sqrt{2}} H_{\max}$$

But, for a low pass filter, $H_{max} = |H(j0)| = 1$

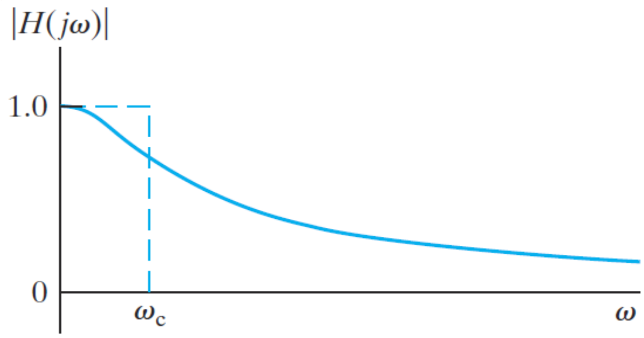
Therefore, the cut-off frequency can be found as,

$$|H(j\omega_c)| = \frac{1}{\sqrt{2}}(1) = \frac{\frac{1}{RC}}{\sqrt{\omega_c^2 + \left(\frac{1}{RC}\right)^2}}$$

Solving for ω_c yields,

$$\omega_c = \frac{1}{RC}$$

ω_c is measured in rad/s, $\left(\omega_c = \frac{1}{\tau}\right)$; thus, $f_c = \frac{\omega_c}{2\pi}$ and is measured in Hz



General Comments:

- For both types of low pass filters (LP), the transfer function can be written as:

$$H(s) = \frac{\omega_c}{s + \omega_c}$$

- For RL LPF, $\omega_c = \frac{R}{L}$
- For RC LPF, $\omega_c = \frac{1}{RC}$
- For both types, $\omega_c = \frac{1}{\tau}$

Example # 3:

Design an RC low pass filter, whose $f_c = 3kHz$.

Solution:

Let $C = 1\mu F$, to find R,

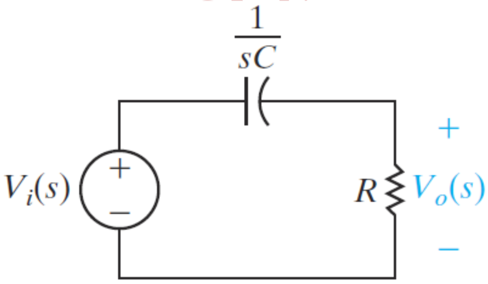
$$\omega_c = 2\pi f_c = 2\pi(3000) = 18{,}849.56\,rad/s$$

$$R = \frac{1}{\omega_c C} = \frac{1}{18{,}849.56(1X10^{-6})} = 53.05\Omega$$

12.3.2 High-Pass Filters (HPF)

12.3.2.1 Series RC Circuits HPF

Assuming zero initial conditions, the voltage transfer function of the circuit shown in the Figure next is obtained by voltage divider as:



$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{R}{R+\frac{1}{sC}}$$

Simplifying yields,

$$H(s) = \frac{s}{s + 1/RC}$$

Thus,

$$H(j\omega) = \frac{j\omega}{j\omega + 1/RC}$$

The magnitude of the transfer function is:

$$|H(j\omega)| = \frac{\omega}{\sqrt{\omega^2 + (1/RC)^2}}$$

and the phase angle of the transfer function is:

$$\theta(j\omega) = 90^\circ - \tan^{-1}\omega RC$$

To plot the magnitude of the Transfer function,

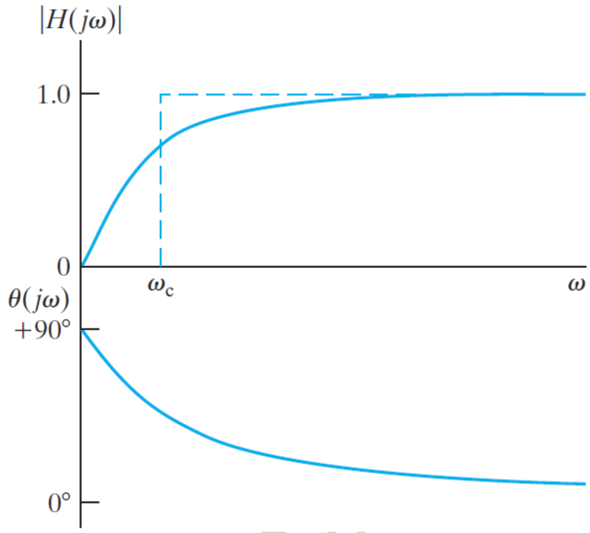
as $\omega \to 0$, and $|H(j\omega)| \to 0$

as $\omega \to \infty$, and $|H(j\omega)| \to 1$

To plot the phase angle of the Transfer function,

as $\omega \to 0$, and $\theta(j\omega) \to 90$

as $\omega \to \infty$, and $\theta(j\omega) \to 0^o$



To find the cut-off frequency,

$$|H(j\omega_c)| = \frac{1}{\sqrt{2}} H_{\max}$$

But, for a high pass filter, $H_{max} = |H(j\infty)| = 1$

Therefore,

$$|H(j\omega_c)| = \frac{1}{\sqrt{2}}(1) = \frac{\omega_c}{\sqrt{\omega_c^2 + (1/RC)^2}}$$

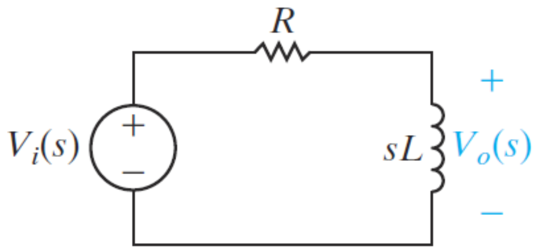
Solving for ω_c yields,

$$\omega_c = \frac{1}{RC}$$

ω_c is measured in rad/s, $\left(\omega_c = \frac{1}{\tau}\right)$; thus, $f_c = \frac{\omega_c}{2\pi}$ and is measured in Hz, which is the same as that for RC LPF!

12.3.2.2 Series RL Circuits HPF

Assuming zero initial conditions, the voltage transfer function of the circuit shown next is:



$$H(s) = \frac{s}{s + R/L}$$

To study the frequency response, substitute for $s = j\omega$,

$$H(j\omega) = \frac{j\omega}{j\omega + R/L}$$

The magnitude of the transfer function is:

$$|H(j\omega)| = \frac{\omega}{\sqrt{\omega^2 + (R/L)^2}}$$

To find the cut-off frequency,

$$\frac{1}{\sqrt{2}} = \frac{\omega_c}{\sqrt{\omega_c^2 + (R/L)^2}}$$

$\therefore \quad \omega_c = \dfrac{R}{L}$ which is the same as that for RL LPF!

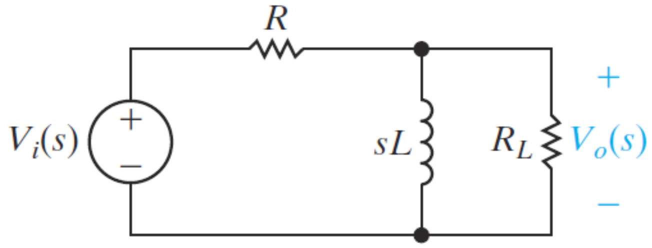
Note that, for a High-Pass Filter the Transfer function is:

$$H(s) = \frac{s}{s + \omega_c}$$

- For RL HPF, $\boldsymbol{\omega_c = \frac{R}{L}}$
- For RC HPF, $\boldsymbol{\omega_c = \frac{1}{RC}}$
- For both types, $\boldsymbol{\omega_c = \frac{1}{\tau}}$

Loading the Series RL HPF

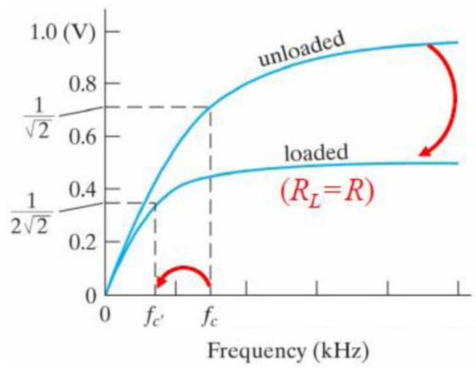
To examine the effect of placing a load resistor in parallel with the inductor in the RL high-pass filter as shown in Figure find the transfer function as:



$$H(s) = \frac{\dfrac{R_L sL}{R_L + sL}}{R + \dfrac{R_L sL}{R_L + sL}} = \frac{\left(\dfrac{R_L}{R + R_L}\right)s}{s + \left(\dfrac{R_L}{R + R_L}\right)\dfrac{R}{L}} = \frac{Ks}{s + \omega_c}$$

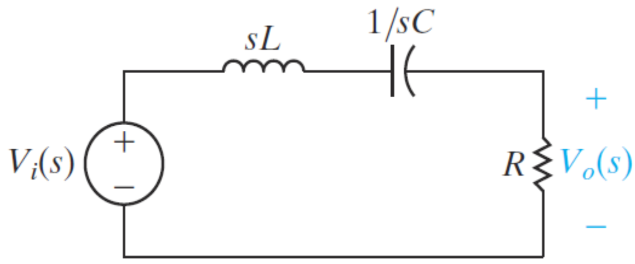
where,

$$K = \frac{R_L}{R + R_L}, \quad \omega_c = KR/L$$



Note that, the effect of the load resistor is to **reduce the passband magnitude by the factor K**, and to **lower the cutoff frequency by the same factor**.

12.3.3 Band-Pass Filters



Assuming zero initial conditions, the voltage transfer function of the circuit shown in the Figure next (**Series RLC Band-Pass Filter**), $H(s) = \frac{V_o(s)}{V_i(s)}$ is obtained by voltage divider as:

$$H(s) = \frac{(R/L)s}{s^2 + (R/L)s + (1/LC)}$$

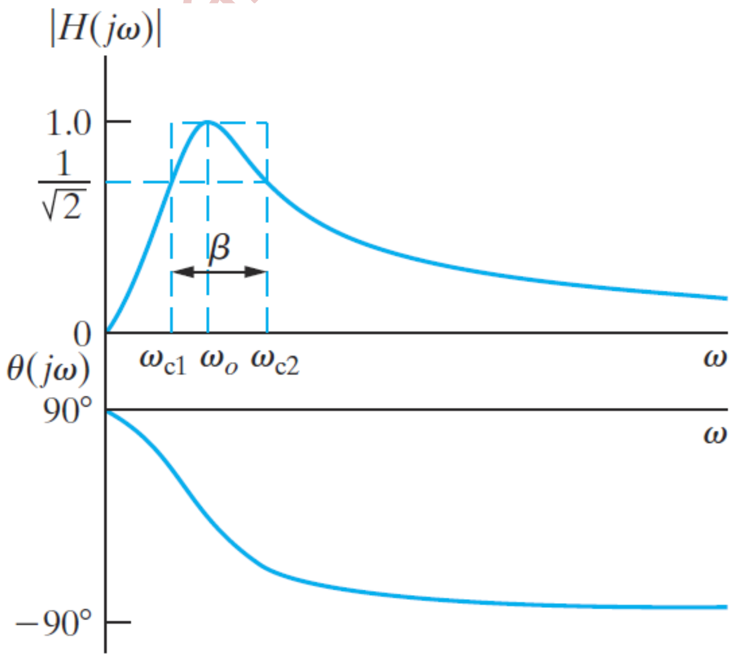
The magnitude of the transfer function is obtained by substituting for $s = j\omega$ and yields,

$$|H(j\omega)| = \frac{\omega(R/L)}{\sqrt{[(1/LC) - \omega^2]^2 + [\omega(R/L)]^2}}$$

And the phase angle of the transfer function is:

$$\theta(j\omega) = 90^\circ - \tan^{-1}\left[\frac{\omega(R/L)}{(1/LC) - \omega^2}\right]$$

A plot of the magnitude and the phase angle of the transfer function are shown in the Figure below,



Define:

The Center (Resonant) frequency (ω_o), is defined as the frequency for which the circuit's transfer function is purely real, $|H(j\omega_o)| = H_{max} = 1$ and $\theta(j\omega_o) = 0$.

$$j\omega_o L + \frac{1}{j\omega_o C} = 0$$

Thus,

$$\omega_o = \sqrt{\frac{1}{LC}}$$

Bandwidth (β): is the width of the passband; $\beta = \omega_{c2} - \omega_{c1}$

Quality Factor (Q): is the ratio of the center frequency to the bandwidth; $Q = \frac{\omega_o}{\beta}$

The Cutoff frequencies (ω_{c1} *and* ω_{c2}): are defined as the frequencies for which

$$|H(j\omega_{c1})| = |H(j\omega_{c2})| = \frac{H_{max}}{\sqrt{2}}$$

To calculate the cutoff frequencies, first find the value of H_{max},

$$\begin{aligned} H_{\max} &= |H(j\omega_o)| \\ &= \frac{\omega_o(R/L)}{\sqrt{[(1/LC) - \omega_o^2]^2 + (\omega_o R/L)^2}} \\ &= \frac{\sqrt{(1/LC)}(R/L)}{\sqrt{[(1/LC) - (1/LC)]^2 + \left[\sqrt{(1/LC)}(R/L)\right]^2}} = 1 \end{aligned}$$

To evaluate the transfer function at the cutoff frequencies,

$$\begin{aligned} \frac{1}{\sqrt{2}} &= \frac{\omega_c(R/L)}{\sqrt{[(1/LC) - \omega_c^2]^2 + (\omega_c R/L)^2}} \\ &= \frac{1}{\sqrt{[(\omega_c L/R) - (1/\omega_c RC)]^2 + 1}} \end{aligned}$$

$$\Rightarrow \qquad \pm 1 = \omega_c \frac{L}{R} - \frac{1}{\omega_c RC}$$

$$\omega_c^2 L \pm \omega_c R - 1/C = 0$$

ω_c has 4 values; 2 positive and 2 negative values, only positive values have physical significance.

$$\omega_{c1} = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \left(\frac{1}{LC}\right)},$$

$$\omega_{c2} = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \left(\frac{1}{LC}\right)}.$$

Note that, $\boldsymbol{\omega_{c2} > \omega_{c1}}$

These cutoff frequencies can be used to prove that,

$$\omega_o = \sqrt{\omega_{c1} \cdot \omega_{c2}}$$

$$= \sqrt{\left[-\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \left(\frac{1}{LC}\right)}\right]\left[\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \left(\frac{1}{LC}\right)}\right]}$$

$$\omega_o = \sqrt{\frac{1}{LC}}$$

The Bandwidth

$$\boldsymbol{\beta = \omega_{c2} - \omega_{c1}}$$

$$= \left[\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \left(\frac{1}{LC}\right)}\right] - \left[-\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \left(\frac{1}{LC}\right)}\right]$$

$$\boldsymbol{\beta = \frac{R}{L}}$$

Therefore, the cutoff frequencies can be written as:

$$\omega_{c1} = -\frac{\beta}{2} + \sqrt{\left(\frac{\beta}{2}\right)^2 + \omega_o^2}$$

$$\omega_{c2} = \frac{\beta}{2} + \sqrt{\left(\frac{\beta}{2}\right)^2 + \omega_o^2}.$$

The Quality Factor (Q):

$$\boldsymbol{Q = \frac{\omega_o}{\beta}}$$

$$Q = \frac{\frac{1}{\sqrt{LC}}}{\frac{R}{L}}$$

$$\boldsymbol{Q = \sqrt{\frac{L}{CR^2}}}$$

Note that,

Recall that, for a series RLC circuit, the natural response was characterized by the neper frequency $\left(\alpha = \frac{R}{2L}\right)$ rad/s and the resonant frequency $\left(\omega_o = \frac{1}{\sqrt{LC}}\right)$ rad/s, then the bandwidth:

$$\boldsymbol{\beta = 2\alpha}$$

The transition from overdamped to under damped response occurs when

$$\omega_o{}^2 = \alpha^2 \Rightarrow \omega_o{}^2 = \frac{\beta^2}{4} \Rightarrow \boldsymbol{\beta = 2\omega_o}$$

Then the quality factor,

$$Q = \frac{\omega_o}{\beta} \Rightarrow \boldsymbol{Q = \frac{\omega_o}{2\omega_o}} \Rightarrow \boldsymbol{Q = \frac{1}{2}}$$

Thus,

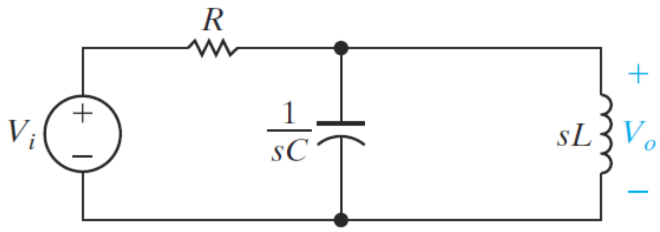
For $> \frac{1}{2}$: Underdamped response; sharp peak at ω_o and narrow bandwidth

For $= \frac{1}{2}$: Critically damped response

For $< \frac{1}{2}$: Overdamped response; broad bandwidth

Another Configuration of a Band-Pass Filter (Parallel RLC)

The transfer function of the circuit shown in the Figure next is:



$$H(s) = \frac{s/RC}{s^2 + s/RC + 1/LC}$$

The Center frequency is also:

$$\omega_o = \sqrt{1/LC}$$

The Bandwidth is

$$\beta = 1/RC$$

In general, the Transfer Function of a Band-Pass Filter can be expressed as:

$$H(s) = \frac{\beta s}{s^2 + \beta s + \omega_o^2}$$

The magnitude of the transfer function can be found, by substituting $s = j\omega$, as:

$$|H(j\omega)| = \frac{\frac{\omega}{RC}}{\sqrt{\left(\frac{1}{LC} - \omega^2\right)^2 + \left(\frac{\omega}{RC}\right)^2}} = \frac{1}{\sqrt{1 + \left(\omega RC - \frac{1}{\omega\frac{L}{R}}\right)^2}}$$

The magnitude is maximum when $\left(\omega RC - \frac{1}{\omega\frac{L}{R}}\right) = 0$ at the center frequency,

$$\omega_o = \sqrt{1/LC}$$

Therefore, $H_{max} = 1$

The cutoff frequencies are:

$$\omega_{c1} = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

and $$\omega_{c2} = \frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

The bandwidth is:

$$\boldsymbol{\beta} = \boldsymbol{\omega_{c2}} - \boldsymbol{\omega_{c1}}$$

$$\boldsymbol{\beta} = \frac{\mathbf{1}}{\boldsymbol{RC}}$$

The quality Factor is:

$$Q = \frac{\omega_o}{\beta} = \sqrt{\frac{R^2C}{L}}$$

The cutoff frequencies can be re-expressed as:

$$\omega_{c1} = -\frac{\beta}{2} + \sqrt{\left(\frac{\beta}{2}\right)^2 - (\omega_o)^2}$$

and $$\omega_{c2} = \frac{\beta}{2} + \sqrt{\left(\frac{\beta}{2}\right)^2 - (\omega_o)^2}$$

Example # 4:

Design a Band-Pass filter (1kHz to 10kHz) using series RLC circuit!

