Key Quiz #1 (part 2)

Exercise#1 [3 marks].

(a) Find all $x \in \mathbb{R}$ that satisfy the inequality $|x-1| > |x+1|$.

1.5 $|x-1| > |x+1| \iff |x-1|^2 > |x+1|^2$

$\iff x^2 - 2x + 1 > x^2 + 2x + 1$

$\iff x < 0$

(b) If $\alpha, \beta \in \mathbb{R}$, show that $|\alpha+\beta| = |\alpha| + |\beta|$ iff $\alpha\beta \geq 0$.

1.5 $|\alpha+\beta| = |\alpha| + |\beta| \iff (\alpha+\beta)^2 = (|\alpha| + |\beta|)^2$

$\iff \alpha^2 + 2\alpha\beta + \beta^2 = \alpha^2 + 2|\alpha||\beta| + \beta^2$

$\iff \alpha\beta = |\alpha\beta|$

$\iff \alpha\beta \geq 0$

Exercise#2 [2 marks]. Let a and b be any two real numbers such that $a < b$. Let p be a fixed positive irrational number. Show that there is a rational number q such that $a < pq < b$.

2 Since $a < b$, $p > 0$, then $\frac{a}{p} < \frac{b}{p}$. By density of rationals $\exists q \in \mathbb{Q} : \frac{a}{p} < q < \frac{b}{p}$ or $a < pq < b$.

Exercise#3 [4 marks]. Let S be a set of nonnegative real numbers that is bounded above and let $T := \{s^2 : s \in S\}$. Prove that if $\sup S = \beta$, then $\sup T = \beta^2$. What about the conclusion if the restriction against negative numbers is removed?
Justify your answer.

If $s \in S$, then $0 \leq s \leq \beta$, so that $s^2 \leq \beta^2$ which

1.5 implies that $\boxed{\sup T \leq \beta^2}$. If γ is any upper bound of T, then $s \in S \Rightarrow s^2 \leq \gamma$ so that $s \leq \sqrt{\gamma}$.

It follows that $\beta \leq \sqrt{\gamma}$ so that $\beta^2 \leq \gamma$.

1.5 Thus $\boxed{\beta^2 \leq \sup T}$. For the second question, is NOT true. For example let $S = [-2, 1]$, $T = [0, 4]$

1 $\sup T = 4 \neq 1^2$.