Birzeit University

Mathematics Department

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Title: Mathematical Analysis I

Chapter 1 The real number system ($\mathbb{R}$).

1.2 Ordered Field Axioms

Postulate 1 (Field Axioms).

There are functions $+$ and $\cdot$ defined on $\mathbb{R}^2 := \mathbb{R} \times \mathbb{R}$, which satisfy the following properties for every $a, b, c \in \mathbb{R}$:

(A) $a+b$ and $a \cdot b \in \mathbb{R}$ (Closure properties)

(B) $a+(b+c) = (a+b)+c$ and

$a \cdot (b \cdot c) = (a \cdot b) \cdot c$. (Associative properties)

(C) $a+b = b+a$ and $a \cdot b = b \cdot a$ (Commutative properties)

(D) $a \cdot (b+c) = a \cdot b + a \cdot c$. (Distributive Law).

(E) There is aunique element $0 \in \mathbb{R}$ such that $0+a = a+0 = a$, $\forall a \in \mathbb{R}$.
(Existence of the Additive Identity).

(F) There is aunique element $1 \in \mathbb{R}$ such that $1 \neq 0$ and $1 \cdot a = a$, $\forall a \in \mathbb{R}$.
(Existence of the Multiplicative Identity).

(G) For every $x \in \mathbb{R}$, there is a unique element $-x \in \mathbb{R}$ such that

$$X + (-X) = 0$$ (Additive Inverses).

(H) $\forall x \in \mathbb{R} \setminus \{0\}$, there is a unique $x^{-1} \in \mathbb{R}$ such that $x \cdot (x^{-1}) = 1$ (Multiplicative Inverses)

(H.W) From postulate 1., we can derive the following

① $(-1)^2 = 1$

② $0 \cdot a = 0$, $-a = (-1) \cdot a$, $-(-a) = a$, $\forall a \in \mathbb{R}$

③ $-(a-b) = b-a$, $\forall a, b \in \mathbb{R}$.

④ If $a, b \in \mathbb{R}$ and $ab = 0 \Rightarrow a = 0$ or $b = 0$.

<u>Postulate 2</u> (<u>Order Axioms</u>)

there is a relation on $\mathbb{R} \times \mathbb{R}$ that has the following properties:

(i) $\forall a, b \in \mathbb{R}$, one and only one of the following statements holds:

$a < b$, $b < a$, or $a = b$ (Trichotomy Property).

(ii) $\forall a, b, c \in \mathbb{R}$,

$a < b$ and $b < c \Rightarrow a < c$ (Transitive Property).

(iii) $\forall a, b, c \in \mathbb{R}$, (The additive Property).

$a < b$ and $c \in \mathbb{R} \Rightarrow a + c < b + c$

(iv) $\forall a, b, c \in \mathbb{R}$,

$a < b$ and $c > 0 \Rightarrow ac < bc$

and $a < b$ and $c < 0 \Rightarrow bc < ac$

(the Multiplicative Properties).

Rmk. • $a \leq b$ means $a < b$ or $a = b$.

• $a < b < c$ means $a < b$ and $b < c$

ex. $2 < x < 1$ makes no sense at all.

- $a \in \mathbb{R}$ is nonnegative if $a \geq 0$ and positive if $a > 0$.

Rmk. the real number system $\mathbb{R}$ contains the following special subsets:

(1) The set of natural numbers

$$\mathbb{N} = \{1, 2, 3, 4, \text{-----}$$

(2) the set of integers

$$\mathbb{Z} = \{ \text{----}, -2, -1, 0, 1, 2, \text{----} \}$$

(3) Rationals (Fractions or quotients)

$$\mathbb{Q} := \left\{ \frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0 \right\}$$

Equality in $\mathbb{Q}$ is defined as

$$\frac{m}{n} = \frac{p}{q} \Leftrightarrow mq = np.$$

Notice that $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$.

(4) Irrationals $\mathbb{Q}^c = \mathbb{R} \setminus \mathbb{Q}$.

Rmk. The sets $\mathbb{N}$ and $\mathbb{Z}$ satisfy the following properties

(i) If $m, n \in \mathbb{Z}$, then $m+n$ and $mn \in \mathbb{Z}$.

(ii) If $n \in \mathbb{Z}$, then $n \in \mathbb{N} \Leftrightarrow n \geq 1$.

(iii) There is no $n \in \mathbb{Z}$ that satisfies $0 < n < 1$.

Ex. (H.W). Prove that $\mathbb{Q}$ satisfies Postulate 1

Ex. If $a \in \mathbb{R}$, prove that

$$a \neq 0 \Rightarrow a^2 > 0.$$

In particular, $-1 < 0 < 1$.

proof. Suppose that $a \neq 0$. By the Trichotomy property, either $a > 0$ or $a < 0$.

Case 1. $a > 0 \Rightarrow a.a > 0.a$
$\Rightarrow a^2 > 0$

Case 2. $a < 0 \Rightarrow a.a > 0.a$
$\Rightarrow a^2 > 0$

This proves $a^2 > 0$, when $a \neq 0$.

Since $1 \neq 0 \Rightarrow 1 = 1^2 > 0 \Rightarrow \boxed{1 > 0}$.

$$\Rightarrow 1 + (-1) > 0 + (-1)$$

$$\Rightarrow \boxed{0 > -1}$$

We conclude that $-1 < 0 < 1$ (by Transitive property)

Ex. (H.W): ① If $a \in \mathbb{R}$, prove that

$$0 < a < 1 \Rightarrow 0 < a^2 < a$$

and $a > 1 \Rightarrow a^2 > a$.

② $0 \leq a < b$ and $0 \leq c < d \Rightarrow ac < bd$.

③ $0 \leq a < b \Rightarrow 0 \leq a^2 \leq b^2$ and $0 \leq \sqrt{a} \leq \sqrt{b}$

④ $0 < a < b \Rightarrow \frac{1}{a} > \frac{1}{b} > 0$.

Df (the absolute value)

$$|a| := \begin{cases} a, & a \geq 0 \\ -a, & a < 0. \end{cases}$$

is called the absolute value of a number $a \in \mathbb{R}$.

Example. Prove that $|ab| = |a||b|$, $\forall a, b \in \mathbb{R}$.
That is, the absolute value is multiplicative

proof. We consider 4 cases.

Case 1. $a = 0$ or $b = 0$. Then $ab = 0$,
so by def'n, $|ab| = 0 = |a|\,|b|$

Case 2. $a > 0$ and $b > 0$. Then
$ab > 0 \cdot b = 0$. Hence by definition,
$|ab| = ab = |a|\,|b|$.

Case 3. $a > 0$ and $b < 0$, or, $b > 0$ and $a < 0$.
we prove it for $a > 0$ and $b < 0$ then by reversing the roles of a and b, we can prove it for $a < 0$ and $b > 0$.

Suppose $a > 0$ and $b < 0$, then $ab < 0 \cdot b = 0$

Hence, by definition, HW ② page 2, associativity, and Commutativity,

$$|ab| = -(ab) = (-1)(ab) = a((-1)b) = a(-b)$$
$$= |a||b|.$$

Case 4. $a < 0$ and $b < 0$. Then

$ab > 0 \cdot b = 0$, Hence, by def'n,

$$|ab| = ab = (-1)^2(ab) = (-a)(-b) = |a||b|$$

Theorem ① (Fundamental thm of Absolute Values).

Let $a \in \mathbb{R}$ and $M \geq 0$. Then $|a| \leq M \Leftrightarrow -M \leq a \leq M$.

Proof. ($\Longrightarrow$) Suppose that $|a| \leq M$. multiplying by -1, we have $-|a| \geq -M$.

Case 1. $a \geq 0$. By def'n, $|a| = a$. Thus by hypothesis $-M \leq 0 \leq a = |a| \leq M$.

Case 2. $a<0$. By def'n, $|a|=-a$. thus by hypothesis,

$$-M \le -|a| = a < 0 \le M.$$

This proves that $-M \le a \le M$ in either case.

($\Leftarrow$) Conversely, if $-M \le a \le M$, then

$a \le M$ and $-M \le a$. multiply the second inequality by -1, we have $-a \le M$.

Consequently $|a| = a \le M$ if $a \geq 0$;

and $|a| = -a \le M$ if $a < 0$.

this proves that $|a| \le M$, $\forall a \in \mathbb{R}$. ∎

Ex. (H.W) One Can prove that

$$|a| < M \iff -M < a < M, \forall a \in \mathbb{R}, M \geq 0.$$

Thm 2. the absolute value satisfies

(i) $\forall a \in \mathbb{R}$, $|a| \geq 0$ with $|a| = 0 \iff a = 0$
(Positive Definite).