Solution:

$$f_{c1} = 1\ kHz$$

$$f_{c2} = 10\ kHz$$

$$f_o = \sqrt{f_{c1} f_{c2}} = \sqrt{(1000)(10{,}000)} = 3162.28Hz$$

$$\omega_o = \frac{1}{\sqrt{LC}}$$

Choose, $C = 1\mu F$, as there are stricter limitations on commercially available capacitors than on inductors or resistors

$$L = \frac{1}{\omega_o{}^2 C} \Rightarrow L = 2.533mH$$

To calculate R, find

$$Q = \frac{f_o}{f_{c2} - f_{c1}} = \frac{3162.28}{9{,}000} = 0.3514$$

But, for series RLC circuit,

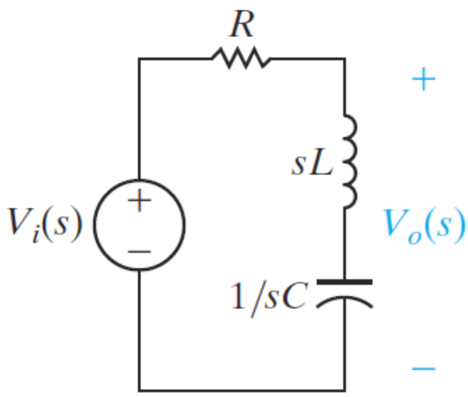
$$Q = \sqrt{\frac{L}{CR^2}} \Rightarrow R = \sqrt{\frac{L}{CQ^2}} \Rightarrow R = \sqrt{\frac{0.002533}{(10^{-6})(0.3514)^2}} = 143.24\Omega$$

Substitute in ω_{c1} *and* ω_{c2} equations and check the answers!

Note that, only two parameters can be specified independently, and the other three parameters depend on these values!

12.3.4 Band-Reject Filters

Assuming zero initial conditions, the voltage transfer function of the circuit shown in the Figure next (**Series RLC Band-Reject Filters**),



$$H(s) = \frac{V_o(s)}{V_i(s)}$$

is obtained by voltage divider as:

$$H(s) = \frac{sL + \dfrac{1}{sC}}{R + sL + \dfrac{1}{sC}}$$

$$H(s) = \frac{s^2 + \dfrac{1}{LC}}{s^2 + \dfrac{R}{L}s + \dfrac{1}{LC}}$$

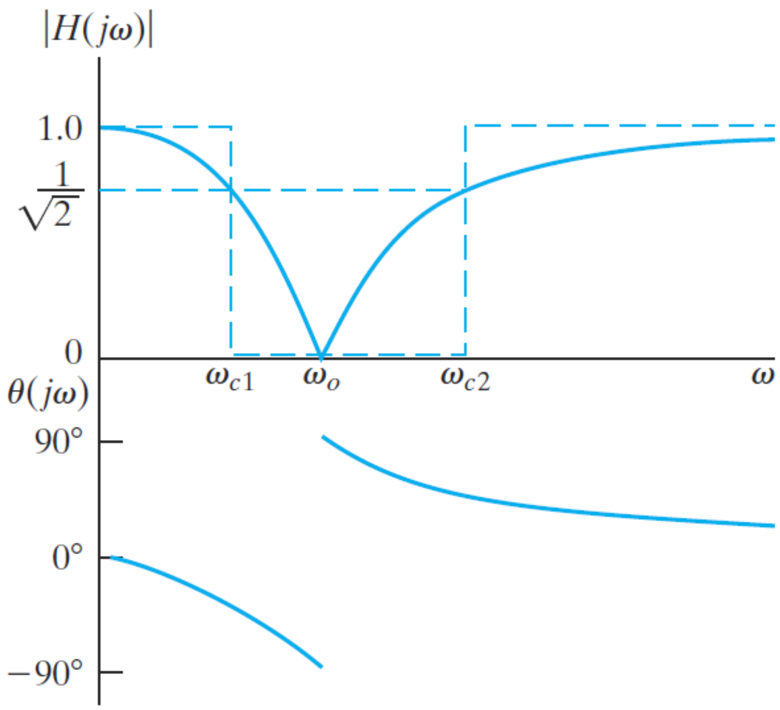
The magnitude of the transfer function is obtained by substituting for $s = j\omega$ and yields,

$$|H(j\omega)| = \frac{\left|\dfrac{1}{LC} - \omega^2\right|}{\sqrt{\left(\dfrac{1}{LC} - \omega^2\right)^2 + \left(\dfrac{\omega R}{L}\right)^2}}$$

And the phase angle of the transfer function is:

$$\theta(j\omega) = -\tan^{-1}\left(\frac{\dfrac{\omega R}{L}}{\dfrac{1}{LC} - \omega^2}\right)$$

A plot of the magnitude and the phase angle of the transfer function are shown in the Figure below,



Define:

The Center (Resonant) frequency ($\boldsymbol{\omega_o}$)**,** is defined as the frequency for which the circuit's transfer function is purely real, $|H(j\omega_o)| = H_{min} = 0$ and $\theta(j\omega_o) = \mp 90.$

$$j\omega_o L + \frac{1}{j\omega_o C} = 0$$

Thus,

$$\omega_o = \sqrt{\frac{1}{LC}}$$

The cutoff frequencies ($\boldsymbol{\omega_{c1}}$ ***and*** $\boldsymbol{\omega_{c2}}$)**:** are defined as the frequencies for which

$$|H(j\omega_{c1})| = |H(j\omega_{c2})| = \frac{H_{max}}{\sqrt{2}}$$

But, $H_{max} = |H(j0)| = |H(j\infty)| = 1$, thus,

$$\omega_{c1} = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \left(\frac{1}{LC}\right)},$$

$$\omega_{c2} = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \left(\frac{1}{LC}\right)}.$$

Note that, $\boldsymbol{\omega_{c2} > \omega_{c1}}$, note also that the cutoff frequencies are the same as those for series Band-Pass filters.

Bandwidth (β): is the width of the stopband;

$$\boldsymbol{\beta = \omega_{c2} - \omega_{c1}}$$

$$\boldsymbol{\beta = \frac{R}{L}}$$

which is the same as that for series Band-Pass filters

Quality Factor (Q): is the ratio of the center frequency to the bandwidth;

$$\boldsymbol{Q = \frac{\omega_o}{\beta}}$$

$$\boldsymbol{Q = \sqrt{\frac{L}{CR^2}}}$$

which is the same as that for series Band-Pass filters

The cutoff frequencies can be expressed as:

$$\omega_{c1} = -\frac{\beta}{2} + \sqrt{\left(\frac{\beta}{2}\right)^2 + \omega_o^2}$$

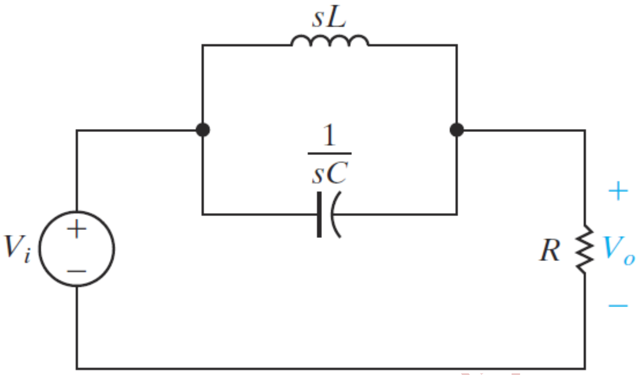
$$\omega_{c2} = \frac{\beta}{2} + \sqrt{\left(\frac{\beta}{2}\right)^2 + \omega_o^2}.$$

or

$$\omega_{c1} = \omega_o \cdot \left[-\frac{1}{2Q} + \sqrt{1 + \left(\frac{1}{2Q}\right)^2}\right],$$

$$\omega_{c2} = \omega_o \cdot \left[\frac{1}{2Q} + \sqrt{1 + \left(\frac{1}{2Q}\right)^2}\right].$$

Another Configuration of a Band-Reject Filter (Parallel RLC)



The transfer function of the circuit shown in the Figure next is:

$$H(s) = \frac{s^2 + 1/LC}{s^2 + s/RC + 1/LC}$$

The Center frequency is also:

$$\omega_o = \sqrt{1/LC}$$

The Bandwidth is

$$\boldsymbol{\beta = \frac{1}{RC}}$$

The quality Factor is:

$$\boldsymbol{Q = \frac{\omega_o}{\beta} = \sqrt{\frac{R^2C}{L}}}$$

In general, the transfer function of Band-reject filter can be written as:

$$H(s) = \frac{s^2 + \omega_o^2}{s^2 + \beta s + \omega_o^2}$$

Again, the cutoff frequencies are:

$$\boldsymbol{\omega_{c1} = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}}$$

and $$\boldsymbol{\omega_{c2} = \frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}}$$

The bandwidth is:

$$\boldsymbol{\beta = \omega_{c2} - \omega_{c1}}$$

$$\boldsymbol{\beta = \frac{1}{RC}}$$

The quality Factor is:

$$Q = \frac{\omega_o}{\beta} = \sqrt{\frac{R^2 C}{L}}$$

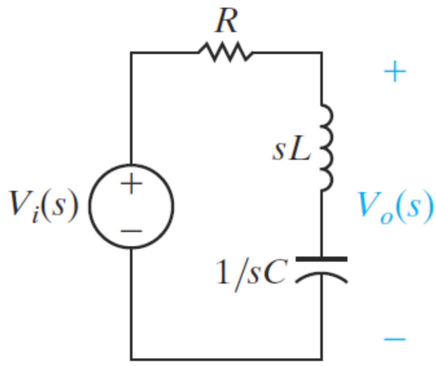
The cutoff frequencies can be re-expressed as:

$$\omega_{c1} = -\frac{\beta}{2} + \sqrt{\left(\frac{\beta}{2}\right)^2 - (\omega_o)^2}$$

and $$\omega_{c2} = \frac{\beta}{2} + \sqrt{\left(\frac{\beta}{2}\right)^2 - (\omega_o)^2}$$

Example # 5:

Design a Band-Reject filter using series RLC circuit, whose bandwidth is 250Hz and center frequency is 750Hz, using C = 100nF to compute the values of Q, L, R, ω_{c1} and ω_{c2}



Solution:

The quality Factor is:

$$Q = \frac{\omega_o}{\beta} = \frac{2\pi(750)}{2\pi(250)} = \mathbf{3}$$

The center frequency is:

$$\omega_o = \frac{1}{\sqrt{LC}}$$

Therefore, $L = \frac{1}{C{\omega_o}^2} = \frac{1}{(10^{-7})(2\pi(750))^2} = \mathbf{450mH}$

The bandwidth,

$$\beta = \frac{R}{L} \Rightarrow R = \beta L \Rightarrow R = 2\pi(250)450X10^{-3} = \mathbf{707}\Omega$$

Substitute in ω_{c1} *and* ω_{c2} equations to find their values!

$$\omega_{c1} = -\frac{\beta}{2} + \sqrt{\left(\frac{\beta}{2}\right)^2 - (\omega_o)^2} = 3992\, rad/s \Rightarrow f_{c1} = 635.3Hz$$

and $$\omega_{c2} = \frac{\beta}{2} + \sqrt{\left(\frac{\beta}{2}\right)^2 - (\omega_o)^2} = 5562.8\, rad/s \Rightarrow f_{c2} = 885.3Hz$$

Check,

The bandwidth is,

$$\beta = 885.3 - 635.3 = 250Hz!$$

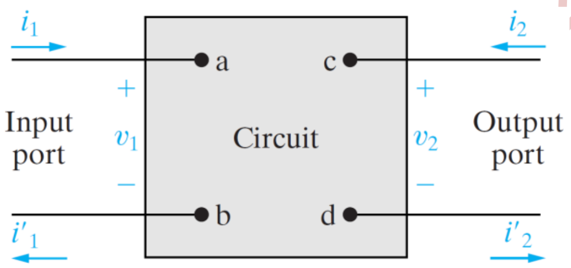
The geometric mean is,

$$f_o = \sqrt{f_{c1}f_{c2}} = \sqrt{(635.3)(885.3)} = 750Hz!$$

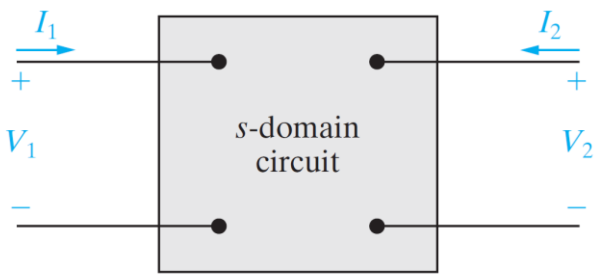
Chapter 13

Two Port Networks

- Analyzing two port networks is convenient, especially when a signal is fed into one pair of terminals and then, after being processed by the system, is extracted at a second pair of terminals.
- A two port building block is shown below,



- The use of two port building block is subject to several restrictions:
  1. No energy is stored within the circuit.
  2. No independent sources within the circuit; dependent sources, however, are permitted.
  3. The current into the port must equal the current out of the port; that is, $i_1 = i_1'$ and $i = i_2'$.
  4. All external connections must be made to either the input port or the output port; no such connections are allowed between ports.
- The fundamental principle underlying two-port modeling of a system is that only the terminal variables (i_1, v_1, i_2, and v_2) are of interest.
- Two variables of the four are independent; the other two are found in terms of the known two variables.
- The most general description of the two port networks is carried out in s-domain.



- A two-port circuit can be modeled by a 2*2 matrix to relate the V/I variables, where the four matrix elements can be obtained by performing 2 experiments.
- There are six different ways in which to combine the four V/I variables;

1- Impedance/Admittance Matrices:

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \times \begin{bmatrix} I_1 \\ I_2 \end{bmatrix};$$ $[Z]$ is the Impedance matrix

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \times \begin{bmatrix} V_1 \\ V_2 \end{bmatrix};$$ $[Y] = [Z]^{-1}$ is the Admittance matrix

2- Hybrid Matrices:

$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \times \begin{bmatrix} I_1 \\ V_2 \end{bmatrix};$$ $[H]$ is the *h*-Hybrid matrix

$$\begin{bmatrix} I_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \times \begin{bmatrix} V_1 \\ I_2 \end{bmatrix};$$ $[G] = [H]^{-1}$ is the *g*-Hybrid matrix

3- Transmission Matrices:

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} a_{11} & -a_{12} \\ a_{21} & -a_{22} \end{bmatrix} \times \begin{bmatrix} V_2 \\ I_2 \end{bmatrix};$$ $[A]$ is the *a*-Transmission matrix

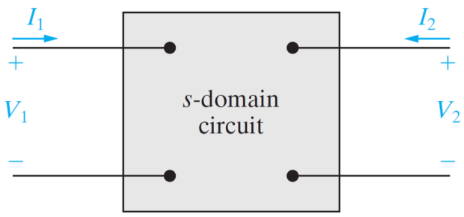
$$\begin{bmatrix} V_2 \\ I_2 \end{bmatrix} = \begin{bmatrix} b_{11} & -b_{12} \\ b_{21} & -b_{22} \end{bmatrix} \times \begin{bmatrix} V_1 \\ I_1 \end{bmatrix};$$ $[B] = [A]^{-1}$ is the *b*-Transmission matrix

- Which set to be choosen depends on which variables are given; e.g., if the source voltage and current are given then choose the transmission matrix [B] in the analysis.

13.1 The Immittance (Impedance or Admittance) Parameters

13.1.1 (Impedance) *z*-Parameters

$$V_1 = z_{11}I_1 + z_{12}I_2$$
$$V_2 = z_{21}I_1 + z_{22}I_2$$



where, the impedance seen when looking into port 1, with port 2 open (with $I_2 = 0$), is:

$$z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0} \Omega$$

The transfer impedances are:

$$z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0} \Omega$$

and

$$z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0} \Omega$$

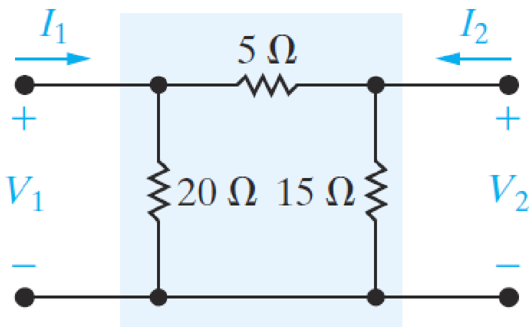
The impedance seen when looking into port 2, with port 1 open (with $I_1 = 0$), is:

$$z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0} \Omega$$

Note that, to set the current to zero, open circuit the terminals of the respective port!

Example # 1:

Find the z-parameters for the circuit shown in the Figure below.



