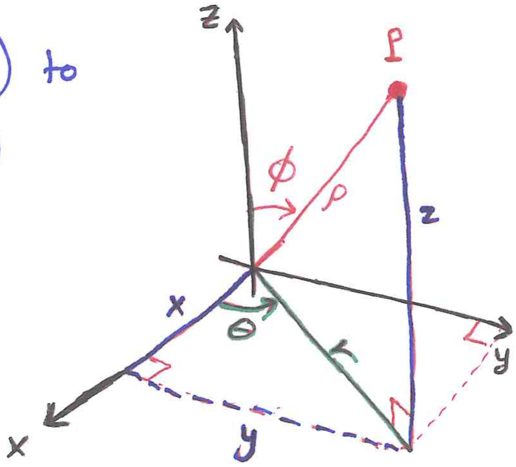
$\boxed{16.5}$ Surfaces and Area $\boxed{16.8}$

- Recall the coordinate conversion formulas in section 15.7 :

$\boxed{1}$ Cylindrical coordinate $P = (r, \theta, z)$ to
Rectangular coordinate $P = (x, y, z)$

$$x = r\cos\theta \ , \quad y = r\sin\theta \ , \quad z = z$$
$$r^2 = x^2 + y^2$$



$\boxed{2}$ Spherical coordinate $P = (\rho, \phi, \theta)$ to
Rectangular coordinate $P = (x, y, z)$

$$x = r\cos\theta = \rho\sin\phi\cos\theta$$
$$y = r\sin\theta = \rho\sin\phi\sin\theta$$
$$z = \rho\cos\phi$$

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{r^2 + z^2}$$

$$0 \le \phi \le \frac{\pi}{2}$$
$$0 \le \theta \le 2\pi$$

$\boxed{3}$ Spherical coordinate $P(\rho, \phi, \theta)$ to
Cylindrical coordinate $P(r, \theta, z)$

$$r = \rho\sin\phi \ , \quad \theta = \theta \ , \quad z = \rho\cos\phi$$

- The curve in the plane can be defined in three different ways:

① Explicit form: $y = f(x)$ ② Implicit form: $F(x, y) = 0$

③ Parametric vector form: $\vec{r}(t) = f(t)\,\vec{i} + g(t)\,\vec{j} \ , \quad a \le t \le b$

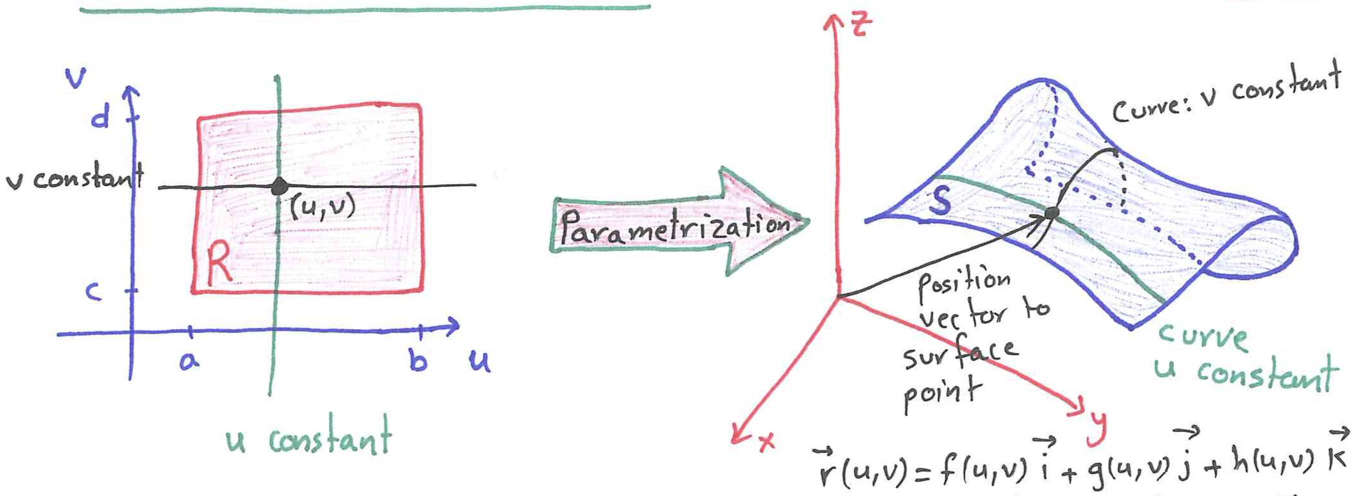
- The surface in space can be defined using:

① Explicit form: $z = f(x, y)$

② Implicit form: $F(x, y, z) = 0$

③ Parametric vector form: we discuss this now.