(ii) $\forall a, b \in \mathbb{R}$, $|a-b| = |b-a|$. (Symmetric).

(iii) $\forall a,b \in \mathbb{R}$, $|a+b| \leq |a| + |b|$ and

$||a| - |b|| \leq |a-b|$. (Triangle Inequalities)

Proof: (i) If $a \geq 0$, then $|a| = a \geq 0$.

If $a < 0$, then by def'n and the second multiplicative property, $|a| = -a = (-1)a > 0$.

Thus, $|a| \geq 0$, $\forall a \in \mathbb{R}$.

If $|a| = 0$, then by def'n $a = |a| = 0$ when $a \geq 0$ and $a = -|a| = 0$ when $a < 0$. Thus, $|a| = 0 \Rightarrow a = 0$. Conversely, $|0| = 0$ by def'n.

(ii) We proved $|ab| = |a||b|$, $\forall a,b \in \mathbb{R}$.

Now, $|a-b| = |(-1)(b-a)| = |-1|\,|b-a|$
$= |b-a|$.

(iii) To prove the first inequality, notice that $|x| \leq |x|$, $\forall x \in \mathbb{R}$.

Thus, $-|x| \leq x \leq |x|$ (Thm 1).

So, we have $-|a| \le a \le |a|$ and $-|b| \le b \le |b|$

Adding these inequalities, we obtain

$$-(|a| + |b|) \le a+b \le |a|+|b|.$$

Hence by thm1 again, $|a+b| \le |a|+|b|$.

To prove the second inequality, apply the first inequality to $(a-b)+b$. We obtain

$$|a|-|b| = |(a-b)+b| - |b|$$
$$\le |a-b| + |b| - |b| = |a-b|. \quad \text{-----}(\alpha)$$

By reversing the roles of a and b and applying part(ii), we obtain

$$|b|-|a| \le |b-a| = |a-b|.$$

$$\Rightarrow \quad |a|-|b| \ge -|a-b| \quad \text{---}(\beta)$$

Combining (α) and (β):

$$-|a-b| \le |a|-|b| \le |a-b|$$

we conclude by Thm1 $\big||a|-|b|\big| \le |a-b|$. ∎

Warning. Some students mistakenly mis absolute values and the Additive property to conclude that $b<c \Rightarrow |a+b| < |a+c|$.

For example, $-5<-1$ but $|-5+2| \not< |-1+2|$

Ex. Prove that if $-2<x<1$, then $|x^2-x|<6$.

Proof. By hypothes, $|x|<2$. Hence by the triangle inequality,

$$|x^2-x| \le |x|^2+|x| < 4+2=6.$$

Thm③. Let $x, y, a \in \mathbb{R}$.

(i) $x<y+\varepsilon, \forall \varepsilon>0 \Leftrightarrow x \le y$.

(ii) $x>y-\varepsilon, \forall \varepsilon>0 \Leftrightarrow x \ge y$.

(iii) $|a|<\varepsilon, \forall \varepsilon>0 \Leftrightarrow a=0$.

Proof. (i)($\Rightarrow$) Suppose to the contrary that $x<y+\varepsilon$, $\forall \varepsilon>0$ but $x>y$. Set $\varepsilon_0 = x-y$ and observe that $y+\varepsilon_0 = x$.

Hence by the Trichotomy property, x cannot be smaller than $y+\varepsilon_0$. This contradicts the hypothesis for $\varepsilon=\varepsilon_0$. Thus, $x \leq y$.

($\Leftarrow$) Conversely, suppose that $x \leq y$ and $\varepsilon > 0$ is given. Either $x<y$ or $x=y$.

Case 1. If $x<y$, then $x+0 < y+0 < y+\varepsilon$ by the additive and Transitive property.

Case 2. If $x=y$, then $x<y+\varepsilon$ by the Additive property. Thus, in either case, $x<y+\varepsilon$, $\forall \varepsilon>0$.

(ii) Suppose that $x>y-\varepsilon$, $\forall \varepsilon>0$. This is equivalent to $-x<-y+\varepsilon$, hence by part(i), equivalent to $-x \leq -y$. This is equivalent to $x \geq y$ by the second Multiplicative Property.

(iii) Suppose that $|a| < \varepsilon = 0+\varepsilon$, $\forall \varepsilon > 0$. By part(i), this is equivalent to $|a| \leq 0$.

Since it is always the case that $|a| \geq 0$, we conclude by the Trichotomy Property that $|a| = 0$. Therefore $a = 0$ by Thm (2) (i).

Conversely, if $a = 0$ and $\varepsilon > 0$, then

$$|a| = |0| = 0 < \varepsilon. \quad \blacksquare$$

<u>Rmk.</u> let $a, b \in \mathbb{R}$. A closed interval is a set of the form

$$[a,b] := \{x \in \mathbb{R} : a \leq x \leq b\}.$$

$$[a,\infty) := \{x \in \mathbb{R} : x \geq a\}.$$

$$(-\infty, b] := \{x \in \mathbb{R} : x \leq b\}.$$

or $(-\infty, \infty) := \mathbb{R}$.

<u>open interval</u>

$$(a,b) := \{x \in \mathbb{R} : a < x < b\}$$

$$(a,\infty) := \{x \in \mathbb{R} : x > a\}$$

$$(-\infty, b) = \{x \in \mathbb{R} : x < b\}.$$

By an <u>interval</u>, we mean a closed interval, an open interval, or a set of the form $[a,b)$, $(a,b]$.

- An interval I is bounded iff it has the form $[a,b]$, (a,b), $[a,b)$, or $(a,b]$ for $-\infty < a \leq b < \infty$. a and b are called the endpoints of I. All other intervals will be unbounded.

If $a=b$, then I is said to be <u>degenerate</u> and it is <u>nondegenerate</u> if $a<b$.

- The Length of a bounded interval I with endpoints a,b is defined to be

$$|I| := |b-a|,$$ and the distance between any two points $a,b \in \mathbb{R}$ is defined by $|a-b|$.

1.3 Completeness Axiom

Df ①. Let $E \subseteq \mathbb{R}$ be a nonempty set. Then

(i) E is said to be bounded above iff $\exists$ an $M \in \mathbb{R}$ such that $x \leq M$, $\forall x \in E$, in this case, M is called an upper bound of E.

(ii) A number β is called a supremum of E iff β is an upperbound of E and $\beta \leq M$ for all upperbounds M of E. Here, we say E has a finite supremum and write $\boxed{\sup E = \beta}$.

Remarks. (1) By the last definition, $\sup E$ (when it exists) is the smallest (least) upperbound of E.

(2) Inorder to prove $\sup E = \beta$ for some $E \subset \mathbb{R}$ we must show two things:

(i) β is an upperbound of E.

(ii) β is the smallest upperbound, that is, if M is any upperbound of E, then $\beta \leq M$).

Ex. Let $E = [0,1]$, prove that $\sup E = 1$.

Proof. By the definition of interval, 1 is an upperbound of E. Let M be any upper bound of E, that is, $x \leq M, \forall x \in E$. Since $1 \in E$ it follows that $M \geq 1$. Thus 1 is the smallest upperbound of E. $\blacksquare$

Ex. Let $E_1 = \mathbb{R}^- = \{x : x < 0\}$, $E_2 = \mathbb{Z}^- = \{\dots, -3, -2, -1\}$ then $\sup E_1 = 0$, $\sup E_2 = -1$.

Question. How many upperbounds and suprema can a given set have?

Ans. ① If a set has one upper bound, it has infinitely many upper bounds.

proof. If M_0 is an upperbound for a set E, then

So is M for any $M > M_0$. ∎

(2) If a set E has a sup β (i.e, $\sup E = \beta$), then it is unique.

Proof. Let β_1 and β_2 be suprema of the same set E, we need to prove that $\beta_1 = \beta_2$.

Then both β_1 and β_2 are upper bounds of E, whence by Def①, $\beta_1 \leq \beta_2$ and $\beta_2 \leq \beta_1$.

We conclude by the Trichotomy property that $\beta_1 = \beta_2$. ∎

Thm①. [Approximation Property for Suprema].

If E has a finite supremum β and $\varepsilon > 0$ is any positive number, then there is a point $x \in E$ such that $\beta - \varepsilon < x \leq \beta$.

Proof. Suppose that the theorem is false. Then there is an $\varepsilon_0 > 0$ such that no element of E lies between $\beta - \varepsilon$ and β.

Since $\beta = \sup E$ is an upper bound of E,

it follows that $x \le \beta - \varepsilon_0$, $\forall x \in E$, i.e,

$\beta - \varepsilon_0$ is an upper bound of E.