Solution:

The resistance seen when looking into port 1, with port 2 open (with $I_2 = 0$), is the parallel combination between 20Ω and (5Ω in series with 5Ω) and equals:

$$z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0} = \frac{(20)(20)}{40} = 10\ \Omega$$

When $I_2 = 0$, then V_2 is:

$$V_2 = \frac{V_1}{15 + 5}(15) = 0.75V_1$$

Therefore,

$$z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0} = \frac{0.75V_1}{V_1/10} = 7.5\ \Omega$$

The resistance seen when looking into port 2, with port 1 open (with $I_1 = 0$), is the parallel combination between 15Ω and (5Ω in series with 20Ω) and equals:

$$z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0} = \frac{(15)(25)}{40} = 9.375\ \Omega$$

When $I_1 = 0$, the V_1 and I_2 are:

$$V_1 = \frac{V_2}{5 + 20}(20) = 0.8V_2$$

$$I_2 = \frac{V_2}{9.375}$$

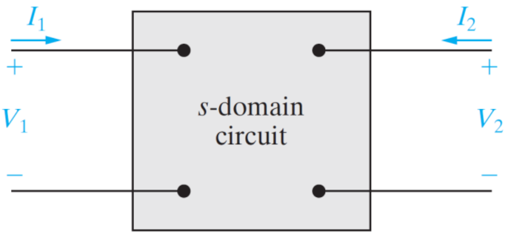
Hence,

$$z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0} = \frac{0.8V_2}{V_2/9.375} = 7.5\ \Omega$$

13.1.2 (Admittance) *y*-Parameters

$$I_1 = y_{11}V_1 + y_{12}V_2$$

$$I_2 = y_{21}V_1 + y_{22}V_2$$



where,

$$y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0} \text{ S}$$

$$y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0} \text{ S}$$

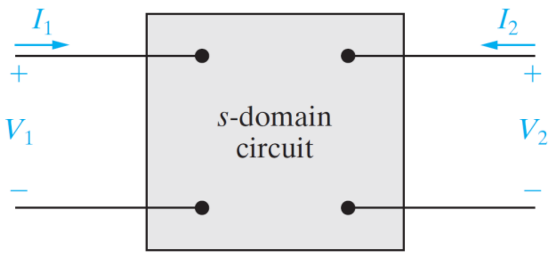
$$y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2=0} \text{ S}$$

$$y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0} \text{ S}$$

Note that, to set the voltage to zero, short circuit the terminals of the respective port!

13.2 The Hybrid Parameters

They relate cross variables; the input voltage and output current (V_1 and I_2) to the input current and output voltage (I_1 and V_2);



13.2.1 *h*-Parameters

$$V_1 = h_{11}I_1 + h_{12}V_2$$
$$I_2 = h_{21}I_1 + h_{22}V_2$$

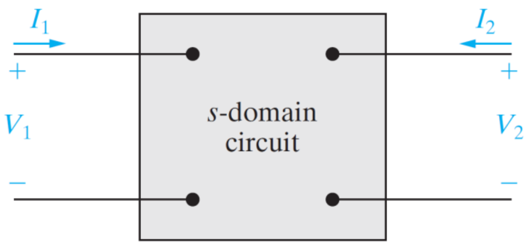
where,

$$h_{11} = \left.\frac{V_1}{I_1}\right|_{V_2=0} \Omega \qquad h_{12} = \left.\frac{V_1}{V_2}\right|_{I_1=0}$$

$$h_{21} = \left.\frac{I_2}{I_1}\right|_{V_2=0} \qquad h_{22} = \left.\frac{I_2}{V_2}\right|_{I_1=0} \text{S}$$

13.2.2 *g*-Parameters

$$I_1 = g_{11}V_1 + g_{12}I_2$$
$$V_2 = g_{21}V_1 + g_{22}I_2$$

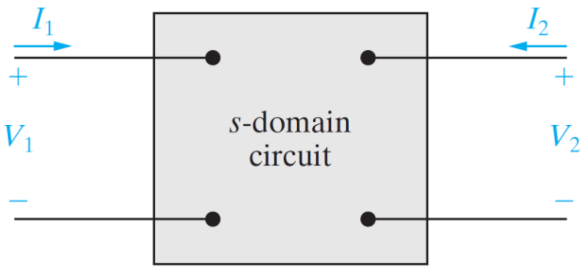


where,

$$g_{11} = \left.\frac{I_1}{V_1}\right|_{I_2=0} \text{S} \qquad g_{12} = \left.\frac{I_1}{I_2}\right|_{V_1=0}$$

$$g_{21} = \left.\frac{V_2}{V_1}\right|_{I_2=0} \qquad g_{22} = \left.\frac{V_2}{I_2}\right|_{V_1=0} \Omega$$

13.3 The Transmission Parameters



They describe the voltage and current at one end of the two-port network in terms of the voltage and current at the other end.

13.3.1 *a*-Parameters

$$V_1 = a_{11}V_2 - a_{12}I_2$$
$$I_1 = a_{21}V_2 - a_{22}I_2$$

where,

$$a_{11} = \left.\frac{V_1}{V_2}\right|_{I_2=0} \qquad a_{12} = -\left.\frac{V_1}{I_2}\right|_{V_2=0}\ \Omega$$

$$a_{21} = \left.\frac{I_1}{V_2}\right|_{I_2=0}\ \text{S} \qquad a_{22} = -\left.\frac{I_1}{I_2}\right|_{V_2=0}$$

13.3.2 *b*-Parameters

$$V_2 = b_{11}V_1 - b_{12}I_1$$
$$I_2 = b_{21}V_1 - b_{22}I_1$$

where,

$$b_{11} = \left.\frac{V_2}{V_1}\right|_{I_1=0} \qquad b_{12} = -\left.\frac{V_2}{I_1}\right|_{V_1=0}\ \Omega$$

$$b_{21} = \left.\frac{I_2}{V_1}\right|_{I_1=0}\ \text{S} \qquad b_{22} = -\left.\frac{I_2}{I_1}\right|_{V_1=0}$$

Example # 2:

The following measurements pertain to a two-port circuit operating in the sinusoidal steady state.

With port 2 open,

The voltage applied to port 1 is:

$$v_1 = 150 \cos 4{,}000t \; V$$

The current into port 1 is:

$$i_1 = 25 \cos(4{,}000t - 45) \; A$$

The voltage measured at port 2 is:

$$v_2 = 100 \cos(4{,}000t + 15) \; V$$

With port 2 short circuited,

The voltage applied to port 1 is:

$$v_1 = 30 \cos 4{,}000t \; V$$

The current into port 1 is:

$$i_1 = 1.5 \cos(4{,}000t + 30) \; A$$

The current into port 2 is:

$$i_2 = 0.25 \cos(4{,}000t + 150) \; A$$

Find the a-parameters that can describe the sinusoidal steady-state behavior of the circuit.

Solution:

- The first set of measurements gives:

$$\mathbf{V}_1 = 150\angle 0^\circ \text{ V}, \qquad \mathbf{I}_1 = 25\angle -45^\circ \text{ A},$$

$$\mathbf{V}_2 = 100\angle 15^\circ \text{ V}, \qquad \mathbf{I}_2 = 0 \text{ A}.$$

Therefore,

$$a_{11} = \left.\frac{\mathbf{V}_1}{\mathbf{V}_2}\right|_{I_2=0} = \frac{150\angle 0^\circ}{100\angle 15^\circ} = 1.5\angle -15^\circ$$

$$a_{21} = \left.\frac{\mathbf{I}_1}{\mathbf{V}_2}\right|_{I_2=0} = \frac{25\angle -45^\circ}{100\angle 15^\circ} = 0.25\angle -60^\circ \text{S}$$

- The second set of measurements gives:

$$\mathbf{V}_1 = 30\angle 0^\circ \text{ V}, \qquad \mathbf{I}_1 = 1.5\angle 30^\circ \text{A},$$
$$\mathbf{V}_2 = 0 \text{ V}, \qquad \mathbf{I}_2 = 0.25\angle 150^\circ \text{ A}.$$

Therefore,

$$a_{12} = -\left.\frac{\mathbf{V}_1}{\mathbf{I}_2}\right|_{V_2=0} = \frac{-30\angle 0^\circ}{0.25\angle 150^\circ} = 120\angle 30^\circ \ \Omega,$$
$$a_{21} = -\left.\frac{\mathbf{I}_1}{\mathbf{I}_2}\right|_{V_2=0} = \frac{-1.5\angle 30^\circ}{0.25\angle 150^\circ} = 6\angle 60^\circ.$$

13.4 Relation Among the Two-Port Parameters

13.4.1 Relation between *z*-Parameters and *y*-Parameters

Given the y-parameters,

$$I_1 = y_{11}V_1 + y_{12}V_2$$

$$I_2 = y_{21}V_1 + y_{22}V_2$$

Solving (using Cramer's rule) for V_1 and V_2 to find the z-parameters yields,

$$V_1 = \frac{\begin{vmatrix} I_1 & y_{12} \\ I_2 & y_{22} \end{vmatrix}}{\begin{vmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{vmatrix}} = \frac{y_{22}}{\Delta y} I_1 - \frac{y_{12}}{\Delta y} I_2$$

$$V_2 = \frac{\begin{vmatrix} y_{11} & I_1 \\ y_{21} & I_2 \end{vmatrix}}{\Delta y} = -\frac{y_{21}}{\Delta y} I_1 + \frac{y_{11}}{\Delta y} I_2$$

where,

$$\Delta y = y_{11}y_{22} - y_{12}y_{21}$$

Recall,

$$V_1 = z_{11}I_1 + z_{12}I_2$$

$$V_2 = z_{21}I_1 + z_{22}I_2$$

Therefore,

$$z_{11} = \frac{y_{22}}{\Delta y}, \quad z_{12} = -\frac{y_{12}}{\Delta y}, \quad z_{21} = -\frac{y_{21}}{\Delta y}, \quad z_{22} = \frac{y_{11}}{\Delta y}.$$

13.4.2 Relation between *z*-Parameters and *a*-Parameters

The a-parameters equations are:

$$V_1 = a_{11}V_2 - a_{12}I_2$$

$$I_1 = a_{21}V_2 - a_{22}I_2$$

Rearranging I_1 equation yields,

$$V_2 = \frac{1}{a_{21}}I_1 + \frac{a_{22}}{a_{21}}I_2$$

Substituting the latter equation (V_2 eq.) in V_1 equation yields,

$$V_1 = \frac{a_{11}}{a_{21}}I_1 + \left(\frac{a_{11}a_{22}}{a_{21}} - a_{12}\right)I_2$$

Thus, from V_1 equation;

$$z_{11} = \frac{a_{11}}{a_{21}}, \qquad z_{12} = \frac{\Delta a}{a_{21}}.$$

and from V_2 equation; $$V_2 = \frac{1}{a_{21}}I_1 + \frac{a_{22}}{a_{21}}I_2,$$

$$z_{21} = \frac{1}{a_{21}}, \quad z_{22} = \frac{a_{22}}{a_{21}}.$$

where, $$\Delta a = a_{11}a_{22} - a_{12}a_{21}$$

For other conversions, refer to the parameters' conversion table next!

Parameters' Conversion Table:

$$z_{11} = \frac{y_{22}}{\Delta y} = \frac{a_{11}}{a_{21}} = \frac{b_{22}}{b_{21}} = \frac{\Delta h}{h_{22}} = \frac{1}{g_{11}}$$

$$z_{12} = -\frac{y_{12}}{\Delta y} = \frac{\Delta a}{a_{21}} = \frac{1}{b_{21}} = \frac{h_{12}}{h_{22}} = -\frac{g_{12}}{g_{11}}$$

$$z_{21} = \frac{-y_{21}}{\Delta y} = \frac{1}{a_{21}} = \frac{\Delta b}{b_{21}} = -\frac{h_{21}}{h_{22}} = \frac{g_{21}}{g_{11}}$$

$$z_{22} = \frac{y_{11}}{\Delta y} = \frac{a_{22}}{a_{21}} = \frac{b_{11}}{b_{21}} = \frac{1}{h_{22}} = \frac{\Delta g}{g_{11}}$$

$$y_{11} = \frac{z_{22}}{\Delta z} = \frac{a_{22}}{a_{12}} = \frac{b_{11}}{b_{12}} = \frac{1}{h_{11}} = \frac{\Delta g}{g_{22}}$$

$$y_{12} = -\frac{z_{12}}{\Delta z} = -\frac{\Delta a}{a_{12}} = -\frac{1}{b_{12}} = -\frac{h_{12}}{h_{11}} = \frac{g_{12}}{g_{22}}$$

$$y_{21} = -\frac{z_{21}}{\Delta z} = -\frac{1}{a_{12}} = -\frac{\Delta b}{b_{12}} = \frac{h_{21}}{h_{11}} = -\frac{g_{21}}{g_{22}}$$

$$y_{22} = \frac{z_{11}}{\Delta z} = \frac{a_{11}}{a_{12}} = \frac{b_{22}}{b_{12}} = \frac{\Delta h}{h_{11}} = \frac{1}{g_{22}}$$

$$h_{11} = \frac{\Delta z}{z_{22}} = \frac{1}{y_{11}} = \frac{a_{12}}{a_{22}} = \frac{b_{12}}{b_{11}} = \frac{g_{22}}{\Delta g}$$

$$h_{12} = \frac{z_{12}}{z_{22}} = -\frac{y_{12}}{y_{11}} = \frac{\Delta a}{a_{22}} = \frac{1}{b_{11}} = -\frac{g_{12}}{\Delta g}$$

$$h_{21} = -\frac{z_{21}}{z_{22}} = \frac{y_{21}}{y_{11}} = -\frac{1}{a_{22}} = -\frac{\Delta b}{b_{11}} = -\frac{g_{21}}{\Delta g}$$

$$h_{22} = \frac{1}{z_{22}} = \frac{\Delta y}{y_{11}} = \frac{a_{21}}{a_{22}} = \frac{b_{21}}{b_{11}} = \frac{g_{11}}{\Delta g}$$

$$g_{11} = \frac{1}{z_{11}} = \frac{\Delta y}{y_{22}} = \frac{a_{21}}{a_{11}} = \frac{b_{21}}{b_{22}} = \frac{h_{22}}{\Delta h}$$

$$g_{12} = -\frac{z_{12}}{z_{11}} = \frac{y_{12}}{y_{22}} = -\frac{\Delta a}{a_{11}} = -\frac{1}{b_{22}} = -\frac{h_{12}}{\Delta h}$$

$$g_{21} = \frac{z_{21}}{z_{11}} = -\frac{y_{21}}{y_{22}} = \frac{1}{a_{11}} = \frac{\Delta b}{b_{22}} = -\frac{h_{21}}{\Delta h}$$

$$g_{22} = \frac{\Delta z}{z_{11}} = \frac{1}{y_{22}} = \frac{a_{12}}{a_{11}} = \frac{b_{12}}{b_{22}} = \frac{h_{11}}{\Delta h}$$

$$a_{11} = \frac{z_{11}}{z_{21}} = -\frac{y_{22}}{y_{21}} = \frac{b_{22}}{\Delta b} = -\frac{\Delta h}{h_{21}} = \frac{1}{g_{21}}$$

$$a_{12} = \frac{\Delta z}{z_{21}} = -\frac{1}{y_{21}} = \frac{b_{12}}{\Delta b} = -\frac{h_{11}}{h_{21}} = \frac{g_{22}}{g_{21}}$$

$$a_{21} = \frac{1}{z_{21}} = -\frac{\Delta y}{y_{21}} = \frac{b_{21}}{\Delta b} = -\frac{h_{22}}{h_{21}} = \frac{g_{11}}{g_{21}}$$

$$a_{22} = \frac{z_{22}}{z_{21}} = -\frac{y_{11}}{y_{21}} = \frac{b_{11}}{\Delta b} = -\frac{1}{h_{21}} = \frac{\Delta g}{g_{21}}$$

$$b_{11} = \frac{z_{22}}{z_{12}} = -\frac{y_{11}}{y_{12}} = \frac{a_{22}}{\Delta a} = \frac{1}{h_{12}} = -\frac{\Delta g}{g_{12}}$$

$$b_{12} = \frac{\Delta z}{z_{12}} = -\frac{1}{y_{12}} = \frac{a_{12}}{\Delta a} = \frac{h_{11}}{h_{12}} = -\frac{g_{22}}{g_{12}}$$

$$b_{21} = \frac{1}{z_{12}} = -\frac{\Delta y}{y_{12}} = \frac{a_{21}}{\Delta a} = \frac{h_{22}}{h_{12}} = -\frac{g_{11}}{g_{12}}$$

$$b_{22} = \frac{z_{11}}{z_{12}} = \frac{y_{22}}{y_{12}} = \frac{a_{11}}{\Delta a} = \frac{\Delta h}{h_{12}} = -\frac{1}{g_{12}}$$

Such that, the respective determinants are:

$$\Delta z = z_{11}z_{22} - z_{12}z_{21}$$

$$\Delta y = y_{11}y_{22} - y_{12}y_{21}$$

$$\Delta a = a_{11}a_{22} - a_{12}a_{21}$$

$$\Delta b = b_{11}b_{22} - b_{12}b_{21}$$

$$\Delta h = h_{11}h_{22} - h_{12}h_{21}$$

$$\Delta g = g_{11}g_{22} - g_{12}g_{21}$$

Example # 3:

Two sets of measurements are made on a two-port resistive circuit. The first set is made with port 2 open, and the second set is made with port 2 short-circuited. The results are as follows:

Port 2 Open	**Port 2 Short-Circuited**
$V_1 = 10\text{ mV}$	$V_1 = 24\text{ mV}$
$I_1 = 10\ \mu\text{A}$	$I_1 = 20\ \mu\text{A}$
$V_2 = -40\text{ V}$	$I_2 = 1\text{ mA}$

Find the h-parameters of the circuit.

Solution:

$$h_{11} = \left.\frac{V_1}{I_1}\right|_{V_2=0}$$

$$= \frac{24 \times 10^{-3}}{20 \times 10^{-6}} = 1.2\text{ k}\Omega$$

$$h_{21} = \left.\frac{I_2}{I_1}\right|_{V_2=0}$$

$$= \frac{10^{-3}}{20 \times 10^{-6}} = 50$$

The parameters h_{12} and h_{22} cannot be obtained directly from the open-circuit test.

$$h_{12} = \left.\frac{V_1}{V_2}\right|_{I_1=0} \qquad h_{22} = \left.\frac{I_2}{V_2}\right|_{I_1=0}\text{S}$$

But, the four a-parameters can be derived from the test data.