Parametrization of surfaces



$\vec{r}(u,v) = f(u,v)\,\vec{i} + g(u,v)\,\vec{j} + h(u,v)\,\vec{k}$
$= x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$

- Suppose $\vec{r}(u,v) = f(u,v)\,\vec{i} + g(u,v)\,\vec{j} + h(u,v)\,\vec{k}$ is continuous vector function on region R in the uv-plane.
- Suppose $\vec{r}$ is 1-1 on the interior of R to ensure that the surface S does not cross itself.
- The variables u and v are called the parameters and R is called the parameter domain.
- To simplify things: we consider R to be a rectangle defined by $a \le u \le b$, $c \le v \le d$.
- The vector function $\vec{r}(u,v)$ together with the parameter domain R form a parametrization for the surface S.

- Note that the vector function $\vec{r}(u,v)$ is equivalent to the following three parametric equations:

$$x = f(u,v), \quad y = g(u,v), \quad z = h(u,v)$$

Exp Find parametrization for the

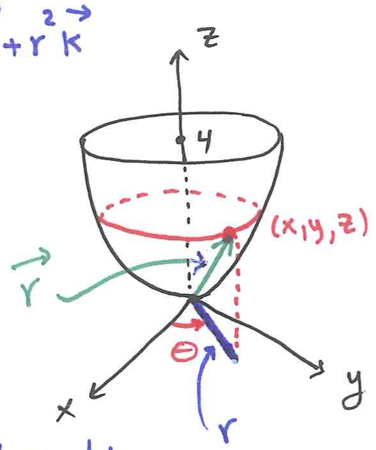
1. The paraboloid $z = x^2 + y^2$, $z \le 4$

Cylindrical coordinates provide a parametrization:

$x = r\cos\theta$ | $\vec{r}(r,\theta) = (r\cos\theta)\vec{i} + (r\sin\theta)\vec{j} + r^2\vec{k}$

$y = r\sin\theta$ | with $0 \le r \le 2$

$z = x^2 + y^2 = r^2$ | $0 \le \theta \le 2\pi$



2. portion of the sphere $x^2 + y^2 + z^2 = 3$ between the planes $z = \frac{\sqrt{3}}{2}$ and $z = -\frac{\sqrt{3}}{2}$

- Spherical coordinates provide a parametrization:

$x = \rho\sin\phi\cos\theta = \sqrt{3}\sin\phi\cos\theta$ | $\rho = \sqrt{x^2+y^2+z^2}$

$y = \rho\sin\phi\sin\theta = \sqrt{3}\sin\phi\sin\theta$ | $= \sqrt{3}$

$z = \rho\cos\phi = \sqrt{3}\cos\phi$

- when $z = \frac{\sqrt{3}}{2} \Rightarrow \frac{\sqrt{3}}{2} = \sqrt{3}\cos\phi \Leftrightarrow \cos\phi = \frac{1}{2} \Leftrightarrow \phi = \frac{\pi}{3}$

  $z = -\frac{\sqrt{3}}{2} \Rightarrow -\frac{\sqrt{3}}{2} = \sqrt{3}\cos\phi \Leftrightarrow \cos\phi = -\frac{1}{2} \Leftrightarrow \phi = \frac{2\pi}{3}$

- Thus, $\vec{r}(\phi,\theta) = x\vec{i} + y\vec{j} + z\vec{k}$

$$= (\sqrt{3}\sin\phi\cos\theta)\vec{i} + (\sqrt{3}\sin\phi\sin\theta)\vec{j} + (\sqrt{3}\cos\phi)\vec{k}$$

with $\frac{\pi}{3} \le \phi \le \frac{2\pi}{3}$ and $0 \le \theta \le 2\pi$

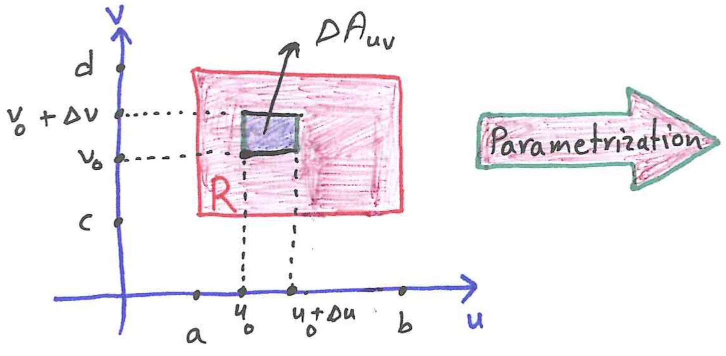
3. the portion of the cone $z = 2\sqrt{x^2+y^2}$ between the planes $z = 2$ and $z = 4$