Thus by def①, $\beta \le \beta - \varepsilon_0$. It follows that

$\varepsilon_0 \le 0$, a contradiction. $\blacksquare$

<u>Thm②.</u> If $E \subset \mathbb{Z}$ has a supremum, then $\sup E \in \mathbb{Z}$. In particular, if the supremum of a set, which contains only integers, exists, that supremum must be an integer.

<u>proof.</u> Suppose that $\sup E := \beta$, and apply the Approximation Property to choose an $x_0 \in E$ such that $\beta - 1 < x_0 \le \beta$.

If $\beta = x_0$, then $\beta \in E$.

If $\beta - 1 < x_0 < \beta$, we can apply the Approximation Property again to choose $x_1 \in E$ such that

$x_0 < x_1 < \beta$.

the last inequality gives $0 < x_1 - x_0 < \beta - x_0$.

Since $-x_0 < 1-\beta$, it follows that

$0 < x_1 - x_0 < \beta + (1-\beta) = 1$. Thus,

$x_1 - x_0 \in \mathbb{Z} \cap (0,1)$, a contradiction. We

conclude that $\beta \in E$. ■

Postulate 3. [Completeness Axiom].

If E is a nonempty subset of $\mathbb{R}$ that is bounded above, then E has a finite supremum.

Rmk. From postulate 1 and 2 (section 1.2), and Postulate 3 (section 1.3), we say that $\mathbb{R}$ is a complete ordered field

Thm ③. (Archimedean property).

Given real numbers a and b, with $a>0$, there is an integer $n \in \mathbb{N}$ such that $b < na$.

Proof. If $b < a$, set $n = 1$.

If $b \geq a$ and $a > 0$. Consider the set

$E = \{ k \in \mathbb{N} : ka \leq b \}$. $E \neq \emptyset$ since

$1 \in E$. Let $k \in E$ (i.e., $ka \leq b$). Since

$a > 0$, it follows $k \leq \frac{b}{a}$. This proves

that E is bounded above by $\frac{b}{a}$.

Thus, by the Completeness Axiom and thm 2,

E has a finite supremum β and $\beta \in E$,

in particular, $\beta \in \mathbb{N}$.

Set $n = \beta + 1$. Then $n \in \mathbb{N}$ and $n > \beta$, it

follows that $n \notin E$. Thus, $na > b$. $\blacksquare$

Rmk: Sup E is not always belong to E.

ex. Let $A = \{ 1, \frac{1}{2}, \frac{1}{4}, \ldots \}$, $B = \{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots \}$

prove that $\sup A = \sup B = 1$.

Proof. It is clear that 1 is an upper bound of both sets. It remains to see that 1 is the smallest upper bound of both sets.

For A, let M be any upper bound of A. This means that $M \geq a$, $\forall a \in A$. In particular, $M \geq 1$. Hence, 1 is the smallest upper bound of A.

For B, let K be any upperbound of B. Suppose that $K < 1$. This gives $1-K > 0$. In particular, $\frac{1}{1-K} \in \mathbb{R}$. By Archimedean Principle, $\exists$ an $n \in \mathbb{N}$ s.t $n > \frac{1}{1-K}$. It follows that $x_0 = 1 - \frac{1}{n} > K$. Since $x_0 \in B$, this contradicts the assumption that K is an upper bound of B. Thus, $K \geq 1$ and hence 1 is the least upper bound of B.

Thm 17. [Density of Rationals]

The rational numbers $\mathbb{Q}$ are dense in the reals $\mathbb{R}$, i.e., if $a, b \in \mathbb{R}$ with $a<b$, then there is a rational $q \in \mathbb{Q}$ such that

$a < q < b$.

Pf. Case 1. Suppose first that $a>0$. Since $b-a>0$, use the Archimedean Principle to choose an $n \in \mathbb{N}$ such that

$$\max\left\{\frac{1}{a}, \frac{1}{b-a}\right\} < n,$$

and observe that $\frac{1}{n} < a$ and $\frac{1}{n} < b-a$.

Consider the set $E := \left\{k \in \mathbb{N} : \frac{k}{n} \leq a\right\}$.

Since $1 \in E$, then $E \neq \phi$. Since $n>0$, E is bounded above by na. Hence, by Thm 2, $k_0 := \sup E$ exists and $k_0 \in E$, in particular, $k_0 \in \mathbb{N}$.

Set $m = k_0+1$ and $q = \frac{m}{n}$. Since $k_0 = \sup E$,

then $m \notin E$. Thus, $q > a$ $\left(q = \frac{m}{n} = \frac{k_0+1}{n} > a \right)$.

On the other hand, since $k_0 \in E$, it follows that

$$b = a + (b-a) > \frac{k_0}{n} + \frac{1}{n} = \frac{k_0+1}{n} = \frac{m}{n} = q$$

Thus, $\exists$ a $q \in \mathbb{Q}$ s.t. $a < q < b$ if $a > 0$, $a < b$.

Case 2.

Suppose $a \leq 0$. By the Archimedean Principle, $\exists$ an integer $k \in \mathbb{N}$ such that $-a < k$.

Then $0 < k+a < k+b$, and by case 1 (the case already proved), there is an $r \in \mathbb{Q}$ s.t., $k+a < r < k+b$. Therefore,

$q := r-k \in \mathbb{Q}$ and satisfies $a < q < b$. ▤

Exercise (H.W). If $a, b \in \mathbb{R}$ with $a < b$, $\exists$ an irrational η such that $a < \eta < b$.

Infimum of a set

Df(27). Let $E \subset \mathbb{R}$ be nonempty.

(i) The set E is said to be bounded below iff $\exists\, m \in \mathbb{R}$ s.t. $m \leq x,\ \forall x \in E$, in this case, m is said to be a lower bound of E

(ii) A number α is called an infimum of the set E iff α is a lower bound of E and $\alpha \geq \gamma$ for all lower bounds γ of E. In this case we shall say that E has an infimum α and write $\boxed{\inf E = \alpha}$

When $\inf E$ exists, it is the greatest lower bound of E.

(iii) E is said to be bounded iff it is bounded above and below. (That is, $\exists\, m, M$ s.t. $m \leq x \leq M,\ \forall x \in E$

OR $\exists M>0$ s.t., $|x|\le M$, $\forall x\in E$).

Ex. prove. A bounded nonempty set E has a unique Supremum and unique infimum. Moreover, $\inf E\le \sup E$. Give necessary and sufficient conditions for equality.

Rmk. When a set E contains its supremum, we write $\max E=\sup E$. Similarly, if $\inf E\in E$, we write $\inf E=\min E$.

Thm 5. [Reflection Principle].

Let $E\subseteq\mathbb{R}$ be nonempty.

(i) E has a supremum iff $-E$ has an infimum, in which case, $\boxed{\inf(-E)=-\sup E}$

(ii) E has an infimum iff $-E$ has a supremum, in which case, $\boxed{\sup(-E)=-\inf(E)}$.

Proof. The proof of these statements are similar, we prove only (i).

($\Rightarrow$) Suppose that $\beta = \sup E$ exists.

Since β is an upper bound of E, $x \le \beta$, $\forall x \in E$.

This gives $-\beta \le -x$, $\forall x \in E$, i.e., $-\beta$ is a lower bound of $-E$.

Suppose that m is any lower bound of $-E$ then $m \le -x$, $\forall x \in E$. This implies,

$x \le -m$, $\forall x \in E$, i.e., $-m$ is an upper bound of E. Since $\sup E = \beta$, then $\beta \le -m$ (i.e., $-\beta \ge m$). Thus, $-\beta$ is the infimum of $-E$ and $\sup E = \beta = -(-\beta) = -\inf(-E)$.

($\Leftarrow$) Conversely, Suppose that $-E$ has an infimum α. By def'n, $\alpha \le -x$, $\forall x \in E$.

thus, $-\alpha$ is an upper bound of E. Since $E \neq \phi$, then E has a supremum by the Completeness Axiom. ▤

Thm 6. [Monotone Property].

Suppose that $A \subseteq B$ are nonempty sets of $\mathbb{R}$.

(i) If B has a supremum, then $\sup A \leq \sup B$,

(ii) If B has an infimum, then $\inf A \geq \inf B$.

Proof. (i) Since $A \subseteq B$, any upper bound of B is an upper bound of A. Therefore, $\sup B$ is an upper bound of A. It follows that by the Completeness Axiom that $\sup A$ exists. thus, by def'n of $\sup A$, $\sup A \leq \sup B$.

(ii) Since $A \subseteq B$, then $-A \subseteq -B$. By part (i), $\sup(-A) \leq \sup(-B)$.