From the conversion table,

$$h_{12} = \frac{\Delta a}{a_{22}}$$

$$h_{22} = \frac{a_{21}}{a_{22}}$$

Hence,

$$a_{11} = \left.\frac{V_1}{V_2}\right|_{I_2=0} = \frac{10 \times 10^{-3}}{-40} = -0.25 \times 10^{-3},$$

$$a_{21} = \left.\frac{I_1}{V_2}\right|_{I_2=0} = \frac{10 \times 10^{-6}}{-40} = -0.25 \times 10^{-6}\,\text{S},$$

$$a_{12} = -\left.\frac{V_1}{I_2}\right|_{V_2=0} = -\frac{24 \times 10^{-3}}{10^{-3}} = -24\ \Omega,$$

$$a_{22} = -\left.\frac{I_1}{I_2}\right|_{V_2=0} = -\frac{20 \times 10^{-6}}{10^{-3}} = -20 \times 10^{-3}.$$

The determinant is:

$$\Delta a = a_{11}a_{22} - a_{12}a_{21}$$

$$= 5 \times 10^{-6} - 6 \times 10^{-6} = -10^{-6}$$

Therefore,

$$h_{12} = \frac{\Delta a}{a_{22}}$$

$$= \frac{-10^{-6}}{-20 \times 10^{-3}} = 5 \times 10^{-5}$$

and

$$h_{22} = \frac{a_{21}}{a_{22}}$$

$$= \frac{-0.25 \times 10^{-6}}{-20 \times 10^{-3}} = 12.5\ \mu\text{S}$$

Chapter 14

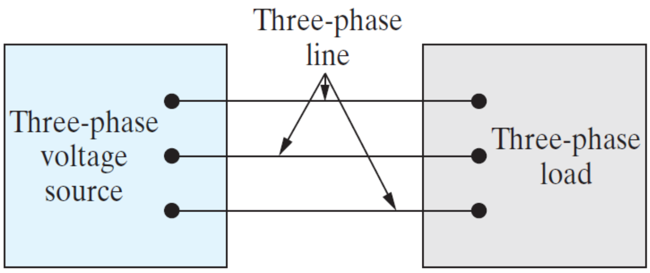
Balanced Three-Phase Circuits

Three-phase circuits are circuits which contain three voltage sources that are a one third of a cycle apart in time (120^o).

It is more *advantageous and economical* to transmit electric power in a three-phase mode.

14.1 Balanced Three-Phase Voltages

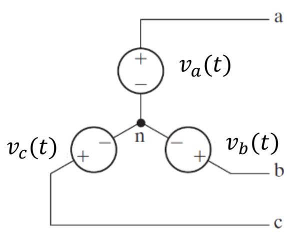
- A set of ***balanced three-phase voltages*** consists of three sinusoidal voltages that have *identical amplitudes and frequencies* but are out of phase with each other by exactly 120^o.
- If the load, connected to the voltage sources, draws balanced currents, the entire circuit is referred to as a balanced three-phase circuit.



- **In Time Domain:**

Balanced positive phase sequence (abc) voltages are defined as:

$$v_a(t) = V_m \cos(\omega t)$$
$$v_b(t) = V_m \cos(\omega t - 120^o)$$
$$v_c(t) = V_m \cos(\omega t - 240^o) = V_m \cos(\omega t + 120^o)$$



For balanced voltages:

$$v_a(t) + v_b(t) + v_c(t) = 0$$

Note that, $v_b(t)$ lags $v_a(t)$ by 120^o

$v_c(t)$ lags $v_b(t)$ by 120^o

If the load is balanced, then the currents for "abc" sequence are:

$$i_a(t) = I_m \cos(\omega t - \theta_i)$$
$$i_b(t) = I_m \cos(\omega t - \theta_i - 120^o)$$
$$i_c(t) = I_m \cos(\omega t - \theta_i + 120^o)$$

where θ_i is the angle between the phase voltage and current.

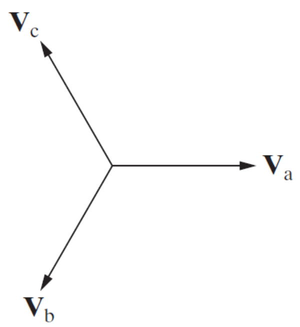
Balanced negative phase sequence (acb) voltages are defined as:

$$v_a(t) = V_m \cos(\omega t)$$
$$v_b(t) = V_m \cos(\omega t + 120^o)$$
$$v_c(t) = V_m \cos(\omega t - 120^o)$$

- **In Frequency Domain:**

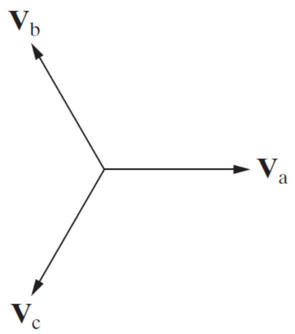
Balanced positive phase sequence (abc) voltages are:

$$\mathbf{V}_a = V_m \angle 0^\circ,$$
$$\mathbf{V}_b = V_m \angle -120^\circ,$$
$$\mathbf{V}_c = V_m \angle +120^\circ,$$



Balanced negative phase sequence (acb) voltages are:

$$\mathbf{V}_a = V_m \angle 0^\circ,$$
$$\mathbf{V}_b = V_m \angle +120^\circ,$$
$$\mathbf{V}_c = V_m \angle -120^\circ.$$



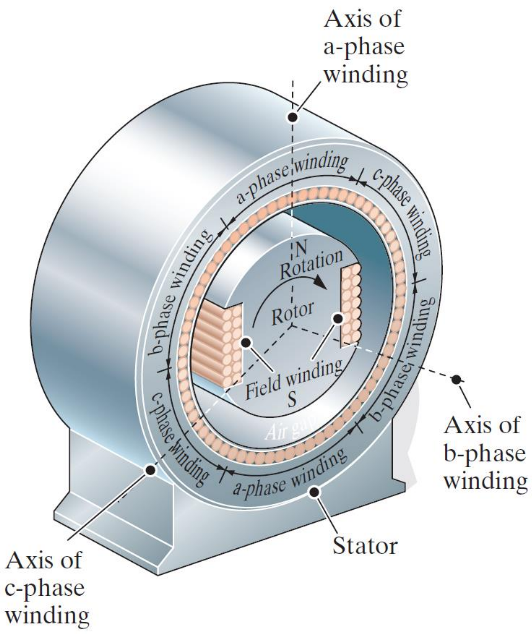
- For a balanced system:

$$\mathbf{V}_a + \mathbf{V}_b + \mathbf{V}_c = 0.$$

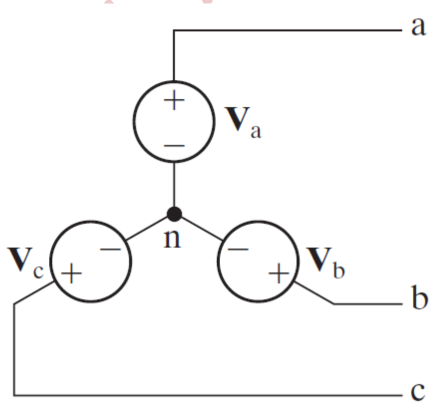
- Note that, for a balanced three phase system, *determining the voltage (or current) in one phase is sufficient* as the quantities for other phases have the same magnitude, but only with a phase shift similar to that between the voltage sources.

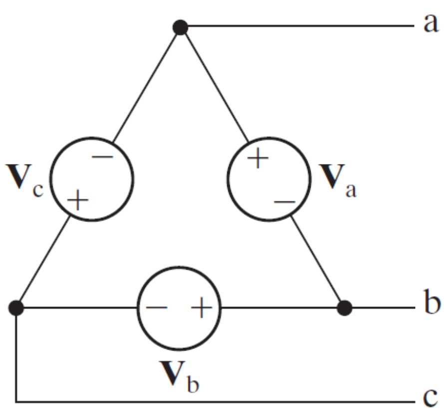
14.2 Three-Phase Voltage Sources

- A three-phase voltage source is a generator with three separate windings distributed around the periphery of the **stator;** *with 120 electrical degree between the windings of any two phases*. Each winding comprises one phase of the generator.
- The **rotor** of the generator is an electromagnet driven at synchronous speed by a prime mover, such as a steam or gas turbine.
- A **Prime mover** provides the mechanical energy to the generator (from steam, gas, coal, hydraulic,…).
- The prime mover rotates the magnetic field at synchronous speed, thus all phases will have the **same frequency** (relative speed).
- If the **windings are identical**, identical voltages result, but out of phase by 120^o.

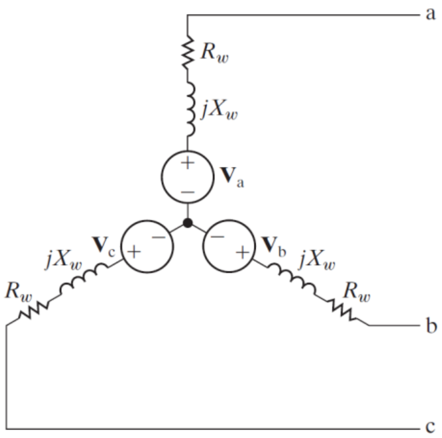


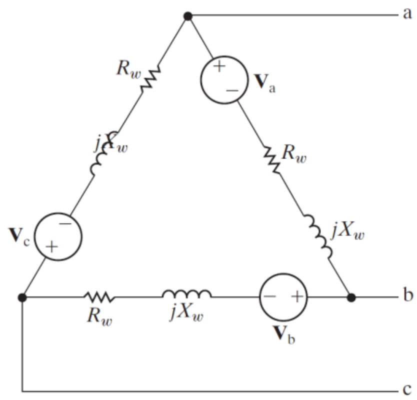
- The three phase voltage are either connected as WYE or Delta (left and right figures below, respectively), but the WYE configuration is more commonly used.
- The common node in WYE configuration is called the Neutral point (n).
- If the internal impedance of each voltage source is small, it can be neglected; an ideal source results.





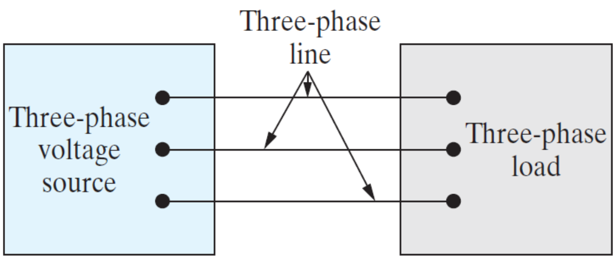
- If the winding impedances were not neglected, then the equivalent three-phase sources model are:





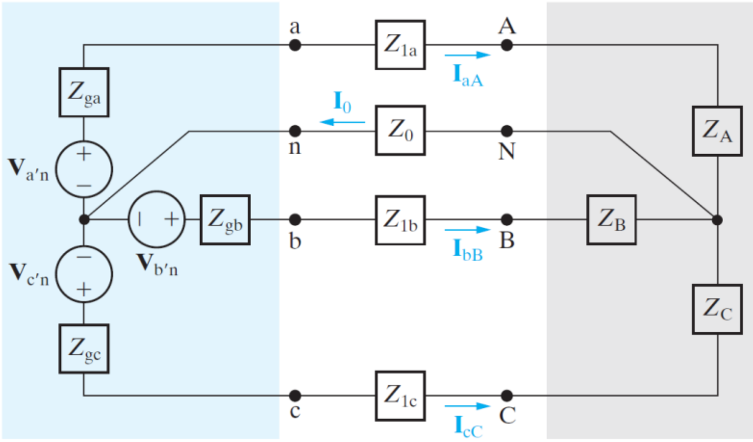
- Since the three-phase load can be also connected as WYE or Delta configuration, then 4 possible combinations exist:

Source	Load
Y	Y
Y	Δ
Δ	Y
Δ	Δ



14.3 Analysis of the Wye-Wye Circuit

- Consider the three-phase circuit shown in the figure below,



- Let the neutral point of the sources be the reference voltage; 0V.
- The *impedances* of the sources, the transmission lines, and loads are shown in the figure.
- Applying KCL at the Neutral point of the load (N), which has a voltage with respect to "n", V_N, yields:

$$\frac{\mathbf{V}_N}{Z_0} + \frac{\mathbf{V}_N - \mathbf{V}_{a'n}}{Z_A + Z_{1a} + Z_{ga}} + \frac{\mathbf{V}_N - \mathbf{V}_{b'n}}{Z_B + Z_{1b} + Z_{gb}} + \frac{\mathbf{V}_N - \mathbf{V}_{c'n}}{Z_C + Z_{1c} + Z_{gc}} = 0$$

- If the three-phase circuit is balanced, then:

1. The source voltages, $v_{a'n}$, $v_{b'n}$ and $v_{c'n}$ are balanced;

$$\mathbf{V}_{a'n} + \mathbf{V}_{b'n} + \mathbf{V}_{c'n} = 0$$

2. The impedance of the each phase source is the same;

$$Z_{ga} = Z_{gb} = Z_{gc}$$

3. The impedance of each transmission line is the same;

$$Z_{la} = Z_{lb} = Z_{lc}$$

4. The impedance of each phase of the load is the same;

$$Z_A = Z_B = Z_C$$

- Thus, the node voltage equation can be rewritten for a balanced circuits as:

$$\mathbf{V}_N\left(\frac{1}{Z_0} + \frac{3}{Z_\phi}\right) = \frac{\mathbf{V}_{a'n} + \mathbf{V}_{b'n} + \mathbf{V}_{c'n}}{Z_\phi}$$

where

$$Z_\phi = Z_A + Z_{1a} + Z_{ga} = Z_B + Z_{1b} + Z_{gb} = Z_C + Z_{1c} + Z_{gc}$$

But, $\mathbf{V_{a'n}} + \mathbf{V_{b'n}} + \mathbf{V_{c'n}} = 0$, the right hand side of the node equation is zero; leading to the fact that *for a balanced system*,

$$\mathbf{V_N} = 0$$

Since no potential difference between the load's and the sources' neutral points;

$$\mathbf{V_N} = \mathbf{V}_{\boldsymbol{n}} = 0,$$

The current in the neutral transmission line is:

$$\mathbf{I_0} = 0$$

Which means that the neutral conductor can be short circuited or even removed (open circuited).

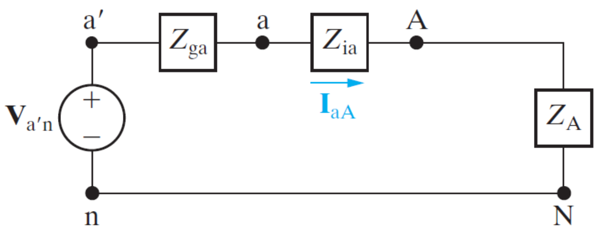
- Therefore, the three line currents are:

$$\mathbf{I}_{aA} = \frac{\mathbf{V}_{a'n} - \mathbf{V}_N}{Z_A + Z_{1a} + Z_{ga}} = \frac{\mathbf{V}_{a'n}}{Z_\phi}$$

$$\mathbf{I}_{bB} = \frac{\mathbf{V}_{b'n} - \mathbf{V}_N}{Z_B + Z_{1b} + Z_{gb}} = \frac{\mathbf{V}_{b'n}}{Z_\phi}$$

$$\mathbf{I}_{cC} = \frac{\mathbf{V}_{c'n} - \mathbf{V}_N}{Z_C + Z_{1c} + Z_{gc}} = \frac{\mathbf{V}_{c'n}}{Z_\phi}$$

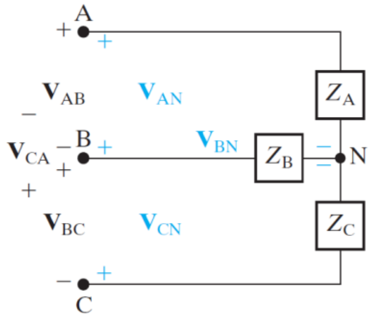
- Thus, the line current of each phase equals that phase's voltage divided by the total impedance of its circuit.
- To analyze a balanced three-phase circuit, *a single (per-phase) equivalent circuit suffices to analyze the system.* Then, the currents (voltages) in the other phases can be found, with an appropriate phase shift.



Note that, the neutral has also the currents from the other phases;

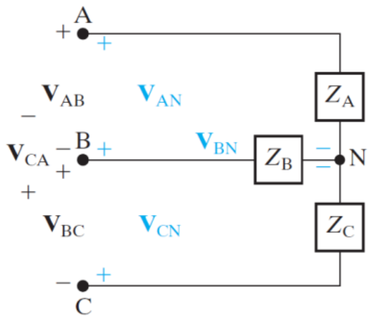
$$\mathbf{I}_o = \mathbf{I}_{aA} + \mathbf{I}_{bB} + \mathbf{I}_{cC}$$

- Once, the line currents are found, then ***the line to neutral (phase) voltages at the load*** can be found.



The line-to-neutral voltages are $\mathbf{V}_{AN}$, $\mathbf{V}_{BN}$, and $\mathbf{V}_{CN}$.

The Line-to-Neutral (phase) voltages are:



$$\mathbf{V_{AN}} = V_{\emptyset}\ \angle 0^o\ \text{V}$$

$$\mathbf{V_{BN}} = V_{\emptyset}\ \angle -120^o\ \text{V}$$

$$\mathbf{V_{CN}} = V_{\emptyset}\ \angle 120^o\ \text{V}$$

The Line-to-Line voltages are:

$$\mathbf{V}_{AB} = \mathbf{V}_{AN} - \mathbf{V}_{BN},$$

$$\mathbf{V}_{BC} = \mathbf{V}_{BN} - \mathbf{V}_{CN},$$

$$\mathbf{V}_{CA} = \mathbf{V}_{CN} - \mathbf{V}_{AN}.$$

Hence, substituting the phase voltages in $\mathbf{V_{AB}}$;

$$\mathbf{V_{AB}} = \mathbf{V_{AN}} - \mathbf{V_{BN}}$$

$$\mathbf{V_{AB}} = V_{\emptyset}\ \angle 0^o\ - V_{\emptyset}\ \angle -120^o$$

$$\mathbf{V_{AB}} = V_{\emptyset} - V_{\emptyset}\left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right)$$

$$\mathbf{V_{AB}} = V_{\emptyset}\left(\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)$$

$$\mathbf{V_{AB}} = \sqrt{3}V_{\emptyset}\ \angle +30^o$$

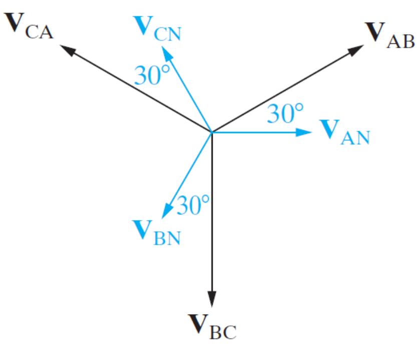
Similarly,

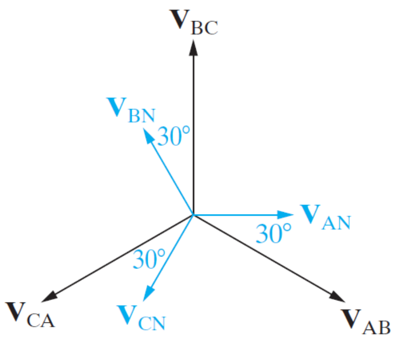
$$\mathbf{V_{BC}} = \sqrt{3}V_{\emptyset}\ \angle -90^o$$

$$\mathbf{V_{CA}} = \sqrt{3}V_{\emptyset}\ \angle -210^o \quad \Rightarrow \quad \mathbf{V_{CA}} = \sqrt{3}V_{\emptyset}\ \angle +150^o$$

Thus, the line-to-line voltages are obtained by multiplying the phase voltages with $\sqrt{3}$ and shifted by $+30^o$; $\mathbf{V_L} = \sqrt{3}V_{\emptyset}\ \angle \theta_{\emptyset}+30^o$.