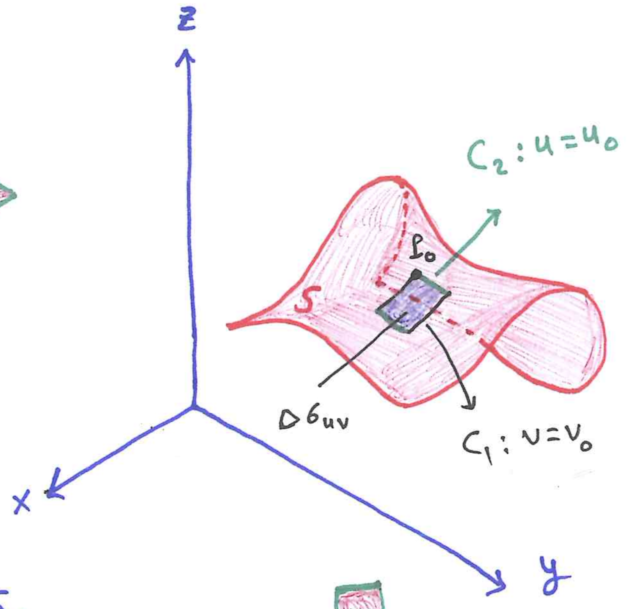
- Using Cylindrical coordinates $x = r\cos\theta$, $y = r\sin\theta$, $z = 2r$
- Thus, $\vec{r}(r,\theta) = (r\cos\theta)\vec{i} + (r\sin\theta)\vec{j} + 2r\vec{k}$ | when $2 \le z \le 4$

  with $0 \le \theta \le 2\pi$ and $1 \le r \le 2$ | we have $1 \le r \le 2$

Surface Area

How to find the surface area S ?

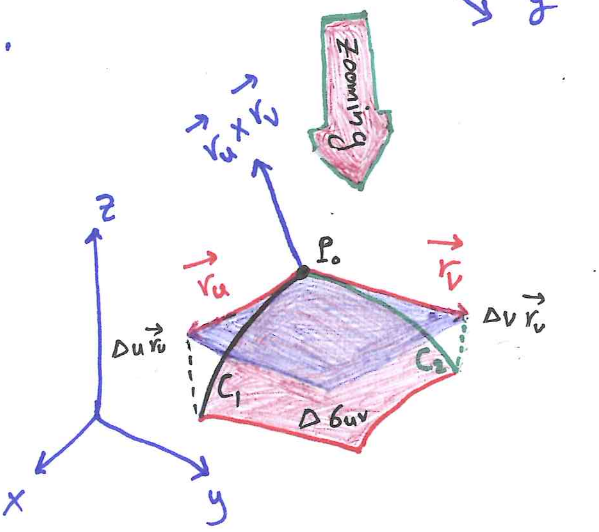




- Small rectangular area element ΔA_{uv} in R (uv-plane) maps onto a curved patch element $\Delta\sigma_{uv}$ on S.
- The vertex (u_0, v_0) maps to P_0.
- The surface S is parametrized by

$$\vec{r}(u,v) = f(u,v)\,\vec{i} + g(u,v)\,\vec{j} + h(u,v)\,\vec{k}$$

$$a \le u \le b \quad , \quad c \le v \le d$$



- $\vec{r}_u = f_u\,\vec{i} + g_u\,\vec{j} + h_u\,\vec{k}$

  $\vec{r}_v = f_v\,\vec{i} + g_v\,\vec{j} + h_v\,\vec{k}$

Def A parametrized surface $\vec{r}(u,v) = f(u,v)\,\vec{i} + g(u,v)\,\vec{j} + h(u,v)\,\vec{k}$ is **smooth** if $\vec{r}_u \times \vec{r}_v \neq \vec{0}$ on the interior of the parameter domain R and $\vec{r}_u$ together with $\vec{r}_v$ are continuous.

- We need $\vec{r}$ to be smooth so that $\vec{r}_u \times \vec{r}_v$ is never zero, which means the two vectors $\vec{r}_u$ and $\vec{r}_v$ are non zero and never lie along the same line. This determine a plane tangent to the surface.
- $\vec{r}_u$ is tangent to C_1 at P_0 and $\vec{r}_v$ is tangent to C_2 at P_0.
- The cross product $\vec{r}_u \times \vec{r}_v$ is normal to the surface at P_0.