By thm (5), we have $-\inf A \leq -\inf B$

$\Rightarrow \inf A \geq \inf B$. ∎

Thm 7. (Approximation Property for infima).

If a set $E \subseteq \mathbb{R}$ has a finite infimum α and $\varepsilon > 0$ is any positive number, then there is a point $x \in E$ such that

$$\alpha + \varepsilon > x \geq \alpha$$

Proof. Exercise 1.3.6 (a).

Completeness Property For Infima

If $E \subseteq \mathbb{R}$ is nonempty and bounded below, then E has a finite infimum.

Proof. Exercise 1.3.6 (b).

The extended real numbers

- $$\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}.$$

Thus, x is an extended real number iff either $x \in \mathbb{R}$, $x = +\infty$, or $x = -\infty$.

- $\phi \neq E \subseteq \mathbb{R}$ is unbounded above if it has no upper bound and unbounded below if it has no lower bound.

- Let $\phi \neq E \subseteq \mathbb{R}$. We define $\sup E = \infty$ if E is unbounded above and $\inf E = -\infty$ if E is unbounded below.
- We define $\sup \phi = -\infty$, $\inf \phi = \infty$.

Ex. Let $E_1 = (-\infty, 2)$, $E_2 = (2, \infty)$.

$\sup E_1 = 2$, $\inf E_1 = -\infty$.

$\sup E_2 = \infty$, $\inf E_2 = 2$.

Ex. $\sup \mathbb{Z} = \infty$, $\inf \mathbb{Z} = -\infty$

$\sup \mathbb{N} = \infty$, $\inf \mathbb{N} = 1$

$\sup \mathbb{R} = \infty$, $\inf \mathbb{R} = -\infty$.

CH2. Sequences in $\mathbb{R}$

2.1 limits of sequences.

- An infinite sequence (briefly, a sequence) is a function whose domain is $\mathbb{N}$.

- A sequence $x_n := f(n)$ will be denoted by

  $x_1, x_2, \text{-----}$ OR $\{x_n\}_{n\in\mathbb{N}}$, OR $\{x_n\}$.

Ex. ① $\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \text{---}\}$ represents the sequence $\{\frac{1}{2^{n-1}}\}_{n\in\mathbb{N}}$

② $\{-1, 1, -1, 1, \text{----}\}$ is the sequence $\{(-1)^n\}_{n\in\mathbb{N}}$.

③ $\{1, 2, 3, 4, \text{---}\}$ " " " $\{n\}_{n\in\mathbb{N}}$.

Important $\{x_n\}_{n\in\mathbb{N}}$ is not the set $\{x_n : n\in\mathbb{N}\}$.

ex. $\{1, 2, 3, \text{-----}\}$ is different from $\{2, 1, 3, \text{-----}\}$ as sequences. But as sets $\{1, 2, 3, \text{----}\}$ is identical with $\{2, 1, 3, \text{-----}\}$.

ex. $\{1, -1, 1, -1, \ldots\}$ is infinite but the set $\{(-1)^n : n \in \mathbb{N}\} = \{-1, 1\}$ only.

Df. ① A sequence of real numbers $\{x_n\}$ is said to converge to $a \in \mathbb{R}$ iff

$\forall \varepsilon > 0$, $\exists$ an $K \in \mathbb{N}$ (in general $K(\varepsilon)$)

such that $\boxed{n \geq K \Rightarrow |x_n - a| < \varepsilon}$

Notations (a) $\{x_n\}$ converges to a.

(b) x_n converges to a.

(c) $\lim_{n \to \infty} x_n = a$

(d) $x_n \to a$ as $n \to \infty$

(e) the limit of $\{x_n\}$ exists and equals a.

Rmks. (1) when $x_n \to a$ as $n \to \infty$, you can think of x_n as a sequence of approximations to a and ε as an upper bound for the error

(2) the number K in Df① is choosen so that the error is less than ε when $n \geq K$

In general, the smaller ε gets, the Larger K must be.

(3) $x_n \longrightarrow a$ iff $|x_n - a| \longrightarrow 0$ as $n \longrightarrow \infty$

In particular, $x_n \longrightarrow 0$ iff $|x_n| \longrightarrow 0$ as $n \longrightarrow \infty$.

(4) K depends on ε CANNOT depend on n

(5) (Summary of Df①). $x_n \longrightarrow a \iff |x_n - a|$ is small for large n.

Ex 1. Prove that $\frac{1}{n} \longrightarrow 0$ as $n \longrightarrow \infty$.

Pf. Let $\varepsilon > 0$ be given. We need to find $K \in \mathbb{N}$ s.t. $n \geq K \Longrightarrow \left|\frac{1}{n} - 0\right| = \frac{1}{n} < \varepsilon$. Use the Archimedean principle $\exists K \in \mathbb{N}$ s.t $K > \frac{1}{\varepsilon}$.

Now, $n \geq K \Rightarrow \frac{1}{n} \leq \frac{1}{K} < \varepsilon$. It follows that $\left|\frac{1}{n} - 0\right| = \frac{1}{n} < \varepsilon$, $\forall n \geq K$. ∎

Ex 2. If $\lim_{n\to\infty} x_n = 2$, prove that

$$\lim_{n\to\infty}\left(\frac{2x_n+1}{x_n}\right) = \frac{5}{2}.$$

Proof. Let $\varepsilon>0$ be given. Since $x_n \to 2$, Apply Df① to this $\varepsilon>0$, $\exists\, K_1\in\mathbb{N}$ such that

$n\geq K_1 \implies |x_n-2|<\varepsilon$. Next, apply Df① with $\varepsilon=1$, $\exists\, K_2\in\mathbb{N}$ such that

$n\geq K_2 \implies |x_n-2|<1$. That is,

$n\geq K_2 \implies x_n>1$ (i.e., $2x_n>2$).

Set $K=\max\{K_1,K_2\}$ and suppose that $n\geq K$. Since $n\geq K_1$, we have

$|2-x_n| = |x_n-2|<\varepsilon$. Since $n\geq K_2$, we have $0<\frac{1}{2x_n}<\frac{1}{2}<1$. It follows that

$$\left|\frac{2x_n+1}{x_n}-\frac{5}{2}\right| = \left|\frac{2-x_n}{2x_n}\right| = \frac{|x_n-2|}{2x_n} < \frac{\varepsilon}{2x_n} < \varepsilon,$$

for all $n\geq K$. ∎

Ex: Show that the sequence $\{(-1)^n\}_{n\in\mathbb{N}}$ has no limit.

Proof: Spse that $(-1)^n \longrightarrow \alpha$ as $n\longrightarrow\infty$ for some $\alpha\in\mathbb{R}$. Given $\varepsilon=1$, $\exists$ a $K\in\mathbb{N}$ s.t.

$$n\geq K \Longrightarrow |(-1)^n-\alpha|<1.$$

For n odd this implies $|1+\alpha|=|-1-\alpha|<1$, and for n even this implies $|1-\alpha|<1$.

Hence, $2=|1+1|=|1-\alpha+\alpha+1|$
$$\leq |1-\alpha|+|1+\alpha|$$
$$<1+1=2,$$ i.e $2<2$, a contradiction ■

Rmk: A sequence can have at most one limit.

Proof: Spse that $x_n\longrightarrow\alpha$ and $x_n\longrightarrow\beta$ as $n\to\infty$

By def'n; $\forall\varepsilon>0$, $\exists$ a $K\in\mathbb{N}$ s.t.

$$n\geq K \Longrightarrow |x_n-\alpha|<\frac{\varepsilon}{2} \text{ and } |x_n-\beta|<\varepsilon/2$$

Thus, it follows from triangle inequality that

$$|\alpha-\beta| = |(\alpha - x_n) + (x_n - \beta)|$$
$$\leq |\alpha - x_n| + |x_n - \beta|$$
$$= |x_n - \alpha| + |x_n - \beta|$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon$$

i.e., $|\alpha-\beta| < \varepsilon$, $\forall \varepsilon > 0$. We conclude that $\alpha = \beta$ (see Thm in section 1.2). ■

Df ②. A subsequence of a sequence $\{x_n\}_{n\in\mathbb{N}}$ is a sequence of the form $\{x_{n_k}\}_{k\in\mathbb{N}}$, where each $n_k \in \mathbb{N}$ and $n_1 < n_2 < \cdots$

Thus, a subsequence $x_{n_1}, x_{n_2}, \ldots$ of $x_1, x_2, \ldots$ is obtained by deleting from $x_1, x_2, \ldots$ all x_n's except those such that $n = n_k$ for some k.

