14.3 Partial Derivatives

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* If $z = f(x, y)$ (or $w = f(x,y,z)$), then the partial derivative with respect to x is denoted by:

$$\frac{\partial f}{\partial x} = f_x = z_x = \frac{\partial z}{\partial x} \quad \left(\text{or } \frac{\partial f}{\partial x} = f_x = w_x = \frac{\partial w}{\partial x}\right)$$

and the partial derivative with respect to y is:

$$\frac{\partial f}{\partial y} = f_y = z_y = \frac{\partial z}{\partial y} \quad \left(\text{or } w_y = \frac{\partial w}{\partial y} = f_y = \frac{\partial f}{\partial y}\right)$$

Exp Find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ at the point $(2,-1)$ for

1. $f(x,y) = x^2 - xy + y^2$

$f_x = 2x - y \Rightarrow f_x(2,-1) = 4+1 = 5$

$f_y = -x + 2y \Rightarrow f_y(2,-1) = -2 + (-2) = -4$

2. $f(x,y,z) = xy + yz + xz$ " find also $\frac{\partial f}{\partial z}$ " at $(2,-1,1)$

$f_x = y + z \Rightarrow f_x(2,-1,1) = -1+1 = 0$

$f_y = x + z \Rightarrow f_y(2,-1,1) = 2+1 = 3$

$f_z = y + x \Rightarrow f_z(2,-1,1) = -1 + 2 = 1$

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* Second-Order Partial Derivatives:

The second-order derivatives of $f(x,y)$ are

$$\frac{\partial^2 f}{\partial x^2} = f_{xx} \quad \text{or} \quad \frac{\partial^2 f}{\partial y^2} = f_{yy} \quad \text{or} \quad \frac{\partial^2 f}{\partial x \partial y} = f_{xy} = f_{yx} = \frac{\partial^2 f}{\partial y \partial x}$$

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Th (Mixed Derivative) If :

[1] $f(x,y)$, f_x, f_y, f_{xy}, f_{yx} are defined on an open region containing a point (a,b) and

[2] all are continuous at (a,b), then

$$f_{xy}(a,b) = f_{yx}(a,b)$$

Exp Let $w = xy + \frac{e^y}{y^2+1}$. Find $\frac{\partial^2 w}{\partial x \partial y}$

$$w_x = y \Rightarrow w_{xy} = 1$$

Exp Find all the second-order partial derivatives of

$g(x,y) = x^2 y + \cos y + y \sin x$

$g_x = y\cos x + 2xy \Rightarrow g_{xx} = -y\sin x + 2y$

$g_y = x^2 - \sin y + \sin x \Rightarrow g_{yy} = -\cos y$

$g_{xy} = \cos x + 2x = g_{yx}$

Exp Let $z = f(x,y)$ with partial derivative exists. Find $\left.\frac{\partial z}{\partial x}\right|_{(1,1,1)}$ if

$xy + z^3 x - 2yz = 0$

$y + z^3 + 3xz^2 \frac{\partial z}{\partial x} - 2y\frac{\partial z}{\partial x} = 0$

$1 + 1 + 3\frac{\partial z}{\partial x} - 2\frac{\partial z}{\partial x} = 0 \Leftrightarrow \frac{\partial z}{\partial x} = -2$

Exp Find f_{yxyz} if $f(x,y,z) = 1 - 2xy^2 z + x^2 y$

$f_z = -2xy^2$

$f_{zx} = -2y^2$

$f_{zxy} = -4y \Rightarrow f_{zxyy} = -4$

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Using Partial Derivative Definition

Def The partial derivative of $f(x,y)$ w.r.t x at (x_0,y_0) is

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = f_x(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0+h,y_0)-f(x_0,y_0)}{h} \quad \text{provided}$$

the limit exists.