- We approximate the area of the surface patch element $\Delta\sigma_{uv}$ by the parallelgram on the tangent plane whose sides are given by the vector $\Delta u\,\vec{r}_u$ and $\Delta v\,\vec{r}_v$.

- Hence, the area of the parallelgram is

$$\left|\Delta u\,\vec{r}_u \times \Delta v\,\vec{r}_v\right| = \left|\vec{r}_u \times \vec{r}_v\right| \Delta u\,\Delta v$$

- Hence, we approximate the surface area of S by

$$\sum_n \left|\vec{r}_u \times \vec{r}_v\right| \Delta u\,\Delta v$$

- $n \longrightarrow \infty$ as Δu and $\Delta v \longrightarrow 0$. Since $\vec{r}_u$ and $\vec{r}_v$ are continuous, this guarantees

$$\lim_{n\to\infty} \sum_n \left|\vec{r}_u \times \vec{r}_v\right| \Delta u\,\Delta v = \int_c^d \int_a^b \left|\vec{r}_u \times \vec{r}_v\right| du\,dv$$

Def The area of the smooth surface

$$\vec{r}(u,v) = f(u,v)\,\vec{i} + g(u,v)\,\vec{j} + h(u,v)\,\vec{k}, \quad a \le u \le b, \quad c \le v \le d$$

is $$A = \iint_R \left|\vec{r}_u \times \vec{r}_v\right| dA = \int_c^d \int_a^b \left|\vec{r}_u \times \vec{r}_v\right| du\,dv$$

Exp Find the surface area for the portion of the cylinder $x^2+y^2=1$ between the planes $z=1$ and $z=4$.

- Cylindrical coordinates provide a parametrization but we use

$x = r\cos\theta = \cos\theta$ | $r^2 = x^2+y^2 = 1$

$y = r\sin\theta = \sin\theta$ | $0 \le \theta \le 2\pi$

$z = x^2+y^2 = r^2 = 1$ | $1 \le z \le 4$

$\vec{r}(z,\theta) = (\cos\theta)\,\vec{i} + (\sin\theta)\,\vec{j} + z\,\vec{k}$ and not $\vec{r}(r,\theta)$ (to avoid $\vec{r}_r = \vec{0}$)

- $\vec{r}_z = \vec{k}$ and $\vec{r}_\theta = (-\sin\theta)\,\vec{i} + (\cos\theta)\,\vec{j}$

- $\vec{r}_\theta \times \vec{r}_z = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = (\cos\theta)\vec{i} + (\sin\theta)\vec{j}$

- Hence, $|\vec{r}_\theta \times \vec{r}_z| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$

- So, $A = \int_0^{2\pi}\int_1^4 1\, dz\, d\theta = \int_0^{2\pi} 3\, d\theta = 6\pi$

Exp Find the surface area for the portion of the cone $z = 2\sqrt{x^2+y^2}$ between the planes $z=2$ and $z=6$

- Cylindrical coordinates provide a parametrization

$x = r\cos\theta$ | $1 \le r \le 3$ and $0 \le \theta \le 2\pi$

$y = r\sin\theta$ | $\vec{r}(r,\theta) = (r\cos\theta)\vec{i} + (r\sin\theta)\vec{j} + 2r\vec{k}$

$z = 2\sqrt{r^2} = 2r$

- $\vec{r}_r = (\cos\theta)\vec{i} + (\sin\theta)\vec{j} + 2\vec{k}$ and $\vec{r}_\theta = (-r\sin\theta)\vec{i} + (r\cos\theta)\vec{j}$

- $\vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & 2 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = (-2r\cos\theta)\vec{i} - (2r\sin\theta)\vec{j} + r\vec{k}$

- $|\vec{r}_r \times \vec{r}_\theta| = \sqrt{4r^2\cos^2\theta + 4r^2\sin^2\theta + r^2} = \sqrt{5r^2} = \sqrt{5}\, r$

- Hence, $A = \int_0^{2\pi}\int_1^3 \sqrt{5}\, r\, dr\, d\theta = \int_0^{2\pi} 4\sqrt{5}\, d\theta = 8\pi\sqrt{5}$