Ex. $\{1, 1, 1, \ldots\}$ is a subsequence of $\{-1, 1, -1, 1, \ldots\}$ by deleting every other terms (set $n_k = 2k$).

توضيح

$$\{-1, 1, -1, 1, \ldots\}$$

$-1 \to a_1$, $1 \to a_2 \to a_{n_1}$, $-1 \to a_3$, $1 \to a_4 = a_{n_2}$

Exercises

① prove that If x_n converges to $a \in \mathbb{R}$, then $\frac{x_n}{n} \to 0$.

Pf. spse that $x_n \longrightarrow a$ as $n \longrightarrow \infty$. Since x_n converges, then it is bdd, i.e., $\exists$ an $M>0$ s.t. $|x_n| \leq M$, $\forall n \in \mathbb{N}$. Let $\varepsilon>0$ be given, we need to find $K \in \mathbb{N}$ s.t

$$n \geq K \Longrightarrow \left|\frac{x_n}{n} - 0\right| = \frac{|x_n|}{n} < \varepsilon.$$

Use the Archimedean principle $\exists$ $K \in \mathbb{N}$ s.t $K > \frac{M}{\varepsilon}$. Then $n \geq K$ implies

$$\left|\frac{x_n}{n} - 0\right| = \frac{|x_n|}{n} \leq \frac{|x_n|}{K} \leq \frac{M}{K} < \varepsilon.$$

∎

② (True) or (False)

a) If x_n conv. & y_n bdd, then $x_n y_n$ conv.

Ans. False Take $x_n = 1$ conv., $y_n = (-1)^n$ bdd but $x_n y_n = (-1)^n$ does not conv.

b) If $x_n \longrightarrow 0$ and $y_n > 0$, $\forall n \in \mathbb{N}$, then $x_n y_n$ converges.

Ans. False. Take $x_n = \frac{1}{n}$ conv., $y_n = n^2 > 0$ but $x_n y_n = n$ div.

(3) Prove that $\lim\limits_{n\to\infty} \frac{2n^2+1}{3n^2} = \frac{2}{3}$

Pf: Let $\varepsilon > 0$ be given. We need to find a $K \in \mathbb{N}$ s.t $n \geq K \Longrightarrow \left| \frac{2n^2+1}{3n^2} - \frac{2}{3} \right| < \varepsilon$. Use the Archimedean principle $\exists K \in \mathbb{N}$ s.t.

$K > \frac{1}{\sqrt{3\varepsilon}}$. Thus $n \geq K$ implies

$$\left| \frac{2n^2+1}{3n^2} - \frac{2}{3} \right| = \left| \frac{6n^2+3-6n^2}{9n^2} \right|$$
$$= \frac{1}{3n^2} \leq \frac{1}{3K^2} < \frac{(\sqrt{3\varepsilon})^2}{3} = \varepsilon$$

(4) Let c be fixed, positive constant. If $\{b_n\}$ is a seq. s.t $b_n \geq 0$ and $b_n \to 0$, and $\{x_n\}$ is a real seq. that satisfies

$$|x_n - a| \leq c\, b_n \text{ for large } n.$$

Prove that $x_n \to a$ as $n \to \infty$.

Proof: Let $\varepsilon > 0$ be given. Since $b_n \to 0$, then $\exists K \in \mathbb{N}$ s.t $n \geq K \Rightarrow |b_n - 0| = b_n < \frac{\varepsilon}{c}$.

Hence by hypothesis, $n \geq K$ implies

$|x_n - a| \leq C b_n < C \cdot \frac{\varepsilon}{C} = \varepsilon$. therefore, by def'n, $x_n \to a$ as $n \to \infty$ ∎

5. prove that if $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \beta$, then

$$\lim_{n\to\infty} (x_n - y_n) = 0$$

Pf: Let $\varepsilon > 0$ be given. Since $x_n \to \beta$ and $y_n \to \beta$, by def'n, $\exists$ a $K \in \mathbb{N}$ s.t.

$$n \geq K \implies |x_n - \beta| < \varepsilon/2 \text{ and } |y_n - \beta| < \varepsilon/2.$$

By the triangle inequality, $n \geq K$ implies

$$|(x_n - y_n) - 0| = |x_n - y_n| = |x_n - \beta + \beta - y_n|$$
$$\leq |x_n - \beta| + |y_n - \beta|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

By def'n, $x_n - y_n \to 0$ as $n \to \infty$ ∎

2.2 limit Thms

Thm ① (Squeeze thm).
Suppose that $\{x_n\}$, $\{y_n\}$, and $\{w_n\}$ are real sequences. then

(i) If $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \alpha$, and if

$\exists$ an $N_0 \in \mathbb{N}$ s.t.

$$x_n \leq w_n \leq y_n \quad , \forall n \geq N_0,$$

then $\lim_{n\to\infty} w_n = \alpha$.

(ii) If $\lim_{n\to\infty} x_n = 0$ and $\{y_n\}$ is bdd,

then $\lim_{n\to\infty} (x_n y_n) = 0$.

Proof. (i) Let $\varepsilon > 0$ be given. Since x_n and y_n converge to α, by def'n, $\exists$ $N_1, N_2 \in \mathbb{N}$

s.t. $n \geq N_1 \implies -\varepsilon < x_n - \alpha < \varepsilon$

$n \geq N_2 \implies -\varepsilon < y_n - \alpha < \varepsilon$.

Set $N = \max\{N_0, N_1, N_2\}$. If $n \geq N$, we have by the hypothesis & the choise of

N_1 and N_2 that

$$\alpha-\varepsilon < x_n \le w_n \le y_n < \alpha+\varepsilon$$

i.e., $\alpha-\varepsilon < w_n < \alpha+\varepsilon$, for $n \ge N$.

or $|w_n-\alpha| < \varepsilon$, for $n \ge N$.

We conclude that $w_n \to \alpha$ as $n \to \infty$.

(ii) Spse that $\lim\limits_{n\to\infty} x_n = 0$ and $\{y_n\}$ is bdd.

Since $\{y_n\}$ is bdd, this means that $\exists$ an $M>0$ s.t. $|y_n| \le M$, $\forall n \in \mathbb{N}$. Let $\varepsilon>0$, $\exists$ an $N \in \mathbb{N}$ s.t. $n \ge N \Longrightarrow |x_n| < \frac{\varepsilon}{M}$ (since $x_n \to 0$).

then $n \ge N$ implies

$$|x_n y_n - 0| = |x_n y_n| = |x_n||y_n| < \frac{\varepsilon}{M} M = \varepsilon.$$

We conclude that $x_n y_n \to 0$ as $n \to \infty$. ∎

Ex. Find $\lim\limits_{n\to\infty} \dfrac{\cos(n^3-n^2+n-13)}{2^n}$

Sol. Since $|\cos x| \le 1$, $\forall x \in \mathbb{R}$, then

$$\left| \frac{\cos(n^3-n^2+n-13)}{2^n} \right| \le 2^{-n}$$

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Since $2^n > n$ (why?), it is clear $2^{-n} < \frac{1}{n}$.

$$\Rightarrow \quad \underbrace{-\frac{1}{n}}_{\to 0 \text{ as } n\to\infty} < \frac{\cos(n^3-n^2+n-13)}{2^n} < \underbrace{\frac{1}{n}}_{\to 0 \text{ as } n\to\infty}$$

$$\Rightarrow \lim_{n\to\infty} \frac{\cos(n^3-n^2+n-13)}{2^n} = 0 \text{ by Squeeze thm}$$

Rmk. The squeeze thm can be used to construct convergent sequences with certain properties.

We now establish a result that connects suprema & infima with convergent sequences.

Thm (2). Let $E \subset \mathbb{R}$. If E has a finite sup (resp., a finite inf), then $\exists$ a seq. $x_n \in E$ s.t. $\lim_{n\to\infty} x_n = \sup E$ (resp., a seq. $y_n \in E$ s.t. $\lim_{n\to\infty} y_n = \inf E$)

proof: Spse that E has a finite sup β.
For each $n \in \mathbb{N}$, $\exists$ (by the Approximation Property for suprema), an $x_n \in E$ s.t

$$\beta - \frac{1}{n} < x_n \leq \beta.$$

Then by the squeeze Thm (Thm ①),

$$\lim_{n\to\infty} x_n = \beta = \sup E.$$

Similarly, $\exists$ a seq. $y_n \in E$ s.t

$$\inf E \leq y_n < \inf E + \frac{1}{n}.$$

Then by the squeeze thm $\lim_{n\to\infty} y_n = \inf E$

Thm ③. Spse that $\{x_n\}$ & $\{y_n\}$ are real sequences and that $\alpha \in \mathbb{R}$. If $\{x_n\}$ and $\{y_n\}$ are convergent, then

(i) $\lim_{n\to\infty}(x_n + y_n) = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n$

(ii) $\lim_{n\to\infty}(\alpha x_n) = \alpha \lim_{n\to\infty} x_n$

and

(iii) $\lim_{n\to\infty}(x_n y_n) = \left(\lim_{n\to\infty} x_n\right)\left(\lim_{n\to\infty} y_n\right)$.