The phasor diagram of these voltages ("abc" for left figure) is shown below, whist the phasor diagram for "acb" is shown in the right figure below.





For Y-Y connection, the line currents are the same as the phase currents; $\mathbf{I_L} = \mathbf{I_\phi}$.

The line currents are:

$$\mathbf{I_{aA}} = \frac{\mathbf{V_{AN}}}{Z_\phi}$$

$$\mathbf{I_{bB}} = \frac{\mathbf{V_{BN}}}{Z_\phi}$$

$$\mathbf{I_{cC}} = \frac{\mathbf{V_{CN}}}{Z_\phi}$$

The neutral current for a balanced load;

$$\mathbf{I}_o = \mathbf{I}_{aA} + \mathbf{I}_{bB} + \mathbf{I}_{cC}$$

Example # 1:

A balanced three-phase Y-connected generator with positive sequence has an impedance of $0.2 + j0.5\ \Omega/\phi$ and an internal voltage of $120\ V/\phi$. The generator feeds a balanced three-phase Y-connected load having an impedance of $39 + j28\ \Omega/\phi$. The impedance of the line connecting the generator to the load is $0.8 + j1.5\ \Omega/\phi$. The a-phase internal voltage of the generator is specified as the reference phasor.

a) Construct the a-phase equivalent circuit of the system.

b) Calculate the three line currents $\mathbf{I}_{aA}$, $\mathbf{I}_{bB}$, and $\mathbf{I}_{cC}$.

c) Calculate the three phase voltages at the load, $\mathbf{V}_{AN}$, $\mathbf{V}_{BN}$, and $\mathbf{V}_{CN}$.

d) Calculate the line voltages $\mathbf{V}_{AB}$, $\mathbf{V}_{BC}$, and $\mathbf{V}_{CA}$ at the terminals of the load.

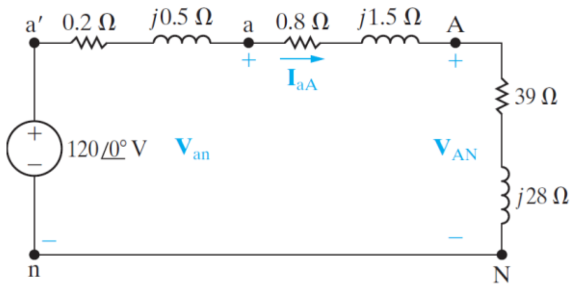
e) Calculate the phase voltages at the terminals of the generator, $\mathbf{V}_{an}$, $\mathbf{V}_{bn}$, and $\mathbf{V}_{cn}$.

f) Calculate the line voltages $\mathbf{V}_{ab}$, $\mathbf{V}_{bc}$, and $\mathbf{V}_{ca}$ at the terminals of the generator.

g) Repeat (a)–(f) for a negative phase sequence.

Solution:

a) The single phase equivalent circuit is:



b) The a-phase line current is

$$\mathbf{I}_{aA} = \frac{120\angle 0^\circ}{(0.2 + 0.8 + 39) + j(0.5 + 1.5 + 28)}$$
$$= \frac{120\angle 0^\circ}{40 + j30}$$
$$= 2.4\angle -36.87^\circ \text{ A.}$$

For a positive phase sequence,

$$\mathbf{I}_{bB} = 2.4\angle -156.87^\circ \text{ A,}$$
$$\mathbf{I}_{cC} = 2.4\angle 83.13^\circ \text{ A.}$$

c) The phase voltage at the A terminal of the load is

$$\mathbf{V}_{AN} = (39 + j28)(2.4\angle -36.87^\circ)$$
$$= 115.22\angle -1.19^\circ \text{ V.}$$

For a positive phase sequence,

$$\mathbf{V}_{BN} = 115.22\angle -121.19^\circ \text{ V,}$$
$$\mathbf{V}_{CN} = 115.22\angle 118.81^\circ \text{ V.}$$

d) For a positive phase sequence, the line voltages lead the phase voltages by 30°; thus

$$\mathbf{V}_{AB} = (\sqrt{3}\angle 30^\circ)\mathbf{V}_{AN}$$
$$= 199.58\angle 28.81^\circ \text{ V,}$$
$$\mathbf{V}_{BC} = 199.58\angle -91.19^\circ \text{ V,}$$
$$\mathbf{V}_{CA} = 199.58\angle 148.81^\circ \text{ V.}$$

e) The phase voltage at the a terminal of the source is

$$\mathbf{V}_{an} = 120 - (0.2 + j0.5)(2.4\angle -36.87^\circ)$$
$$= 120 - 1.29\angle 31.33^\circ$$
$$= 118.90 - j0.67$$
$$= 118.90\angle -0.32^\circ \text{ V.}$$

For a positive phase sequence,

$$\mathbf{V}_{bn} = 118.90\angle -120.32^\circ \text{ V,}$$
$$\mathbf{V}_{cn} = 118.90\angle 119.68^\circ \text{ V.}$$

f) The line voltages at the source terminals are

$$\mathbf{V}_{ab} = (\sqrt{3}\angle 30^\circ)\mathbf{V}_{an}$$
$$= 205.94\angle 29.68^\circ \text{ V,}$$
$$\mathbf{V}_{bc} = 205.94\angle -90.32^\circ \text{ V,}$$
$$\mathbf{V}_{ca} = 205.94\angle 149.68^\circ \text{ V.}$$

g) Changing the phase sequence has no effect on the single-phase equivalent circuit. The three line currents are

$$\mathbf{I}_{aA} = 2.4\,\angle -36.87^\circ \text{ A},$$
$$\mathbf{I}_{bB} = 2.4\,\angle 83.13^\circ \text{ A},$$
$$\mathbf{I}_{cC} = 2.4\,\angle -156.87^\circ \text{ A}.$$

The phase voltages at the load are

$$\mathbf{V}_{AN} = 115.22\,\angle -1.19^\circ \text{ V},$$
$$\mathbf{V}_{BN} = 115.22\,\angle 118.81^\circ \text{ V},$$
$$\mathbf{V}_{CN} = 115.22\,\angle -121.19^\circ \text{ V}.$$

For a negative phase sequence, the line voltages lag the phase voltages by 30°:

$$\mathbf{V}_{AB} = (\sqrt{3}\,\angle -30^\circ)\mathbf{V}_{AN}$$
$$= 199.58\,\angle -31.19^\circ \text{ V},$$
$$\mathbf{V}_{BC} = 199.58\,\angle 88.81^\circ \text{ V},$$
$$\mathbf{V}_{CA} = 199.58\,\angle -151.19^\circ \text{ V}.$$

The phase voltages at the terminals of the generator are

$$\mathbf{V}_{an} = 118.90\,\angle -0.32^\circ \text{ V},$$
$$\mathbf{V}_{bn} = 118.90\,\angle 119.68^\circ \text{ V},$$
$$\mathbf{V}_{cn} = 118.90\,\angle -120.32^\circ \text{ V}.$$

The line voltages at the terminals of the generator are

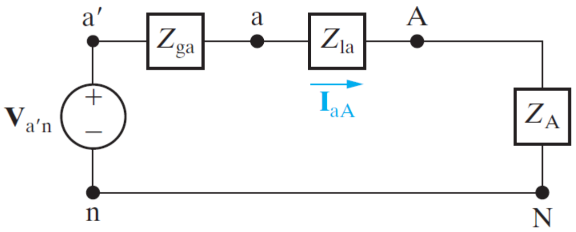
$$\mathbf{V}_{ab} = (\sqrt{3}\,\angle -30^\circ)\mathbf{V}_{an}$$
$$= 205.94\,\angle -30.32^\circ \text{ V},$$
$$\mathbf{V}_{bc} = 205.94\,\angle 89.68^\circ \text{ V},$$
$$\mathbf{V}_{ca} = 205.94\,\angle -150.32^\circ \text{ V}.$$

14.4 Analysis of the Wye-Delta Circuit

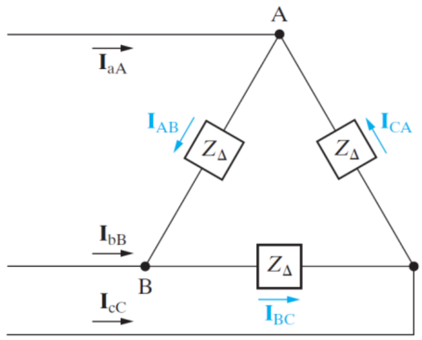
- *For a Δ-load, the phase voltages are the same as the line voltages*.
- The Δ-load can be transformed into a Wye by using the delta-to-wye transformation.
- When the Δ-load is balanced, the impedance of each leg of the wye is one third the impedance of each leg of the delta, or

$$Z_Y = \frac{Z_\Delta}{3}$$

- After the Δ-load has been replaced by its equivalent Y, the a-phase can be modeled by the single phase equivalent circuit, as shown in the figure below.



- This circuit is used to calculate the line currents, and then the line currents are used to find the currents in each leg of the original Δ-load.
- The relationship between the line currents and the currents in each leg of the delta can be derived using the circuit shown in the figure next.
- Assuming a positive sequence (abc), and assuming the phase currents are:



$$\mathbf{I}_{AB} = I_\phi \angle 0^\circ,$$

$$\mathbf{I}_{BC} = I_\phi \angle -120^\circ,$$

$$\mathbf{I}_{CA} = I_\phi \angle 120^\circ.$$

where $I_\emptyset$ is the magnitude of each phase current.

- The line currents can be derived in terms of the phase currents by applying KCL at each node. KCL at node A:

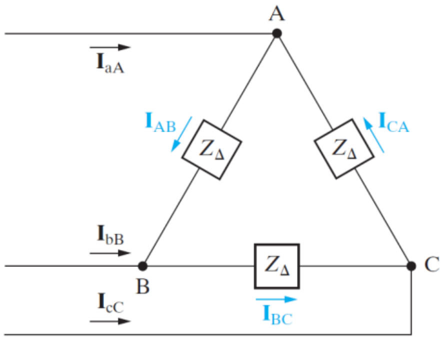
$$\mathbf{I}_{aA} = \mathbf{I}_{AB} - \mathbf{I}_{CA}$$

$$= I_\phi \angle 0^\circ - I_\phi \angle 120^\circ$$

$$= \sqrt{3} I_\phi \angle -30^\circ,$$

- KCL at nodes B and C, respectively;

$$\mathbf{I}_{bB} = \mathbf{I}_{BC} - \mathbf{I}_{AB}$$
$$= I_{\phi}\angle -120^{\circ} - I_{\phi}\angle 0^{\circ}$$
$$= \sqrt{3}I_{\phi}\angle -150^{\circ},$$
$$\mathbf{I}_{cC} = \mathbf{I}_{CA} - \mathbf{I}_{BC}$$
$$= I_{\phi}\angle 120^{\circ} - I_{\phi}\angle -120^{\circ}$$
$$= \sqrt{3}I_{\phi}\angle 90^{\circ}.$$



Notes:

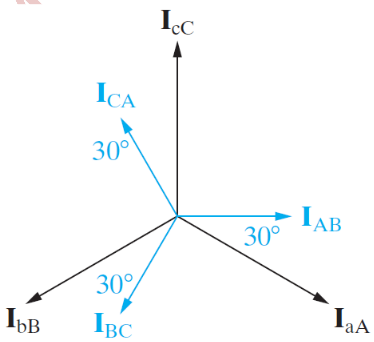
For ***a positive sequence***, the magnitude of the line currents is $\sqrt{3}$ times the magnitude of the phase currents, and that the set of line currents lags the set of phase currents by 30^o;

$$\mathbf{I_L} = \sqrt{3}I_{\emptyset}\ \angle\theta_{\emptyset}-30^o.$$

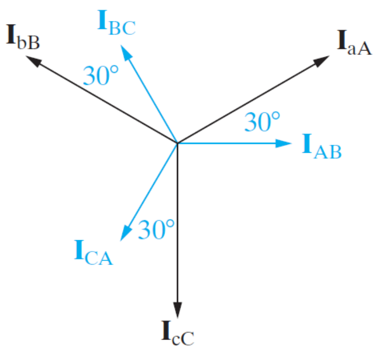
Similarly, it can be shown that, for ***a negative sequence*** the magnitude of the line currents is $\sqrt{3}$ times the magnitude of the phase currents, and that the set of line currents leads the set of phase currents by 30^o;

$$\mathbf{I_L} = \sqrt{3}I_{\emptyset}\ \angle\theta_{\emptyset}+30^o.$$

- The phasor diagrams for the currents of a Δ-load are shown for positive sequence in figure (a), and for a negative sequence in figure (b) below.



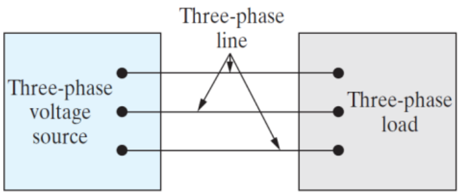
(a)



(b)

Example # 2:

The Y-connected source in Example # 1 feeds a Δ-connected load through a distribution line having an impedance of $0.3 + j0.9\ \Omega/\phi$. The load impedance is $118.5 + j85.8\ \Omega/\phi$. Use the a-phase internal voltage of the generator as the reference.



a) Construct a single-phase equivalent circuit of the three-phase system.
b) Calculate the line currents $\mathbf{I}_{aA}$, $\mathbf{I}_{bB}$, and $\mathbf{I}_{cC}$.
c) Calculate the phase voltages at the load terminals.
d) Calculate the phase currents of the load.
e) Calculate the line voltages at the source terminals.

Solution:

a) The load impedance of the Y equivalent is

$$Z_Y = \frac{Z_\Delta}{3}$$

$$Z_Y = \frac{118.5 + j85.8}{3} = 39.5 + j28.6\ \Omega/\phi$$

The single-phase equivalent circuit is shown in the figure.

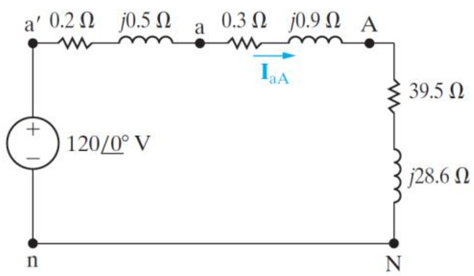


Figure 11.14 ▲ The single-phase equivalent circuit for Example 11.2.

b) The a-phase line current is

$$\mathbf{I}_{aA} = \frac{120\,\angle 0^\circ}{(0.2 + 0.3 + 39.5) + j(0.5 + 0.9 + 28.6)}$$

$$= \frac{120\,\angle 0^\circ}{40 + j30} = 2.4\,\angle -36.87^\circ \text{ A}.$$

Hence

$$\mathbf{I}_{bB} = 2.4\,\angle -156.87^\circ \text{ A},$$

$$\mathbf{I}_{cC} = 2.4\,\angle 83.13^\circ \text{ A}.$$

c) Because the load is Δ connected, the phase voltages are the same as the line voltages. To calculate the line voltages, we first calculate $\mathbf{V}_{AN}$:

$$\mathbf{V}_{AN} = (39.5 + j28.6)(2.4 \angle -36.87^\circ)$$

$$= 117.04 \angle -0.96^\circ \text{ V}.$$

Because the phase sequence is positive, the line voltage $\mathbf{V}_{AB}$ is

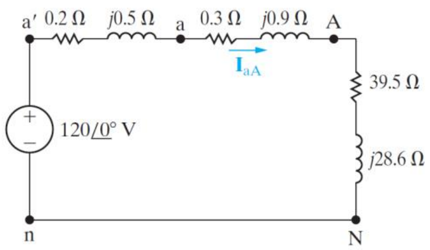
$$\mathbf{V}_{AB} = (\sqrt{3} \angle 30^\circ)\, \mathbf{V}_{AN}$$

$$= 202.72 \angle 29.04^\circ \text{ V}.$$

Therefore

$$\mathbf{V}_{BC} = 202.72 \angle -90.96^\circ \text{ V},$$

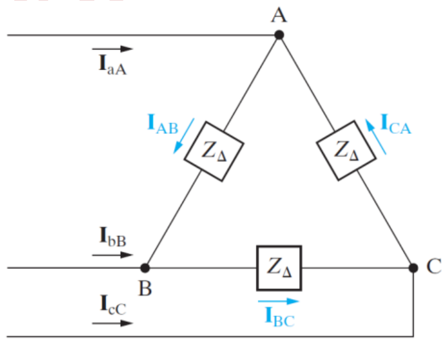
$$\mathbf{V}_{CA} = 202.72 \angle 149.04^\circ \text{ V}.$$



d) The phase currents of the load may be calculated directly from the line currents:

$$\mathbf{I}_{AB} = \left(\frac{1}{\sqrt{3}} \angle 30^\circ\right) \mathbf{I}_{aA}$$

$$= 1.39 \angle -6.87^\circ \text{ A}.$$



Once we know $\mathbf{I}_{AB}$, we also know the other load phase currents:

$$\mathbf{I}_{BC} = 1.39 \angle -126.87^\circ \text{ A},$$

$$\mathbf{I}_{CA} = 1.39 \angle 113.13^\circ \text{ A}.$$

Note that we can check the calculation of $\mathbf{I}_{AB}$ by using the previously calculated $\mathbf{V}_{AB}$ and the impedance of the Δ-connected load; that is,

$$\mathbf{I}_{AB} = \frac{\mathbf{V}_{AB}}{Z_\phi} = \frac{202.72 \angle 29.04^\circ}{118.5 + j85.8}$$

$$= 1.39 \angle -6.87^\circ \text{ A}.$$

e) To calculate the line voltage at the terminals of the source, we first calculate $\mathbf{V}_{an}$. Figure 11.14 shows that $\mathbf{V}_{an}$ is the voltage drop across the line impedance plus the load impedance, so

$$\mathbf{V}_{an} = (39.8 + j29.5)(2.4 \angle -36.87^\circ)$$

$$= 118.90 \angle -0.32^\circ \text{ V}.$$

The line voltage $\mathbf{V}_{ab}$ is

$$\mathbf{V}_{ab} = (\sqrt{3} \angle 30^\circ)\mathbf{V}_{an},$$

or

$$\mathbf{V}_{ab} = 205.94 \angle 29.68^\circ \text{ V}.$$

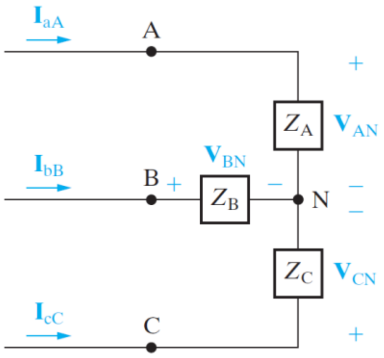
Therefore

$$\mathbf{V}_{bc} = 205.94 \angle -90.32^\circ \text{ V},$$

$$\mathbf{V}_{ca} = 205.94 \angle 149.68^\circ \text{ V}.$$

14.5 Power Calculations in Balanced Three-Phase Circuits

14.5.1 Average Power in a Balanced Wye Load



- For the Y- connected load, shown in the figure next, where *all voltages and currents are given in their rms values*, the average power associated with a-phase is:

$$P_A = |\mathbf{V}_{AN}||\mathbf{I}_{aA}| \cos(\theta_{vA} - \theta_{iA}),$$

where θ_{vA} and θ_{iA} denote the phase angles of $\mathbf{V}_{AN}$ and $\mathbf{I}_{aA}$, respectively.