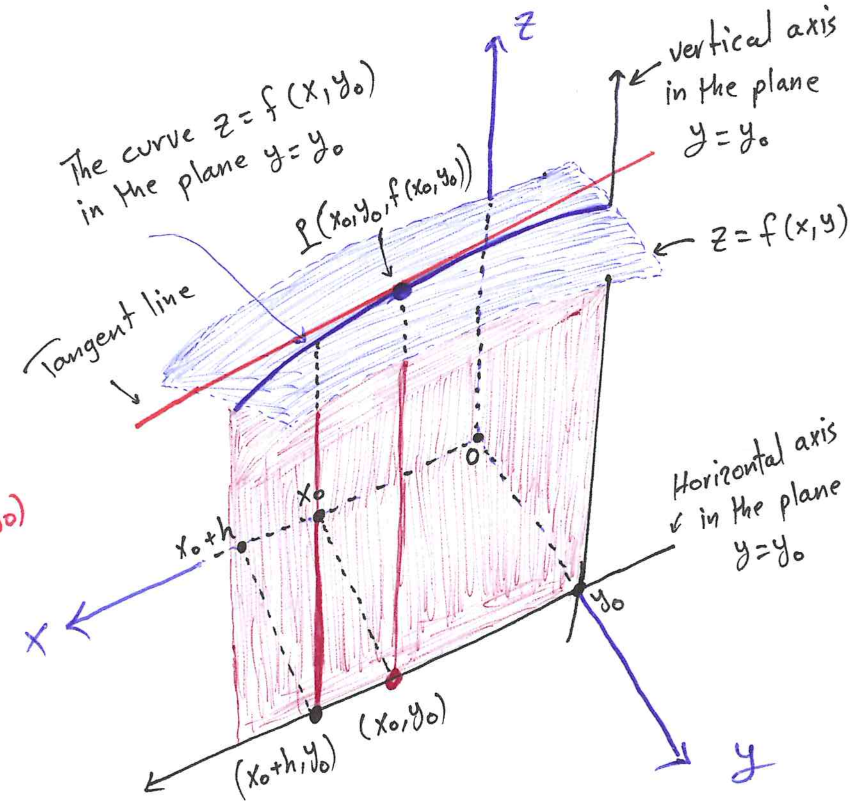

Fig 1

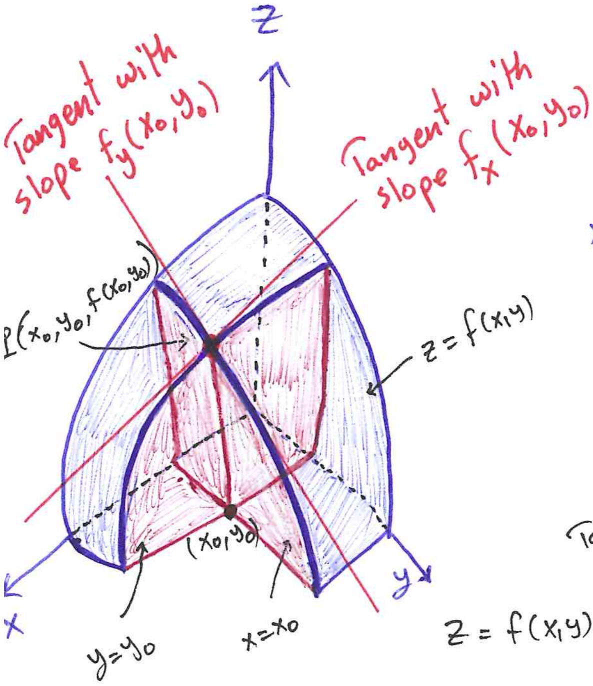

Fig 3

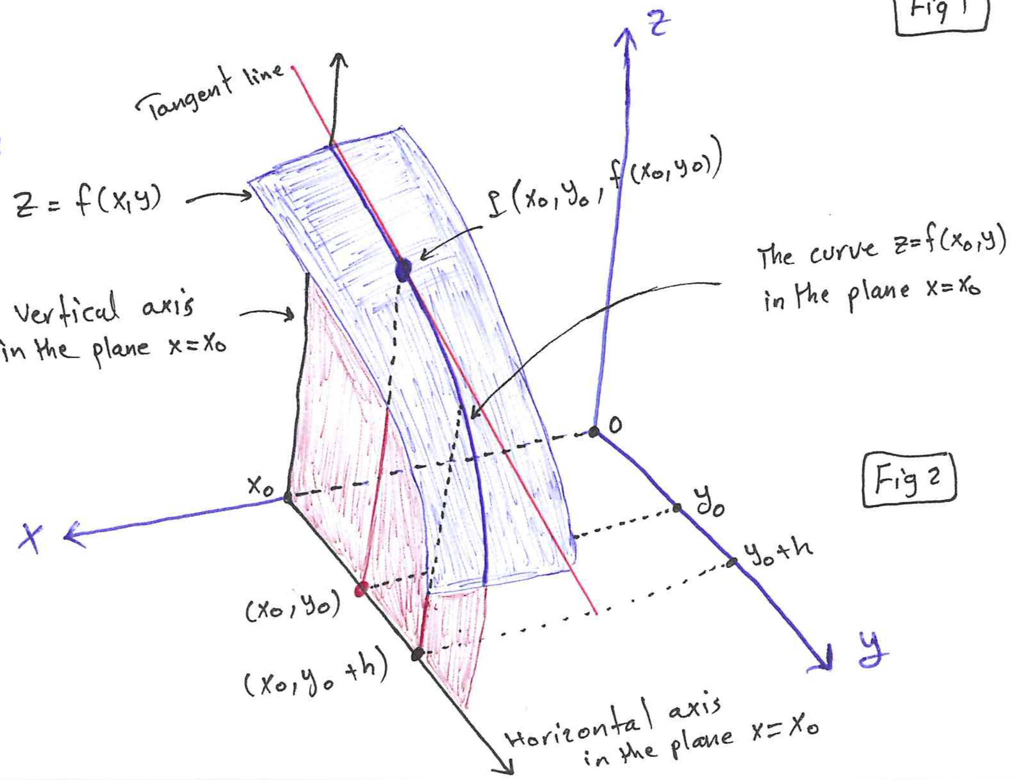

Fig 2

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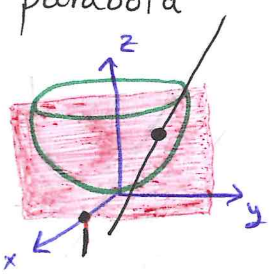
Def The partial derivative of $f(x,y)$ w.r.t y at (x_0, y_0) is

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = f_y(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0, y_0+h) - f(x_0,y_0)}{h}$$

provided the limit exists.