If, in addition, $y_n \neq 0$ and $\lim_{n\to\infty} y_n \neq 0$, then

(iv) $\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{\lim_{n\to\infty} x_n}{\lim_{n\to\infty} y_n}$.

Proof: Spse that $x_n \to x$ and $y_n \to y$ as $n \to \infty$

(i) let $\varepsilon > 0$ be given. then $\exists K \in \mathbb{N}$ s.t

$n \geq K \implies |x_n - x| < \frac{\varepsilon}{2}$ and $|y_n - y| < \frac{\varepsilon}{2}$.

Thus $n \geq K$ implies

$$\begin{aligned} |(x_n+y_n) - (x+y)| &= |x_n - x + y_n - y| \\ &\leq |x_n - x| + |y_n - y| \quad \text{(triangle inequality)} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

We conclude that $x_n + y_n \to x+y$ as $n \to \infty$

(ii) & (iv) (Exercises)

(iii) since $\{x_n\}$ conv., then it is bdd.

Hence by the sequence Thm (ii), then sequences

$$\underbrace{x_n}_{\text{bdd}}\underbrace{(y_n - y)}_{\to 0} \longrightarrow 0 \quad \text{as } n \to \infty$$

and $\underbrace{(x_n - x)}_{\to 0}\underbrace{y}_{\text{bdd}} \longrightarrow 0$ as $n \to \infty$. thus,

$$\begin{aligned} &\Rightarrow \lim_{n\to\infty} (x_n y_n - xy) \\ &= \lim_{n\to\infty} \left[x_n(y_n - y) + (x_n - x)y\right] \\ &= \lim_{n\to\infty} x_n(y_n - y) + \lim_{n\to\infty}(x_n - x)y \quad \text{(part (i))} \\ &= 0 \end{aligned}$$

. We conclude that $x_n y_n \to xy$ as $n \to \infty$

Df ① Let $\{x_n\}$ be a sequence of real numbers.

(i) $\{x_n\}$ is said to be diverge to $+\infty$

(notation: $x_n \to +\infty$ as $n \to \infty$ or $\lim_{n\to\infty} x_n = +\infty$)

iff $\forall M \in \mathbb{R}$, $\exists$ an $N \in \mathbb{N}$ s.t.

$$n \geq N \implies x_n > M.$$

(ii) $\{x_n\}$ is said to be diverge to $-\infty$

(notation: $x_n \to -\infty$ as $n \to \infty$ or $\lim_{n\to\infty} x_n = -\infty$)

iff $\forall M \in \mathbb{R}$, $\exists$ an $N \in \mathbb{N}$ s.t.

$$n \geq N \implies x_n < M.$$

Rmk. (Def ①) $x_n \to +\infty$ iff given $M \in \mathbb{R}$, x_n is greater than M for sufficiently large n, i.e., eventually x_n exceeds every number M (no matter how large and positive M is).

(2) $x_n \to -\infty$ iff x_n eventually is less than every number M (no matter how large and negative M).

Examples [page 12]

Thm (4) (An extension of th (3))

Spse that $\{x_n\}$ and $\{y_n\}$ are real sequences s.t. $x_n \to +\infty$ (resp. $x_n \to -\infty$) as $n \to \infty$.

(1) If y_n is bdd below (resp., y_n is bdd above) then $\lim_{n\to\infty}(x_n+y_n) = +\infty$ (resp. $\lim_{n\to\infty}(x_n+y_n) = -\infty$)

(2) If $\alpha > 0$, then

$$\lim_{n\to\infty}(\alpha x_n) = +\infty \left(\text{resp. } \lim_{n\to\infty}(\alpha x_n) = -\infty \right).$$

(3) If $y_n > M_0$, for some $M_0 > 0$ and all $n \in \mathbb{N}$, then

$$\lim_{n\to\infty}(x_n y_n) = +\infty \quad \left(\text{resp. } \lim_{n\to\infty}(x_n y_n) = -\infty \right)$$

(4) If $\{y_n\}$ is bdd and $x_n \neq 0$, then $\lim_{n\to\infty} \frac{y_n}{x_n} = 0$.

Proof: Spse for simplicity that $x_n \to +\infty$ as $n \to \infty$.

(1) By hypothesis, $y_n \geq M_0$ for some $M_0 \in \mathbb{R}$.

Let $M \in \mathbb{R}$ and set $M_1 = M - M_0$.

Since $x_n \to +\infty$, $\exists N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow x_n > M_1$.

Then $n \geq N \Rightarrow x_n + y_n > M_1 + M_0 = M$.

(2) Let $M \in \mathbb{R}$ and set $M_1 = \frac{M}{\alpha}$.

Since $x_n \to +\infty$, $\exists N \in \mathbb{N}$ s.t

$n \geq N \Rightarrow x_n > M_1$. Since $\alpha > 0$,

we conclude that $\alpha x_n > \alpha M_1 = M$, $\forall n \geq N$.

(3) Let $M \in \mathbb{R}$ and set $M_1 = \frac{M}{M_0}$. Since

$x_n \to +\infty$, $\exists N \in \mathbb{N}$ s.t $n \geq N \Rightarrow x_n > M_1$

then $n \geq N \Rightarrow x_n y_n > M_1 y_n > M_1 M_0 = M$.

(4) Let $\varepsilon > 0$. Since $\{y_n\}$ is bdd, then $\exists M_0 > 0$

s.t. $|y_n| \leq M_0$. Since $x_n \to \infty$ as $n \to \infty$,

then $\exists N \in \mathbb{N}$ s.t $n \geq N \Rightarrow x_n > M_1$.

Choose M_1 so large that $\frac{M_0}{M_1} < \varepsilon$.

Then $n \geq N \Longrightarrow \left|\frac{y_n}{x_n}\right| = \frac{|y_n|}{x_n} < \frac{M_0}{M} < \varepsilon$. ∎

Rmk. We use the conventions

1) $x + \infty = \infty$, $x - \infty = -\infty$, $x \in \mathbb{R}$.

2) $x \cdot \infty = \infty$, $x \cdot (-\infty) = -\infty$, $x > 0$.

3) $x \cdot \infty = -\infty$, $x \cdot (-\infty) = \infty$, $x < 0$.

4) $\infty + \infty = \infty$, $-\infty - \infty = -\infty$.

5) $\infty \cdot \infty = (-\infty) \cdot (-\infty) = \infty$, and

$\infty \cdot (-\infty) = (-\infty) \cdot \infty = -\infty$.

If we use the above conventions, then Thm 4 contains the following corollary.

Corollary Let $\{x_n\}, \{y_n\}$ be real sequences and α, x, y be extended real numbers. If $x_n \longrightarrow x$ and $y_n \longrightarrow y$, as $n \longrightarrow \infty$, then

$\lim_{n\to\infty} (x_n + y_n) = x + y$, provided $x + y$ is not of the form $\infty - \infty$ and

$\lim_{n\to\infty}(\alpha x_n) = \alpha x$, $\lim_{n\to\infty}(x_n y_n) = xy$

provided that none of these products is of the form $0 \cdot \pm\infty$.

Thm5 [Comparison Thm]
Spse that $\{x_n\}$ & $\{y_n\}$ are convergent sequences. If there is an $N_0 \in \mathbb{N}$ s.t.

(*) $\boxed{x_n \le y_n \text{ for } n \ge N_0}$, then

$$\lim_{n\to\infty} x_n \le \lim_{n\to\infty} y_n.$$

In particular, if $x_n \in [a,b]$, converges to some point c, then c must belong to $[a,b]$.

Proof: Spse that the first statement is false, i.e., that (*) holds but $x := \lim_{n\to\infty} x_n > y := \lim_{n\to\infty} y_n$.

Set $\varepsilon = \frac{x-y}{2}$. Choose $N > N_0$ s.t.
$|x_n - x| < \varepsilon$ and $|y_n - y| < \varepsilon$ for all $n \ge N_1$.

Then for such an $n \geq N$,

$$x_n > x - \varepsilon = x - \left(\frac{x-y}{2}\right) = y + \left(\frac{x-y}{2}\right) = y + \varepsilon > y_n,$$

$\Rightarrow$ $x_n > y_n$, which contradicts (*). This proves the first statement.