- The average power associated with b – and c-phases are:

$$P_B = |\mathbf{V}_{BN}||\mathbf{I}_{bB}| \cos(\theta_{vB} - \theta_{iB})$$
$$P_C = |\mathbf{V}_{CN}||\mathbf{I}_{cC}| \cos(\theta_{vC} - \theta_{iC})$$

- In a balanced three-phase system, the *magnitude of each line-to-neutral voltage is the same, as is the magnitude of each phase current*.
- Also, the argument for the cosine function is the same for the three phases;

$$V_\phi = |\mathbf{V}_{AN}| = |\mathbf{V}_{BN}| = |\mathbf{V}_{CN}|,$$
$$I_\phi = |\mathbf{I}_{aA}| = |\mathbf{I}_{bB}| = |\mathbf{I}_{cC}|,$$

and

$$\theta_\phi = \theta_{vA} - \theta_{iA} = \theta_{vB} - \theta_{iB} = \theta_{vC} - \theta_{iC}.$$

- Therefore, the average power of each phase is equal to the other phases' power;

$$P_A = P_B = P_C = P_\phi = V_\phi I_\phi \cos\theta_\phi,$$

- The total average power delivered to the balanced Y-connected load in phase quantities is:

$$P_T = 3P_\phi = 3V_\phi I_\phi \cos\theta_\phi.$$

In terms of the line quantities, V_L & I_L, the total average (real) power is:

$$P_T = 3\left(\frac{V_L}{\sqrt{3}}\right) I_L \cos\theta_\phi$$

$$\boldsymbol{P_T = \sqrt{3} V_L I_L \cos\theta_\emptyset}$$

where $\theta_\emptyset$ is the phase angle between the phase voltage and current; $\theta_\emptyset = \theta_v - \theta_i$

14.5.2 Complex Power in a Balanced Wye Load

- The ***reactive power*** of the load,

  For a single phase is: $Q_\phi = V_\phi I_\phi \sin\theta_\phi,$

  For the three phases is: $Q_T = 3Q_\phi = \sqrt{3}V_L I_L \sin\theta_\phi.$

- For a balanced load, the complex power associated with any phase is:

$$S_\phi = \mathbf{V}_{AN}\mathbf{I}_{aA}^{*} = \mathbf{V}_{BN}\mathbf{I}_{bB}^{*} = \mathbf{V}_{CN}\mathbf{I}_{cC}^{*} = \mathbf{V}_\phi\mathbf{I}_\phi^{*}$$

  where $\mathbf{V}_\phi$ and $\mathbf{I}_\phi$ represent a phase voltage and current

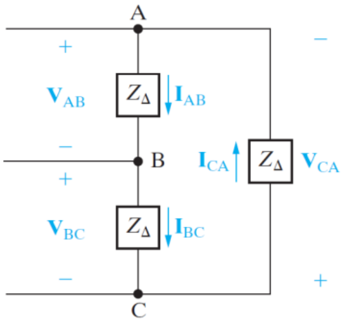
- In general, the ***complex power*** for a Y-connected load

  For a single phase is: $S_\phi = P_\phi + jQ_\phi = \mathbf{V}_\phi\mathbf{I}_\phi^{*},$

  For the three phases is: $S_T = 3S_\phi = \sqrt{3}V_L I_L \angle\theta_\phi^\circ.$

14.5.3 Power Calculations in a Balanced Delta Load

➢ For the Δ- connected load, shown in the figure next, where *all voltages and currents are given in their rms values*, the average power associated with each phase are:



$$P_A = |\mathbf{V}_{AB}||\mathbf{I}_{AB}|\cos(\theta_{vAB} - \theta_{iAB}),$$

$$P_B = |\mathbf{V}_{BC}||\mathbf{I}_{BC}|\cos(\theta_{vBC} - \theta_{iBC}),$$

$$P_C = |\mathbf{V}_{CA}||\mathbf{I}_{CA}|\cos(\theta_{vCA} - \theta_{iCA}).$$

➢ In a balanced three-phase system, the *magnitude of each phase voltage is the same, as is the magnitude of each phase current*. Also, the argument for the cosine function is the same for the three phases;

$$|\mathbf{V}_{AB}| = |\mathbf{V}_{BC}| = |\mathbf{V}_{CA}| = V_\phi,$$

$$|\mathbf{I}_{AB}| = |\mathbf{I}_{BC}| = |\mathbf{I}_{CA}| = I_\phi,$$

$$\theta_{vAB} - \theta_{iAB} = \theta_{vBC} - \theta_{iBC} = \theta_{vCA} - \theta_{iCA} = \theta_\phi,$$

and

$$P_A = P_B = P_C = P_\phi = V_\phi I_\phi \cos\theta_\phi.$$

Thus, in a balanced load, regardless of whether it is Y- or Δ-connected, the phase average power is the same.

The total power delivered to a balanced Δ-connected load is

$$P_T = 3P_\phi = 3V_\phi I_\phi \cos\theta_\phi$$

$$= 3V_L\left(\frac{I_L}{\sqrt{3}}\right)\cos\theta_\phi$$

$$= \sqrt{3}V_L I_L \cos\theta_\phi.$$

$\boldsymbol{P_T = \sqrt{3}V_L I_L \cos\theta_\emptyset}$; which is the same as of Y-load

where $\theta_\emptyset$ is the phase angle between the phase voltage and current; $\theta_\emptyset = \theta_v - \theta_i$

The expressions for ***reactive power*** and ***complex power*** also have the same form as those developed for the Y load:

$$Q_\phi = V_\phi I_\phi \sin\theta_\phi;$$

$$Q_T = 3Q_\phi = 3V_\phi I_\phi \sin\theta_\phi;$$

$$S_\phi = P_\phi + jQ_\phi = \mathbf{V}_\phi \mathbf{I}_\phi^*;$$

$$S_T = 3S_\phi = \sqrt{3}V_L I_L \angle\theta_\phi.$$

where $\mathbf{V}_\emptyset$ and $\mathbf{I}_\emptyset$ represent the phase voltage and current, respectively.

14.5.4 Instantaneous Power in Three-Phase Circuits

The instantaneous power of each phase of a Y-connected load is:

$$p_A = v_{AN}i_{aA} = V_m I_m \cos \omega t \cos(\omega t - \theta_\phi),$$

$$p_B = v_{BN}i_{bB} = V_m I_m \cos(\omega t - 120^\circ)\cos(\omega t - \theta_\phi - 120^\circ),$$

$$p_C = v_{CN}i_{cC} = V_m I_m \cos(\omega t + 120^\circ)\cos(\omega t - \theta_\phi + 120^\circ),$$

where V_m and I_m represent the maximum amplitude of the phase voltage and line current, respectively.

The total instantaneous power is the sum of the instantaneous phase powers, which reduces to:

$$p_T = p_A + p_B + p_C = 1.5 V_m I_m \cos \theta_\phi$$

But, $V_m = \sqrt{2}V_\emptyset$ and $I_m = \sqrt{2}I_\emptyset$, therefore the instantaneous power becomes:

$$p_T = 1.5(\sqrt{2}V_\emptyset)(\sqrt{2}I_\emptyset)\cos\theta_\emptyset$$

$$\boldsymbol{p_T = 3V_\emptyset I_\emptyset \cos\theta_\emptyset}$$

which is the same as the average three-phase power; ***the instantaneous power is constant!***

Note: *In a balanced three-phase circuit,* ***the instantaneous power is time invariant****! Thus, the torque developed at the shaft of a three-phase motor is constant, which in turn* ***means less vibration in machinery*** *powered by three-phase motors.*

Example # 3:

a) Calculate the average power per phase delivered to the Y-connected load of Example # 1.

b) Calculate the total average power delivered to the load.

c) Calculate the total average power lost in the line.

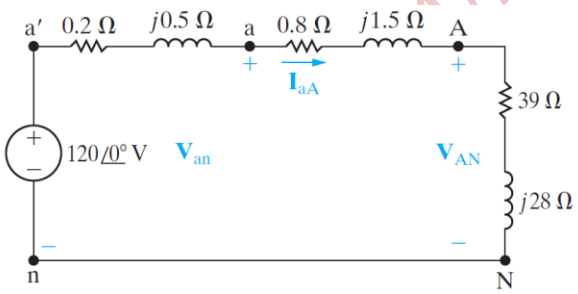
d) Calculate the total average power lost in the generator.

e) Calculate the total number of magnetizing VARs absorbed by the load.

f) Calculate the total complex power delivered by the source.

Solution:

a) From example # 1, the per phase equivalent circuit is shown in the figure next. It was calculated that,



$$\mathbf{I_{aA}} = 2.4 \angle -36.87^\circ \text{ A.}$$

$$\mathbf{V_{AN}} = 115.22 \angle -1.19^\circ \text{ V.}$$

Therefore, the mag. $V_\phi = 115.22$ V, $I_\phi = 2.4$ A, and $\theta_\phi = -1.19 - (-36.87) = 35.68^\circ$. Therefore

$$P_\phi = (115.22)(2.4)\cos 35.68^\circ$$
$$= 224.64 \text{ W.}$$

The power per phase may also be calculated from $I_\phi^2 R_\phi$, or

$$P_\phi = (2.4)^2(39) = 224.64 \text{ W.}$$

b) The total power delivered to the load is either

$$P_T = 3P_\emptyset = 3(224.64) = 673.92W$$

or $V_L = \sqrt{3}V_\emptyset = \sqrt{3}(115.22) = 199.58\text{V}$

$$I_L = I_\emptyset = 2.4A$$

and $P_T = \sqrt{3}V_L I_L \cos\theta_\emptyset$

$$P_T = \sqrt{3}(199.58)(2.4)\cos 35.68 = 673.92W$$

c) The total power lost in the lines is:

$$P_{\text{line}} = 3(2.4)^2(0.8) = 13.824 \text{ W.}$$

d) The total internal power lost in the generator is:

$$P_{gen} = 3(2.4)^2(0.2) = 3.456 \text{ W}$$

e) The total number of magnetizing VARs absorbed by the load is:

$$Q_T = \sqrt{3}(199.58)(2.4)\sin 35.68^\circ$$
$$= 483.84 \text{ VAR.}$$

f) The total complex power associated with the source is:

$$S_T = 3S_\phi = -3(120)(2.4)\angle 36.87^\circ$$
$$= -691.20 - j518.40 \text{ VA.}$$

The minus sign indicates that the internal power and magnetizing reactive power are being delivered to the circuit.

To *check the balance of powers:*

$$P = P_L + P_{Line} + P_{genLoss} \quad \text{and} \quad Q = Q_L + Q_{Line} + Q_{genLoss}$$

$$P = 673.92 + 13.824 + 3.456$$
$$= 691.20 \text{ W (check),}$$
$$Q = 483.84 + 3(2.4)^2(1.5) + 3(2.4)^2(0.5)$$
$$= 483.84 + 25.92 + 8.64$$
$$= 518.40 \text{ VAR(check).}$$

Example # 4:

a) Calculate the total complex power delivered to the Δ-connected load of Example # 2.

b) What percentage of the average power at the sending end of the line is delivered to the load?

Solution:

a) Using the a-phase values from the solution of Example # 2, parts c) and d),

$$\mathbf{V}_\phi = \mathbf{V}_{AB} = 202.72\angle 29.04^\circ \text{ V,}$$
$$\mathbf{I}_\phi = \mathbf{I}_{AB} = 1.39\angle -6.87^\circ \text{ A.}$$

The total complex power is:

$$S_T = 3(202.72\angle 29.04^\circ)(1.39\angle 6.87^\circ)$$
$$= 682.56 + j494.21 \text{ VA.}$$

b) The total power at the sending end of the distribution line equals the total power delivered to the load plus the total power lost in the line;

$$P_{\text{input}} = 682.56 + 3(2.4)^2(0.3)$$

$$= 687.74 \text{ W.}$$

$$Precentage\ of\ power\ reaching\ the\ load = \frac{P_L}{P_{input}} X100\%$$

$$= \frac{682.56}{687.74} X100\% = 99.25\%$$

Example # 5:

A balanced three-phase load requires 480 kW at a lagging power factor of 0.8. The load is fed from a line having an impedance of $0.005 + j0.025\Omega/\emptyset$. The line voltage at the terminals of the load is 600V.

b) Construct a single-phase equivalent circuit of the system.

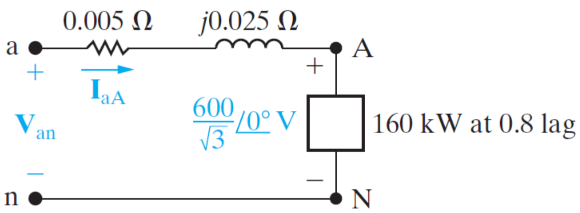
c) Calculate the magnitude of the line current.

d) Calculate the magnitude of the line voltage at the sending end of the line.

e) Calculate the power factor at the sending end of the line.

Solution:

a) The single-phase equivalent circuit is shown in the figure next. Take the phase to neutral voltage as the reference;



$$\mathbf{V_{AN}} = V_{\emptyset} \angle 0$$

$$\mathbf{V_{AN}} = \frac{V_L}{\sqrt{3}} \angle 0 = 346.41 \angle 0V$$

b) The line current can be found from:

$$P_T = \sqrt{3} V_L I_L \cos\theta_p$$

$$= \sqrt{3}(600) I_L (0.8)$$

$$= 480{,}000 \text{ W;}$$

$$I_L = \frac{480{,}000}{\sqrt{3}(600)(0.8)}$$

$$= \frac{1000}{\sqrt{3}}$$

$$= 577.35 \text{ A.}$$

It has the same angle as $\angle -(\cos^{-1} 0.8)$; which is $\angle -36.87$, because of lagging pf.

Alternative solution to find the line current:

The reactive power per phase is:

$$Q_{\emptyset} = P_{\emptyset} \tan(\cos^{-1} 0.8)$$

$$Q_{\emptyset} = 160k \tan(36.87)$$

$$Q_{\emptyset} = 160k(0.75) = 120kVARs$$

Thus, the load per-phase complex power is:

$$\mathbf{S}_{\emptyset} = P_{\emptyset} + jQ_{\emptyset} = 160{,}000 + j120{,}000\ VA = \mathbf{V}_{\emptyset}\mathbf{I}_{\emptyset}^{*}$$

$$160{,}000 + j120{,}000 = \left(\frac{\mathrm{V_L}}{\sqrt{3}}\right)\mathbf{I_{aA}}^{*}$$

$$\mathbf{I_{aA}}^{*} = \frac{160{,}000 + j120{,}000}{\frac{\mathrm{V_L}}{\sqrt{3}}}$$

$$\mathbf{I_{aA}}^{*} = \frac{160{,}000 + j120{,}000}{346.41}$$

$$\mathbf{I_{aA}}^{*} = 577.35\ \llcorner 36.87\ A$$

Therefore, the phase and the line currents are the same;

$$\mathbf{I_{aA}} = 577.35\ \llcorner -36.87\ A$$

c) The magnitude of the line voltage at the sending end is:

$$\mathbf{V}_{\mathrm{an}} = \mathbf{V}_{\mathrm{AN}} + Z_{\ell}\mathbf{I}_{\mathrm{aA}}$$

$$= \frac{600}{\sqrt{3}} + (0.005 + j0.025)(577.35\ \angle -36.87^{\circ})$$

$$= 357.51\ \angle 1.57^{\circ}\ \mathrm{V}.$$

Thus,

$$V_{\mathrm{L}} = \sqrt{3}|\mathbf{V}_{\mathrm{an}}|$$

$$= 619.23\ \mathrm{V}.$$

d) The power factor at the sending end is cosine the angle between $\mathbf{V_{An}}$ and $\mathbf{I_{aA}}$,

$$\mathrm{pf} = \cos[1.57^{\circ} - (-36.87^{\circ})]$$

$$= \cos 38.44^{\circ}$$

$$= 0.783 \text{ lagging}.$$

An alternative way to calculate power factor at the sending end;

$$S_\phi = (160 + j120)10^3 + (577.35)^2(0.005 + j0.025)$$

$$= 161.67 + j128.33 \text{ kVA}$$

$$= 206.41 \angle 38.44^\circ \text{ kVA}.$$

The power factor is

$$\text{pf} = \cos 38.44^\circ$$

$$= 0.783 \text{ lagging}.$$

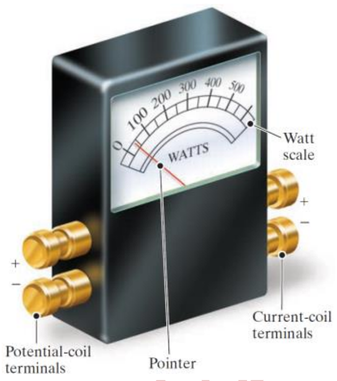
Note that, if the total complex power at the sending end was calculated, after first calculating the magnitude of the line current, it may be used to calculate the magnitude of line voltage;

$$\sqrt{3} V_L I_L = 3(206.41) \times 10^3,$$

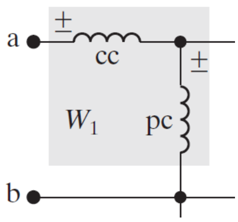
$$V_L = \frac{3(206.41) \times 10^3}{\sqrt{3}(577.35)},$$

$$= 619.23 \text{ V}.$$

14.6 Measuring Average Power in Three-Phase Circuits

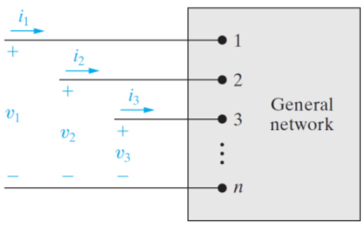


- An ***Electrodynamometer Wattmeter*** is an instrument used to measure the real (average) power. It has two coils; a current coil (low impedance) and a potential coil (high impedance).
- The average (upscale) deflection of the pointer attached to the movable coil is proportional to the **product of the effective** (rms) value of the ***current*** in the current coil, the effective (rms) value of the **voltage** impressed on the potential coil, **and the cosine of the phase angle** between the voltage and current.
- The direction in which the pointer deflects depends on the instantaneous polarity of the current-coil current and the potential-coil voltage.