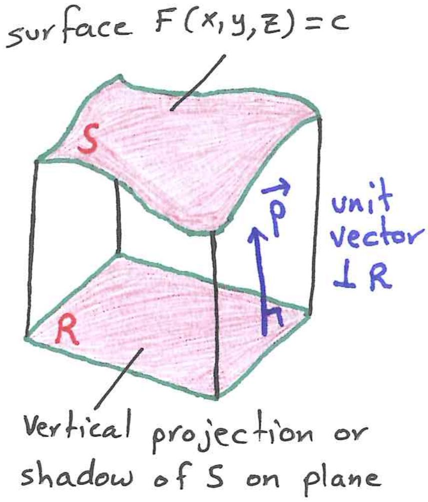
Remark: We can use the following abbreviation:

1. Surface area differential $d\sigma = |\vec{r}_u \times \vec{r}_v|\, du\, dv$
2. Differential formula for surface area $\iint_S d\sigma$

Implicit Surfaces

- Some surfaces does not come with an explicit parametrization and given by

$$F(x,y,z) = c \quad , \quad c \in \mathbb{R}$$



- Such surfaces are called an implicitly defined surfaces.
- Thus, the functions f, g, h are difficult to compute in order to parametrize the surface:

$$\vec{r}(u,v) = f(u,v)\,\vec{i} + g(u,v)\,\vec{j} + h(u,v)\,\vec{k}$$

- So, we need to compute the surface area differential $d\sigma$ for implicit surfaces. To do that, we assume that the surface is smooth (F is diff and ∇F is nonzero and continuous on S) and $\nabla F \cdot \vec{p} \neq 0$ so that F never folds back over itself.
- Simple calculation (see the book) will lead to

$$d\sigma = |\vec{r}_u \times \vec{r}_v|\, du\, dv = \frac{|\nabla F|}{|\nabla F \cdot \vec{p}|}\, dx\, dy \quad , \quad \begin{matrix} u = x \\ v = y \end{matrix}$$

assuming $\vec{p} = \vec{k}$ so that R lies in the xy-plane.

Result (Surface Area of an Implicit Surface)

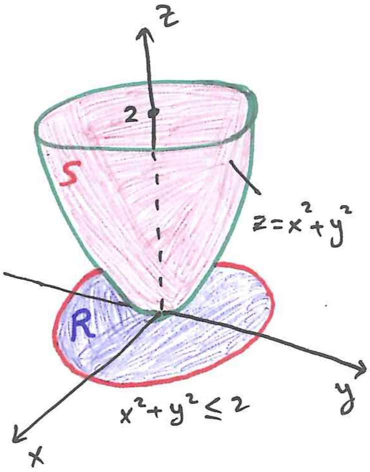
The area of the surface $F(x,y,z) = c$ over a closed and bounded plane region R is

$$\text{Surface area} = \iint_R \frac{|\nabla F|}{|\nabla F \cdot \vec{p}|}\, dA \,,$$

where $\vec{p} = \vec{i}, \vec{j}$ or $\vec{k}$ is normal to R and $\nabla F \cdot \vec{p} \neq 0$.

Exp Find the area of the surface cut from the paraboloid $x^2+y^2-z=0$ by the plane $z=2$

- sketch the surface S and the region R in the xy-plane.



- $F(x,y,z) = x^2+y^2-z=0$ and the disk $x^2+y^2 \le 2$ is R in xy-plane.
- We can take $\vec{p}=\vec{k}$ as a unit vector $\perp R$.
- $\nabla f = 2x\vec{i}+2y\vec{j}-\vec{k}$ and $\nabla f\cdot\vec{p}=-1$
- $|\nabla f| = \sqrt{4x^2+4y^2+1}$ and $|\nabla f\cdot\vec{p}|=1$
- In the region R we have $dA = dx\,dy$.
- when $z=2 \Rightarrow x^2+y^2=2 \Rightarrow r^2=2 \Rightarrow r=\sqrt{2}$
- Surface area $= \iint\limits_R \frac{|\nabla f|}{|\nabla f\cdot\vec{p}|}\,dA = \iint\limits_{x^2+y^2\le 2} \sqrt{4x^2+4y^2+1}\,dx\,dy$

$$= \int_0^{2\pi}\int_0^{\sqrt{2}} \sqrt{4r^2+1}\; r\,dr\,d\theta$$

Polar coordinates (see page 855)

$$= \int_0^{2\pi} \frac{13}{6}\,d\theta = \frac{13}{3}\pi$$

Remark For a graph $z=f(x,y)$ over a region R in the xy-plane, the surface area is

$$\text{Surface area} = \iint\limits_R \sqrt{f_x^2+f_y^2+1}\;dx\,dy$$

(see exp above: red color)