To prove the second statement, we conclude $a \leq x_n \leq b$, then by the first statement $\lim_{n\to\infty} a \leq \lim_{n\to\infty} x_n \leq \lim_{n\to\infty} b$. This implies $a \leq c \leq b$. ∎

انتبه!!
Rmk. $x_n < y_n$, $n \geq N_0$ **Does Not imply** that $\lim_{n\to\infty} x_n < \lim_{n\to\infty} y_n$.

counter example, $\frac{1}{n^2} < \frac{1}{n}$, but $\lim_{n\to\infty} \frac{1}{n^2} \not< \lim_{n\to\infty} \frac{1}{n}$

2.2.2 (a) prove that $\lim_{n\to\infty}(n^2-n)=\infty$.

Pf. Let $M\in\mathbb{R}$. Use Archimedean principle $\exists$ an $N\in\mathbb{N}$ s.t. $N>\max\{M,2\}$. Then

$n\geq N \Rightarrow x_n=n^2-n=n(n-1)>N(N-1)>M(2-1)=M$ ⊠

(b) $\lim_{n\to\infty}(n-3n^2)=-\infty$

Pf. Let $M\in\mathbb{R}$, by Archimedean principle, $\exists$ an $N\in\mathbb{N}$ s.t $N>-\frac{M}{2}$. Notice that

$n\geq 1 \Rightarrow -3n\leq -3$ so $1-3n<-2$. Thus,

$n\geq N \Rightarrow x_n=n-3n^2=n(1-3n)\leq -2n\leq -2N<M$ ⊠

Ex. show that $\lim_{n\to\infty}\left(\frac{n}{2}+\frac{1}{n}\right)=\infty$.

Pf. Let $M\in\mathbb{R}$, by Archimedean principle, $\exists$ an $N\in\mathbb{N}$ s.t $N>2M$. Then

$n\geq N \Rightarrow x_n=\frac{n}{2}+\frac{1}{n}\geq\frac{N}{2}+\frac{1}{n}>\frac{N}{2}>\frac{2M}{2}=M.$ ⊠

$\boxed{\text{H.w's}\quad 0,1,2,3,4,5,6,7,8}$

2.3 Bolzano – Weirstrass theorem

Notice that the seq $\{(-1)^n\}$ does not converse but it has convergent subsequence. In this section, we will prove that this is a general principle. That is every bounded seq. has a convergent subsequence.

Df ①. Let $\{x_n\}_{n\in\mathbb{N}}$ be a seq. of real numbers

(i) $\{x_n\}$ is said to be increasing (resp., strictly increasing) iff $x_1 \leq x_2 \leq \cdots$ (resp. $x_1 < x_2 < \cdots$)

(ii) $\{x_n\}$ is said to be decreasing (resp., strictly decreasing) iff $x_1 \geq x_2 \geq \cdots$ (resp. $x_1 > x_2 > \cdots$)

(iii) $\{x_n\}$ is said to be monotone iff it is either increasing or decreasing.

Rmk. ① Some times, we call decreasing seq. nonincreasing and increasing seq. nondecreasing.

(2) If $\{x_n\}$ is increasing (resp., decreasing) and $x_n \longrightarrow a$ as $n \longrightarrow \infty$, we shall write $x_n \uparrow a$ (resp. $x_n \downarrow a$), as $n \longrightarrow \infty$.

(3) Every strictly increasing seq. is increasing and every strictly decreasing seq. is decreasing.

(4) $\{x_n\}$ is increasing iff the sequence $\{-x_n\}$ is decreasing.

We know that any convergent seq. is bounded. We now establish the converse for ~~the~~ monotone sequences.

Thm (1). [Monotone convergence thm]
If $\{x_n\}$ is increasing and bounded above, or $\{x_n\}$ is decreasing and bounded below, then $\{x_n\}$ converges to a finite limit.

proof. Spse first that $\{x_n\}$ is increasing and bounded above. By the Completeness

Axiom, the supremum $\beta := \sup\{x_n : n \in \mathbb{N}\}$ exists and is finite. Let $\varepsilon > 0$. By the Approximation property for Suprema, choose $N \in \mathbb{N}$ s.t

$$\beta - \varepsilon < x_N \leq \beta.$$

Since $x_N \leq x_n$ for $n \geq N$ and $x_n \leq \beta$ for all $n \in \mathbb{N}$, it follows that

$$\beta - \varepsilon < x_n \leq \beta, \text{ for all } n \geq N.$$

In particular, $x_n \uparrow a$ as $n \to \infty$.

If $\{x_n\}$ is decreasing with $\alpha := \inf\{x_n : n \in \mathbb{N}\}$, then $\{-x_n\}$ is increasing with supremum $-\alpha$. Hence, by the first case,

$$\alpha = -(-\alpha) = -\left(\lim_{n\to\infty} -x_n\right) = \lim_{n\to\infty} x_n$$

$\blacksquare$

Ex. If $|x| < 1$, then $\lim_{n\to\infty} x^n = 0$.

Pf. It suffices to prove that

$$|x|^n \to 0 \quad \text{as } n \to \infty.$$

First, we notice that $|x|^n$ is monotone decreasing since, $|x|<1$ implies $|x|^{n+1}<|x|^n, \forall n\in\mathbb{N}$.

Next, notice that $|x|^n$ is bounded below (by 0). Hence by the Monotone Convergence Thm, $\lim |x|^n := L$ exists. To find this limit, $\lim\limits_{n\to\infty} |x|^{n+1} = \lim\limits_{n\to\infty} |x|^n \cdot |x|$.

this implies $L = |x| \cdot L$, thus either $L=0$ or $|x|=1$. since $|x|<1$, we conclude that $L=0$. ∎

ex. If $x>0$, then $\lim\limits_{n\to\infty} x^{\frac{1}{n}} = 1$.

proof. We consider three cases:

Case 1: $x=1$. Then $x^{\frac{1}{n}}=1, \forall n\in\mathbb{N}$ and it follows that $\lim\limits_{n\to\infty} x^{\frac{1}{n}} = \lim\limits_{n\to\infty} 1 = 1$.

Case 2: $x>1$, we shall apply the MCT. We shall show that $\{x^{\frac{1}{n}}\}$ is decreasing and bdd below. Indeed, Since $x>1$, then $x^{n+1} > x^n$. Taking the $n(n+1)$st root of this inequality, we obtain $x^{\frac{1}{n}} > x^{\frac{1}{n+1}}$, i.e., $\{x^{\frac{1}{n}}\}$ is <u>decreasing</u>. Since $x>1$ implies $x^{\frac{1}{n}}>1$, it follows that $\{x^{\frac{1}{n}}\}$ is <u>bounded below</u>. Hence, by the MCT, $L := \lim_{n\to\infty} x^{\frac{1}{n}}$ exists. To find its value L, we have

$$\lim_{n\to\infty} x^{\frac{1}{n}} = \lim_{n\to\infty} \left(x^{\frac{1}{2n}}\right)^2 = \left(\lim_{n\to\infty} x^{\frac{1}{2n}}\right)^2$$

$$\Rightarrow L = L^2, \text{ i.e., } L=0 \text{ or } L=1.$$

Since $x^{\frac{1}{n}}>1$, the comparison then shows that $\lim_{n\to\infty} x^{\frac{1}{n}} \geq \lim_{n\to\infty} 1$, i.e., $L \geq 1$.

Hence $L=1$.

<u>Case 3</u>. $0<x<1$. Then $\frac{1}{x}>1$. It follows from Case 2 that

$$\lim_{n\to\infty} x^{\frac{1}{n}} = \frac{1}{\lim_{n\to\infty}\left(\frac{1}{x}\right)^{\frac{1}{n}}} = 1$$

Df. ② A sequence of sets $\{I_n\}_{n\in\mathbb{N}}$ is said to be nested iff $I_1 \supseteq I_2 \supseteq \cdots$

Rmk. this is a monotone property for sequence of sets.

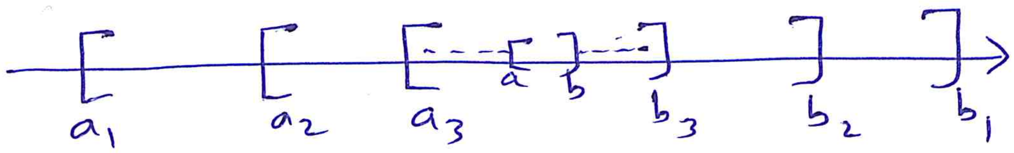
Thm ②. [Nested Interval property]

If $\{I_n\}_{n\in\mathbb{N}}$ is a nested sequence of nonempty closed bdd intervals, then

$E := \bigcap_{n=1}^{\infty} I_n \neq \phi$. Moreover, if the lengths of these intervals satisfy

$|I_n| \to 0$ as $n \to \infty$, then E is a single point.

proof: Let $I_n = [a_n, b_n]$. Since $\{I_n\}$ is nested, then the real seq. $\{a_n\}$ is increasing and bdd above by b_1, and $\{b_n\}$ is decreasing and bdd below by a_1 (see Fig.)