The Two-Wattmeter Method:

➢ To measure the total power at the terminals of the box of n conductors, $(n-1)$ currents and voltages need to be known. This follows because one terminal is chosen as a reference, there are only $(n-1)$ independent voltages/currents.



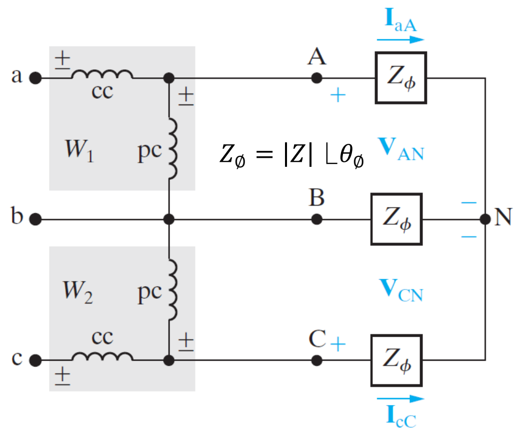
Thus, the total average power is:

$$p = v_1 i_1 + v_2 i_2 + \cdots + v_{n-1} i_{n-1}$$

➢ For a three-conductor circuit, whether balanced or not, only two wattmeters are needed to measure the total average power.

➢ For a balanced three-phase circuit, only three conductors are needed (the neutral in a Y-connection has a zero current, and can be removed), therefore two wattmeters are needed.

The reading of **Wattmeter # 1** is:



$$W_1 = |\mathbf{V}_{AB}||\mathbf{I}_{aA}| \cos\theta_1$$

$$= V_L I_L \cos\theta_1.$$

where θ_1 is the angle between $\mathbf{V_{AB}}$ & $\mathbf{I}_{aA}$.

The reading of **Wattmeter # 2** is:

$$W_2 = |\mathbf{V}_{CB}||\mathbf{I}_{cC}| \cos\theta_2$$

$$= V_L I_L \cos\theta_2.$$

where θ_2 is the angle between $\mathbf{V_{CB}}$ & $\mathbf{I}_{cC}$.

Note that, the load impedance is:

$$Z_\emptyset = |Z| \llcorner \theta_\emptyset$$

where $-90^o < \theta_\emptyset < 90^o$.

θ_1 & θ_2 can be found in terms of the load impedance angle, $\theta_\emptyset$, and for positive sequence;

The current for W1: $\mathbf{I}_{aA} = \frac{|V_{AN}|}{|Z| \llcorner \theta_\emptyset} = \frac{|V_{AN}|}{|Z|} \llcorner -\theta_\emptyset$

The voltage for W1: $\mathbf{V_{AB}} = \sqrt{3}|V_{AN}| \llcorner +30$

$$\theta_1 = \theta_{V_{AB}} - \theta_{I_{aA}}$$

$$\theta_1 = 30 - (-\theta_\emptyset)$$

$$\boldsymbol{\theta_1 = \theta_\emptyset + 30}$$

Similarly,

The current for W2: $\mathbf{I}_{cC} = \frac{|V_{CN}| \llcorner 120}{|Z| \llcorner \theta_\emptyset} = \frac{|V_{CN}|}{|Z|} \llcorner 120 - \theta_\emptyset$

The voltage for W2: $\mathbf{V_{CB}} = \sqrt{3}|V_{AN}| \llcorner +90$; because $\mathbf{V_{BC}} = \sqrt{3}|V_{AN}| \llcorner -90$!

$$\theta_2 = \theta_{V_{CB}} - \theta_{I_{cC}}$$

$$\theta_2 = 90 - (120 - \theta_\emptyset)$$

$$\boldsymbol{\theta_2 = \theta_\emptyset - 30}$$

Thus, the Wattmeters' readings are:

$$W_1 = V_L I_L \cos(\theta_\emptyset + 30)$$

$$W_2 = V_L I_L \cos(\theta_\emptyset - 30)$$

The total power is:

$$P_T = W_1 + W_2$$

$$P_T = V_L I_L \cos(\theta_\emptyset + 30) + V_L I_L \cos(\theta_\emptyset - 30)$$

Using the trigonometric identity,

$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

Therefore,

$$P_T = V_L I_L \left(\cos(\theta_\emptyset + 30) + \cos(\theta_\emptyset - 30)\right)$$

Let $A = \theta_\emptyset + 30$ and $B = \theta_\emptyset - 30$

$$P_T = V_L I_L \left(2\cos\left(\frac{(\theta_\emptyset + 30) + (\theta_\emptyset - 30)}{2}\right)\cos\left(\frac{(\theta_\emptyset + 30) - (\theta_\emptyset - 30)}{2}\right)\right)$$

$$P_T = 2V_L I_L \left(\cos\left(\frac{2\theta_\emptyset}{2}\right)\cos\left(\frac{60}{2}\right)\right)$$

$$P_T = 2V_L I_L \left(\cos(\theta_\emptyset)\cos(30)\right)$$

$$P_T = 2V_L I_L \left(\cos(\theta_\emptyset)\frac{\sqrt{3}}{2}\right)$$

$P_T = \sqrt{3} V_L I_L \cos(\theta_\emptyset)$, which the same as the three-phase power!

Notes,

Recall, $W_1 = V_L I_L \cos(\theta_\emptyset + 30)$

$W_2 = V_L I_L \cos(\theta_\emptyset - 30)$

1. If the load power factor is greater than 0.5 ($\theta_\emptyset < |60^o|$), both wattmeters read positive.

2. If the load power factor equals 0.5 ($\theta_\emptyset = |60^o|$), one wattmeter ($W_1$) reads zero.

if ($\theta_\emptyset = 60^o$), W_1 reads zero.

if ($\theta_\emptyset = -60^o$), W_2 reads zero.

3. If the load power factor is less than 0.5 ($\theta_\emptyset > |60^o|$), one wattmeter reads negative;

if ($\theta_\emptyset > 60^o$), W_1 reads negative.

if ($\theta_\emptyset < -60^o$), W_2 reads negative.

4. Reversing the phase sequence will interchange the readings on the two wattmeters.

Example # 6:

Calculate the reading of each wattmeter in the circuit in figure if the phase voltage at the load is 120 V and

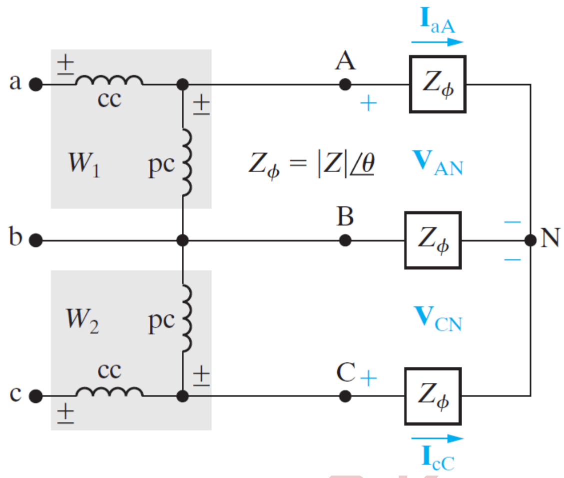
(a) $Z_\phi = 8 + j6\ \Omega$;

(b) $Z_\phi = 8 - j6\ \Omega$;

(c) $Z_\phi = 5 + j5\sqrt{3}\ \Omega$; and

(d) $Z_\phi = 10 \angle -75\ \Omega$;

(e) Verify for (a)–(d) that the sum of the wattmeter readings equals the total power delivered to the load.



Solution:

a) $Z_\phi = 10\,\angle 36.87^\circ\ \Omega$, $V_L = 120\sqrt{3}$ V, and $I_L = 120/10 = 12$ A.

$$W_1 = (120\sqrt{3})(12)\cos(36.87^\circ + 30^\circ)$$

$$= 979.75 \text{ W},$$

$$W_2 = (120\sqrt{3})(12)\cos(36.87^\circ - 30^\circ)$$

$$= 2476.25 \text{ W}.$$

b) $Z_\phi = 10\,\angle -36.87^\circ\ \Omega$, $V_L = 120\sqrt{3}$ V, and $I_L = 120/10 = 12$ A.

$$W_1 = (120\sqrt{3})(12)\cos(-36.87^\circ + 30^\circ)$$

$$= 2476.25 \text{ W},$$

$$W_2 = (120\sqrt{3})(12)\cos(-36.87^\circ - 30^\circ)$$

$$= 979.75 \text{ W}.$$

c) $Z_\phi = 5(1 + j\sqrt{3}) = 10\,\angle 60^\circ\ \Omega$, $V_L = 120\sqrt{3}$ V, and $I_L = 12$ A.

$$W_1 = (120\sqrt{3})(12)\cos(60^\circ + 30^\circ) = 0,$$

$$W_2 = (120\sqrt{3})(12)\cos(60^\circ - 30^\circ)$$

$$= 2160 \text{ W}.$$

d) $Z_\phi = 10\underline{/-75^\circ}\ \Omega$, $V_L = 120\sqrt{3}$ V, and $I_L = 12$ A.

$$W_1 = (120\sqrt{3})(12)\cos(-75^\circ + 30^\circ) = 1763.63 \text{ W},$$

$$W_2 = (120\sqrt{3})(12)\cos(-75^\circ - 30^\circ) = -645.53 \text{ W}.$$

e) $P_T(\text{a}) = 3(12)^2(8) = 3456$ W,

$$W_1 + W_2 = 979.75 + 2476.25 = 3456 \text{ W},$$

$$P_T(\text{b}) = P_T(\text{a}) = 3456 \text{ W},$$

$$W_1 + W_2 = 2476.25 + 979.75 = 3456 \text{ W},$$

$$P_T(\text{c}) = 3(12)^2(5) = 2160 \text{ W},$$

$$W_1 + W_2 = 0 + 2160 = 2160 \text{ W},$$

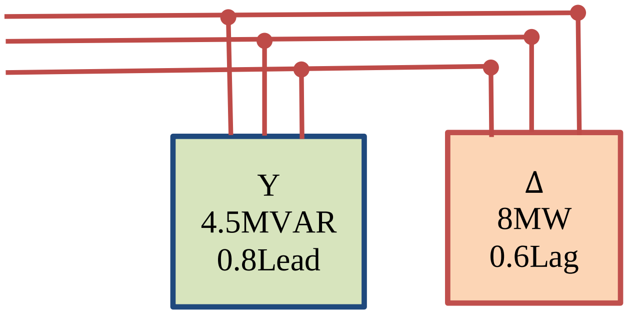
$$P_T(\text{d}) = 3(12)^2(2.5882) = 1118.10 \text{ W},$$

$$W_1 + W_2 = 1763.63 - 645.53 = 1118.10 \text{ W}.$$

14.7 Combined Loads and Power Factor Improvement

Example # 7:

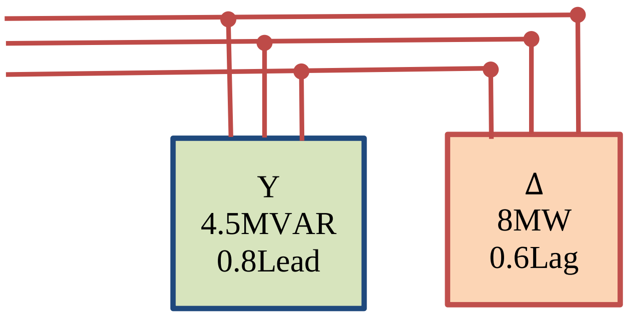
An 11kV, 50 Hz, three phase power line feeds two isolated factories, which are effectively connected in parallel with each other. The first factory load is Y-connected and is rated at 4.5MVARs at a power factor of 0.8 leading. The second factory is Δ-connected, and it is rated at 8MW at a power factor of 0.6 lagging. Then,



a) Calculate the supply line current and the combined power factor of the two rated loads
b) Determine the phase voltage and current of the two factory loads at rated conditions
c) What is the value of the capacitor bank to be connected across the two rated loads as WYE and to improve the power factor to 0.95 lagging?

Solution:

$\boldsymbol{V_L} = 11kV \angle 0V$



<u>For Y-connected load:</u>

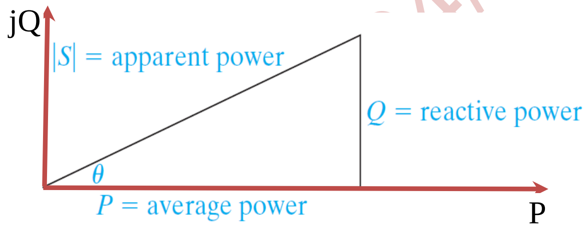
The power can be found from the power triangle:

$$P_Y = \frac{Q_Y}{\tan(\theta_Y)}$$

But, $\theta_Y = \cos^{-1} 0.8 = 36.9$; since the power factor is leading, the load is capacitive;

$$\theta_Y = -36.9$$

And $\quad Q_Y = \boldsymbol{-4.5MVARs}$



Thus, $\quad P_Y = \frac{-4.5M}{\tan(-36.9)} = 6MW$

$$P_Y = \frac{4.5}{3/4} = \boldsymbol{6MW}$$

<u>For Δ-connected load:</u>

$$P_\Delta = \boldsymbol{8MW}$$

And $\quad Q_\Delta = P_\Delta \tan \theta_\Delta$

But $\theta_\Delta = \cos^{-1} 0.6 = 53.13$, since the power factor is lagging, the load is inductive, and

$$\theta_\Delta = +53.13$$

Thus, $\quad Q_\Delta = 8M \tan 53.13$

$$Q_\Delta = 8M \tan 53.13 = \boldsymbol{10.67MVARs}$$

Therefore, the total load power is:

$$P_{in} = P_Y + P_\Delta$$

$$P_{in} = 6M + 8M = \boldsymbol{14MW}$$

The total reactive power is:

$$Q_{in} = Q_Y + Q_\Delta$$

$$Q_{in} = (-)4.5M + 10.67M = \boldsymbol{6.17MVARs}$$

The input (combined) power factor is:

$$\cos \theta_{in} = \cos\left(\tan^{-1} \frac{Q_{in}}{P_{in}}\right)$$

$$\cos \theta_{in} = \cos\left(\tan^{-1} \frac{6.17}{14}\right)$$

$$\cos \theta_{in} = \cos(23.78) = 0.915$$

⇨ $\text{pf}_{\text{in}} = \cos \theta_{in} = \boldsymbol{0.915}$ **(lagging)**

Since, $P_{in} = \sqrt{3}|\mathrm{V}_L||\mathrm{I}_L| \cos\theta_{in}$, then the line current is:

$$|\mathrm{I}_L| = \frac{P_{in}}{\sqrt{3}|\mathrm{V}_L| \cos\theta_{in}}$$

$$|\mathrm{I}_L| = \frac{14M}{\sqrt{3}|11\mathrm{k}|0.915} = 801.1A$$

$$\mathbf{I}_L = \mathbf{801.1} \llcorner -\mathbf{23.8} - \mathbf{30A}$$

Notes,

The phase current lags the phase voltage of the combined load by $\theta_{in} = 23.78^o$.

The line current lags the line voltage by 30^o, regardless of the load connection!