* Note that the tangent lines at the point $(x_0, y_0, f(x_0,y_0))$ determine a plane which is tangent to the surface. Fig 3

Exp The plane $x=1$ intersects the paraboloid $z = x^2+y^2$ in a parabola. Find the slope of the tangent to the parabola at $(1,2,5)$.



The slope $= Z_y(1,2) = \left. 2y \right|_{(1,2)} = 2(2) = 4$

or The parabola is $z = 1+y^2$ which has slope $z'(y=2)$

$z'(y) = 2y \Rightarrow z'(2) = 4$

Exp Use the partial derivative definition to find $\frac{\partial f}{\partial x}$ at $(1,2)$ for

$f(x,y) = 1 - x + y - 3x^2 y$

$$f_x(1,2) = \lim_{h\to 0} \frac{f(1+h,2) - f(1,2)}{h} = \lim_{h\to 0} \frac{\cancel{1} - \cancel{1} - h + 2 - 3(1+h)^2(2) - -4}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{6} - h - \cancel{6} - 6(2h) - 6h^2}{h} = \lim_{h\to 0}(-13 - 6h) = -13$$

$f_x = -1 - 6xy \Rightarrow f_x(1,2) = -1 - 6(1)(2) = -1 - 12 = -13$

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Def $*^1$ A function $z = f(x,y)$ is **differentiable** at (x_0, y_0) (80)

if $\boxed{1}$ $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist and

$\boxed{2}$ $\Delta z = f(x_0+\Delta x, y_0+\Delta y) - f(x_0, y_0)$ satisfies

$$\Delta z = f_x(x_0, y_0)\,\Delta x + f_y(x_0, y_0)\,\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

in which each of $\epsilon_1, \epsilon_2 \to 0$ as both $\Delta x, \Delta y \to 0$

$*^2$ The function f is **differentiable** if it is differentiable at every point in its domain, and we say it's graph is a smooth surface.

Note that the definition above similar to that of single variable:
if $y = f(x)$ is diff. at x_0 then from section 3.9 : $\Delta y = f'(x_0)\Delta x + \epsilon \Delta x$
in which $\epsilon \to 0$ as $\Delta x \to 0$.

Th $*^1$: If the partial derivatives f_x and f_y of $f(x,y)$ are continuous on an open region R, then f is differentiable at every point of R.

Th $*^2$ **(Differentiability $\Rightarrow$ Continuity)**
If $f(x,y)$ is differentiable at (x_0, y_0), then f is continuous at (x_0, y_0).

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* Remember fig 3 when reading Th^{*1} and Th^{*2}:
* The function $f(x,y)$ still possible to be discontinuous at (x_0, y_0) but $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist.

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81

Exp Let $f(x,y)=\begin{cases} 0 & , \quad xy \neq 0 \\ 1 & , \quad xy = 0 \end{cases}$

[a] Prove that f is not continuous at origin

[b] Show that f_x and f_y exist at the origin.

[a] • $f(0,0) = 1$

• $\lim\limits_{\substack{(x,y)\to(0,0) \\ \text{along } y=x}} f(x,y) = \lim\limits_{x\to 0} 0 = 0$

$f(0,0) \neq \lim\limits_{(x,y)\to(0,0)} f(x,y)$

Hence, f is not continuous at $(0,0)$

[b] $f_x(0,0) = \lim\limits_{h\to 0} \dfrac{f(0+h,0)-f(0,0)}{h} = \lim\limits_{h\to 0} \dfrac{f(h,0)-1}{h}$

$= \lim\limits_{h\to 0} \dfrac{1-1}{h} = \lim\limits_{h\to 0} 0 = 0$

$f_y(0,0) = \lim\limits_{h\to 0} \dfrac{f(0,h+0)-f(0,0)}{h} = \lim\limits_{h\to 0} \dfrac{f(0,h)-1}{h}$

$= \lim\limits_{h\to 0} \dfrac{1-1}{h} = \lim\limits_{h\to 0} 0 = 0$

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so this because f is not differentiable at $(0,0)$ even if item [1] holds in Def *' , it is still missing item [2]

or by Th*' , f_x and f_y must be continuous at region contains $(0,0)$.

To see that: look to item [2] ⇒ we have:

$\Delta z = f(0+\Delta x, 0+\Delta y) - f(0,0)$
$= f(\Delta x, \Delta y) - f(0,0)$
$= 0 - 1$
$= -1$

so $\Delta z = -1$
$\neq f_x(0,0)\Delta x + f_y(0,0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$
$= 0 + 0 + 0 + 0$

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