For the Y-connected load:

The phase voltage is:

$$\mathbf{V}_{\mathbf{\emptyset Y}} = \frac{\mathbf{V_L}}{\sqrt{3}} \llcorner -30^o$$

$$\mathbf{V}_{\mathbf{\emptyset Y}} = \frac{11{,}000}{\sqrt{3}} \llcorner 0 - 30^o$$

$$\mathbf{V}_{\mathbf{\emptyset Y}} = 6.35 \llcorner -30^o \text{ kV}$$

The phase current is:

$$|\mathrm{I}_{\emptyset \mathrm{Y}}| = \frac{P_Y}{3|\mathrm{V}_{\emptyset \mathrm{Y}}| \cos\theta_Y}$$

$$|\mathrm{I}_{\emptyset \mathrm{Y}}| = \frac{6M}{3(6.35k)0.8} = 393.7$$

$$\mathbf{I}_{\mathbf{\emptyset Y}} = 393.7 \llcorner -30 + 36.9$$

Note that, the phase current leads the phase voltage by $\cos^{-1}(0.8)$

$$\mathbf{I}_{\mathbf{\emptyset Y}} = 393.7 \llcorner 6.9 \text{ A}$$

For the Δ-connected load:

The phase voltage is:

$$\mathbf{V}_{\mathbf{\emptyset \Delta}} = \mathbf{V_L}$$

$$\mathbf{V}_{\mathbf{\emptyset \Delta}} = 11{,}000 \llcorner 0^o \text{ V}$$

$$\mathbf{V}_{\mathbf{\emptyset \Delta}} = 11 \llcorner 0^o \text{ kV}$$

The phase current is:

$$|\mathrm{I}_{\emptyset \Delta}| = \frac{P_\Delta}{3|\mathrm{V}_{\emptyset \Delta}| \cos\theta_\Delta}$$

$$|\mathrm{I}_{\emptyset \Delta}| = \frac{8M}{3(11k)0.6} = 404.04$$

$$\mathbf{I}_{\mathbf{\emptyset \Delta}} = 404.04 \llcorner -0 - 53.13 \text{ A}$$

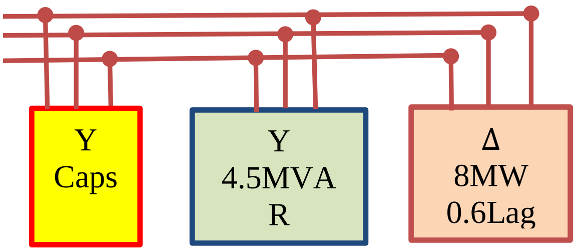
Note that, the phase current lags the phase voltage by $\cos^{-1}(0.6)$

$$\mathbf{I}_{\mathbf{\emptyset \Delta}} = 404.04 \llcorner -53.13 \text{ A}$$

- The original complex load power is:

$$\mathbf{S_{old}} = P_{old} + jQ_{old}$$

$$\mathbf{S_{old}} = 14 + j6.17 \text{ MVA}$$



- The new complex power that results from adding a capacitor in parallel with the load is:

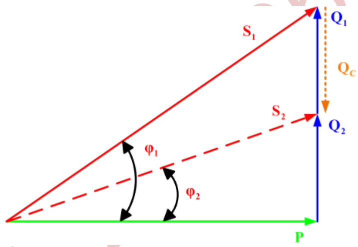
$$\mathbf{S_{new}} = P_{old} + jQ_{new}$$

$$Q_{new} = P_{old} \tan(\cos^{-1} \text{pf}_{new})$$

$$Q_{new} = 14M \tan(\cos^{-1} 0.95)$$

$$Q_{new} = 4.6MVARs$$

$$\mathbf{S_{new}} = 14 + j4.6 \text{ MVA}$$



- Since $\mathbf{S_{new}} = \mathbf{S_{old}} + \mathbf{S_{cap}}$, the added capacitor complex power is:

$$\mathbf{S_{cap}} = \mathbf{S_{new}} - \mathbf{S_{old}}$$

- But, the capacitor is purely reactive;

$$\mathbf{S_{cap}} = 14 + j4.6 - (14 + j6.17)\text{MVA} = -j1.57 \text{ MVA (for 3 phase)}$$

and,

$$\mathbf{S_{cap}} = -3j\omega C_Y\, V_{\emptyset rms}^2$$

$$-\text{j}1.57\text{M} = -3j\omega C\, V_{\emptyset rms}^2$$

$$C_Y = \frac{1{,}570{,}000}{3(2\pi(50))(6{,}350)^2}$$

$$C_Y = 41.31\mu F$$

Note that, if the capacitors are to be connected in Δ configuration, the required capacitors value is:

$$\mathbf{S_{cap}} = -3j\omega C_\Delta\, V_{Lrms}^2$$

$$C_\Delta = \frac{1{,}570{,}000}{3(2\pi(50))(11{,}000)^2}$$

$$C_\Delta = 13.77\mu F \;;$$

$C_\Delta = \frac{C_Y}{3}$; C_Δ is smaller than C_Y but it has to tolerate the line-to-line voltage!

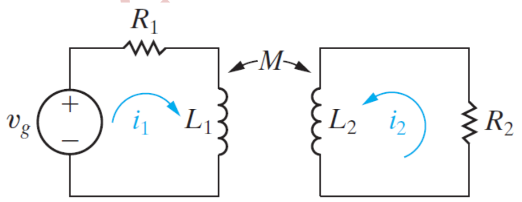
Chapter 15

Mutual Inductance and Transformers

15.1 Mutual Inductance

Recall that, the self inductance is the parameter that relates a voltage to a time-varying current in the same element.

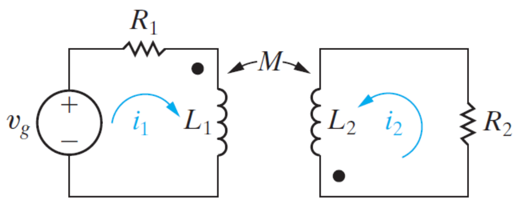
Mutual Inductance ($\boldsymbol{M}$) is a circuit parameter relating the voltage induced in one circuit to the time-varying current in another circuit. The two circuits are linked together by a magnetic field.



- The circuit has *two magnetically coupled coils*.
- The self induced voltage in the first coil is $L_1 \frac{di_1}{dt}$
- The mutually induced voltage in the first coil is $M \frac{di_2}{dt}$

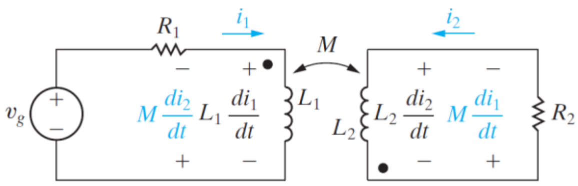
- **Dot Convention:** It is used to determine the mutually induced voltages.

  When the reference direction for *a current enters the dotted terminal* of a coil, the reference polarity of *the voltage* that it induces in the other coil *is positive* at its dotted terminal.



  Alternatively, when the reference direction for *a current leaves the dotted terminal* of a coil, the reference polarity of *the voltage* that it induces in the other coil *is negative* at its dotted terminal.

The voltages induced across each inductor are assigned on the figure next; applying KVL for each loop yields:

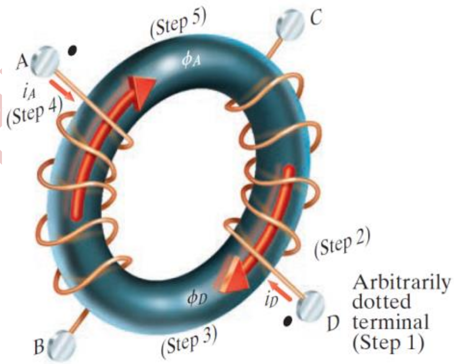


$$-v_g + i_1R_1 + L_1\frac{di_1}{dt} - M\frac{di_2}{dt} = 0,$$

$$i_2R_2 + L_2\frac{di_2}{dt} - M\frac{di_1}{dt} = 0.$$

Note that, $M = k\sqrt{L_1L_2}$, where k is *the coefficient of coupling*, and $0 \le k \le 1$.

Procedure for Determining The Dots Markings

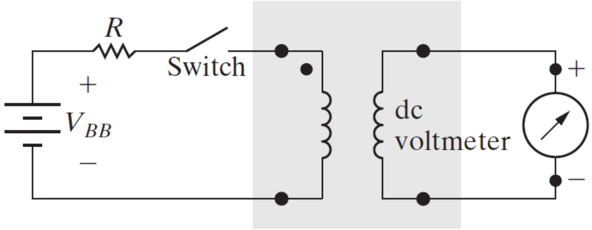


1- Arbitrarily select one terminal of one coil, say D, and mark it with a dot.
2- Assign a current into the dotted terminal and label it, i_D.
3- Use the right-hand rule to determine the direction of the magnetic field established by i_D inside the coupled coils and label this field $\emptyset_D$.
4- Arbitrarily pick one terminal of the second coil, say A, and assign a current into this terminal, showing the current as i_A.
5- Use the right-hand rule to determine the direction of the flux established by i_A inside the coupled coils and label this flux $\emptyset_A$.
6- Compare the directions of the two fluxes $\emptyset_A$ and $\emptyset_D$, if the fluxes have the same reference direction, place a dot on the terminal of the second coil where the test current i_A enters. (In the figure, the fluxes $\emptyset_A$ and $\emptyset_D$ have the same reference direction, and therefore a dot goes on terminal A.)

If the fluxes have different reference directions, place a dot on the terminal of the second coil where the test current i_A leaves.

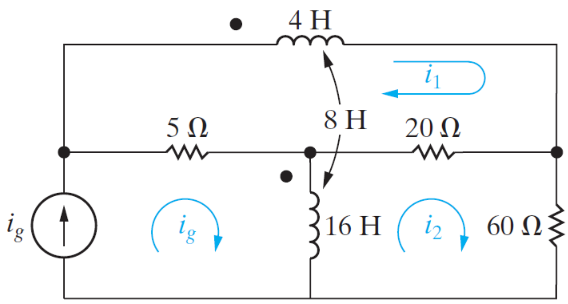
Experimentally,

- The terminal connected to the positive supply terminal is given a dot.
- If the voltmeter deflection is upscale, the coil terminal connected to the positive terminal of the voltmeter is given a dot, and vice versa.



Example #1:

Write the mesh current equations that describe the circuit in tehfigure in terms of i_1 and i_2.



Solution:

Applying KVL for mesh 1:

$$4\frac{di_1}{dt} + 8\frac{d}{dt}(i_g - i_2) + 20(i_1 - i_2) + 5(i_1 - i_g) = 0.$$

Applying KVL for mesh 2:

$$20(i_2 - i_1) + 60i_2 + 16\frac{d}{dt}(i_2 - i_g) - 8\frac{di_1}{dt} = 0.$$

15.2 Transformers

A transformer is a device that is based on magnetic coupling. Transformers are used in both communication and power circuits. *In communication circuits*, the transformer is used *to match impedances and eliminate dc signals* from portions of the system. *In power circuits*, transformers are used to establish ac voltage levels that *facilitate the transmission, distribution, and consumption of electrical power*.

An ideal transformer consists of two magnetically coupled coils having N_1 and N_2 turns, respectively.

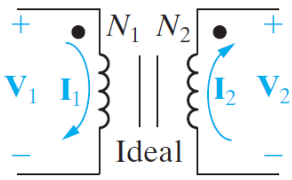
Recalling that, $M = k\sqrt{L_1 L_2}$, where k is *the coefficient of coupling*, and $0 \le k \le 1$, and exhibiting these three properties:

1. The coefficient of coupling is unity; $k = 1$
2. The self-inductance of each coil is infinite; $L_1 = L_2 = \infty$
3. The coil losses, due to parasitic resistance, are negligible; $R_1 = R_2 = 0$.

Determining the Voltage and Current Ratios

The voltage ratio,

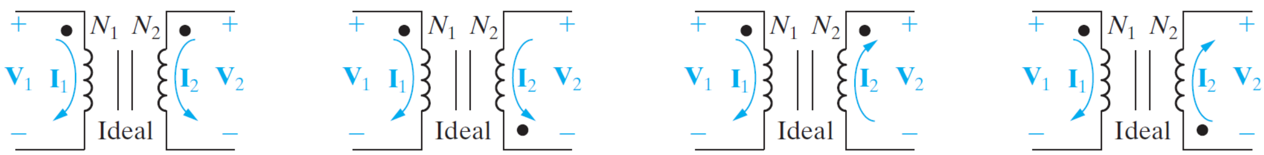
$$\frac{\mathbf{V}_1}{N_1} = \frac{\mathbf{V}_2}{N_2} \qquad (1)$$



The current ratio,

$$\mathbf{I}_1 N_1 = \mathbf{I}_2 N_2. \qquad (2)$$

For the polarity, note the dot convention;



(a)	(b)	(c)	(d)
$\frac{\mathbf{V}_1}{N_1} = \frac{\mathbf{V}_2}{N_2}$, $N_1\mathbf{I}_1 = -N_2\mathbf{I}_2$	$\frac{\mathbf{V}_1}{N_1} = -\frac{\mathbf{V}_2}{N_2}$, $N_1\mathbf{I}_1 = N_2\mathbf{I}_2$	$\frac{\mathbf{V}_1}{N_1} = \frac{\mathbf{V}_2}{N_2}$, $N_1\mathbf{I}_1 = N_2\mathbf{I}_2$	$\frac{\mathbf{V}_1}{N_1} = -\frac{\mathbf{V}_2}{N_2}$, $N_1\mathbf{I}_1 = -N_2\mathbf{I}_2$

If the coil voltages $\mathbf{V}_1$ and $\mathbf{V}_2$ are both positive or negative at the dot-marked terminal, use a plus sign in eq. (1). Otherwise, use a negative sign.

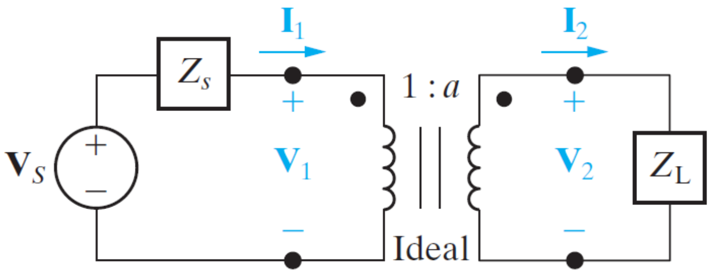
If the coil currents $\mathbf{I}_1$ and $\mathbf{I}_2$ are both directed into or out of the dot-marked terminal, use a minus sign in eq. (2). Otherwise, use a plus sign.

Define the transformer turns' ratio as:

$$a = \frac{N_2}{N_1}$$

The Use of an Ideal Transformer for Impedance Matching

Ideal transformers can also be used to raise or lower the impedance level of a load.



$$\mathbf{V}_1 = \frac{\mathbf{V}_2}{a},$$

and

$$\mathbf{I}_1 = a\mathbf{I}_2.$$

Therefore the impedance seen by the practical source is

$$Z_{IN} = \frac{\mathbf{V}_1}{\mathbf{I}_1} = \frac{1}{a^2}\frac{\mathbf{V}_2}{\mathbf{I}_2},$$

but the ratio $\mathbf{V}_2/\mathbf{I}_2$ is the load impedance Z_L, so

$$Z_{IN} = \frac{1}{a^2} Z_L.$$

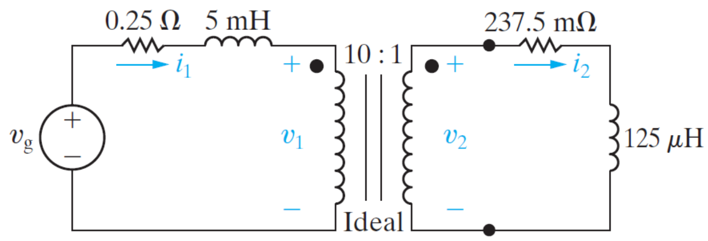
Thus, the ideal transformer's secondary coil reflects the load impedance back to the primary coil, with the scaling factor $1/a^2$.

Note that the ideal transformer changes the magnitude of Z_L but does not affect its phase angle.

This can be used for achieving maximum power transfer to the load.

Example # 2:

The load impedance connected to the secondary winding of the ideal transformer, in the figure shown next, consists of a $237.5 m\Omega$ resistor in series with a $125\mu H$ inductor.



If the sinusoidal voltage source (v_g) is generating the voltage $2500\cos 400t\ V$, find the steady-state expression for: (a) i_1 (b) v_1 (c) i_2 and (d) v_2

Solution:

a) to convert the circuit into the frequency domain;

The voltage source becomes:

$$2500\angle 0^\circ \text{ V}$$

Since $\omega = 400$ rad/s, and the inductor's impedance is:

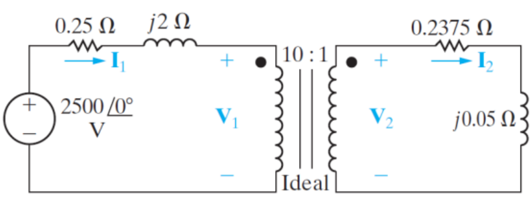
$$Z_L = j\omega L$$

The $5mH$ inductor converts to:

$$Z_L = j400(0.005) = 2\Omega$$

The $125\mu H$ inductor converts to:

$$Z_L = j400(0.000125) = 0.05\Omega$$



Applying KVL for mesh 1:

$$2500\angle 0^\circ = (0.25 + j2)\mathbf{I}_1 + \mathbf{V}_1$$

And

$$\mathbf{V}_1 = 10\mathbf{V}_2 = 10[(0.2375 + j0.05)\mathbf{I}_2].$$

Because

$$\mathbf{I}_2 = 10\mathbf{I}_1$$

Then

$$\mathbf{V}_1 = 10(0.2375 + j0.05)10\mathbf{I}_1$$
$$= (23.75 + j5)\mathbf{I}_1.$$

Therefore,

$$2500\,\underline{/0^\circ} = (24 + j7)\mathbf{I}_1$$

or

$$\mathbf{I}_1 = 100\,\underline{/-16.26^\circ}\text{ A.}$$

Thus the steady-state expression for i_1 is

$$i_1 = 100\cos(400t - 16.26^\circ)\text{ A.}$$

b) $\mathbf{V}_1 = 2500\underline{/0^\circ} - (100\,\underline{/-16.26^\circ})(0.25 + j2)$

$= 2500 - 80 - j185$

$= 2420 - j185 = 2427.06\,\underline{/-4.37^\circ}\text{ V.}$

Hence

$$v_1 = 2427.06\cos(400t - 4.37^\circ)\text{ V.}$$

c) $\mathbf{I}_2 = 10\mathbf{I}_1 = 1000\,\underline{/-16.26^\circ}\text{ A.}$

Therefore

$$i_2 = 1000\cos(400t - 16.26^\circ)\text{ A.}$$

d) $\mathbf{V}_2 = 0.1\mathbf{V}_1 = 242.71\,\underline{/-4.37^\circ}\text{ V,}$

giving

$$v_2 = 242.71\cos(400t - 4.37^\circ)\text{ V.}$$

References

[1] James Nilsson and Susan Riedel, "Electric Circuits", 10th ed., Pearson, 2015.

[2] Robert L. Boylestad and Louis Nashelsky, "Electronic Devices and Circuit Theory", 11th ed. Pearson, 2012.

[3] Roland E. Thomas, Albert J. Rosa, and Gregory J. Toussaint, "The Analysis and Design of Linear Circuits", 7th ed. Wiley and Sons Inc. 2012.