Thus, by MCT, $\exists$ $a, b \in \mathbb{R}$ such that $a_n \uparrow a$ and $b_n \downarrow b$ as $n \to \infty$. Since $a_n \leq b_n, \forall n \in \mathbb{N}$, it follows from the Comparison thm that $a_n \leq a \leq b \leq b_n$.

Hence, $x \in I_n, \forall n \in \mathbb{N}$ iff $x \in [a,b]$.

In partical, any $x \in [a,b]$ belongs to all the I_n's i.e., any $x \in [a,b]$, $x \in \bigcap_{n=1}^{\infty} I_n$ (i.e., $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$).

Next, If $|I_n| \to 0$ as $n \to \infty$, then

$$\lim_{n\to\infty} (b_n - a_n) = 0 \Rightarrow \lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n$$

$$b = a.$$

But we have proved that $x \in \bigcap_{n=1}^{\infty} I_n$ iff $x \in [a,b]$.

Hence, £ is a single point if $\lim_{n\to\infty} |I_n| = 0$ ■

<u>Rmk ①</u>. The Nested Interval property (Thm 2) might not hold if "closed" is <u>omitted</u>.

Proof: $I_n = (0, \frac{1}{n})$, $n \in \mathbb{N}$ are bdd and nested ($I_1 = (0,1) \supset I_2 = (0, \frac{1}{2}) \supset \cdots$). but not closed. If there were an $x \in I_n$, $\forall n \in \mathbb{N}$, then $0 < x < \frac{1}{n}$, i.e., $n < \frac{1}{x}$, for all $n \in \mathbb{N}$. Since this contradicts the Archimedean Principle, it follows that

$$\bigcap_{n=1}^{\infty} I_n = \phi. \blacksquare$$

(2) the Nested Interval Property (thm 2) might not hold if "bounded" is omitted.

proof: The intervals $I_n = [n, \infty)$, $n \in \mathbb{N}$ are closed and nested but not bdd.

Again, $\bigcap_{n=1}^{\infty} I_n = \phi$. $\blacksquare$

We are now prepared to establish the main result of this section.

Thm ③. [Bolzano–Weirstrass Thm]

Every bounded sequence of real numbers has a convergent subsequence

proof: Let $\{x_n\}$ be a bdd sequence. Choose $a, b \in \mathbb{R}$ such that $x_n \in [a, b]$, $\forall n \in \mathbb{N}$, and set $I_0 = [a, b]$. Divide I_0 into two halves, say $I' = [a, \frac{a+b}{2}]$ and $I'' = [\frac{a+b}{2}, b]$. Since $I_0 = I' \cup I''$, then $x_n \in I'$ or $x_n \in I''$, for infinitely many n. Say $x_n \in I_1$ and choose $n_1 > 1$ such that $x_{n_1} \in I_1$. Notice that $|I_1| = \frac{|I_0|}{2} = \frac{b-a}{2}$.

Spse that closed intervals $I_0 \supset I_1 \supset \dots \supset I_m$ and natural numbers $n_1 < n_2 < \dots < n_m$ have been choosen s.t, for each $0 \le k \le m$,

$|I_k| = \frac{b-a}{2^k}$, $n_k \in I_k$ and $x_n \in I_k$ for infinitely many n. (2)

To choose I_{m+1}, divide $I_m = [a_m, b_m]$ into two halves, say $I' = [a_m, \frac{a_m+b_m}{2}]$ and $I'' = [\frac{a_m+b_m}{2}, b_m]$. Since $I_m = I' \cup I''$, at least one of these halves contains x_n for infinitely many n. Call it I_{m+1}, and choose $n_{m+1} > n_m$ s.t. $x_{n_{m+1}} \in I_{m+1}$.

Since $|I_{m+1}| = \frac{|I_m|}{2} = \frac{b-a}{2^{m+1}}$,

it follows by induction that there is a nested seq. $\{I_k\}_{k\in\mathbb{N}}$ of nonempty closed bdd intervals that satify (2) for all $k \in \mathbb{N}$. By the Nested Interval property, there is an $x \in \mathbb{R}$ that belongs to I_k, $\forall k \in \mathbb{N}$. Since $x \in I_k$, we have by (2) that

$$0 \le |x_{n_k} - x| \le |I_k| \le \frac{b-a}{2^k}, \forall k \in \mathbb{N}$$

Hence by the squeeze, $x_{n_k} \to x$ as $k \to \infty$ ∎

H.W's 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 (i.e; All).

2.4 Cauchy sequences

Df ① A sequence of points $x_n \in \mathbb{R}$ is said to be Cauchy (in $\mathbb{R}$) iff $\forall \varepsilon > 0$, $\exists$ an $N \in \mathbb{N}$ s.t. $n, m \geq N \Longrightarrow |x_n - x_m| < \varepsilon$.

Rmk. ① If $\{x_n\}$ is convergent, then $\{x_n\}$ is Cauchy.

Proof. Spse that $x_n \to a$ as $n \to \infty$. then, given $\varepsilon > 0$, $\exists$ an $N \in \mathbb{N}$ s.t. $|x_n - a| < \frac{\varepsilon}{2}$ for all $n \geq N$. Hence, if $n, m \geq N$, then

$$|x_n - x_m| = |x_n - a + a - x_m|$$
$$\leq |x_n - a| + |x_m - a| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \blacksquare$$

the following result shows that the converse of the above remark is also true (for real sequences).

Thm ① [Cauchy]. Let $\{x_n\}$ be a sequence of real numbers. Then $\{x_n\}$ is Cauchy iff $\{x_n\}$ converges (to some point a in $\mathbb{R}$).

Proof: By Rmk①, we need only show that every Cauchy sequence Converges.

Suppose that $\{x_n\}$ is Cauchy. Given $\varepsilon = 1$, $\exists N \in \mathbb{N}$ s.t. $|x_N - x_m| < 1$, for all $m \geq N$.

By the triangle inequality,

$$|x_m| = |x_m - x_N + x_N|$$
$$\leq |x_N - x_m| + |x_N|$$
$$< 1 + |x_N| \quad , \text{ for } m \geq N.$$

Also, $|x_m| \leq \max\{|x_1|, |x_2|, --, |x_{N-1}|\} := M$, for $m = 1, 2, ---, N-1$

Therefore, $|x_n| \leq \max\{M, 1 + |x_N|\}$, $\forall n \in \mathbb{N}$

this means $\{x_n\}$ is bounded. By the Bolzano-Weierstrass theorem, $\{x_n\}$ has a convergent subsequence, say $x_{n_k} \longrightarrow a$ as $k \to \infty$. Let $\varepsilon > 0$. Since $\{x_n\}$ is Cauchy, $\exists N_1 \in \mathbb{N}$ s.t. $n, m \geq N_1 \implies |x_n - x_m| < \frac{\varepsilon}{2}$

Since $x_{n_k} \longrightarrow a$ as $k \to \infty$, $\exists\, N_2 \in \mathbb{N}$ s.t.

$$k \geq N_2 \implies |x_{n_k} - a| < \frac{\varepsilon}{2}.$$

Fix $k \geq N_2$ s.t. $n_k \geq N_1$. then

$$|x_n - a| = |x_n - x_{n_k} + x_{n_k} - a|$$

$$\leq |x_n - x_{n_k}| + |x_{n_k} - a|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \text{ for all } n \geq N_1$$

Thus, $x_n \to a$ as $n \to \infty$ ■

__Rmk ②__ this result is extremely useful because it is often ~~easier~~ to show that a sequence is Cauchy than to show that it converges. Let us see the following example.

__Example__ Prove that any real sequence $\{x_n\}$ satisfies $|x_n - x_{n+1}| \leq \frac{1}{2^n}$, $n \in \mathbb{N}$, is __convergent__.

Proof: If $m > n$, then

$$|x_n - x_m| = |x_n - x_{n+1} + x_{n+1} - x_{n+2} + \cdots + x_{m-1} - x_m|$$

$$\leq |x_n - x_{n+1}| + |x_{n+1} - x_{n+2}| + \cdots + |x_{m-1} - x_m|$$

$$\leq \frac{1}{2^n} + \cdots + \frac{1}{2^{m-1}}$$

$$= \frac{1}{2^{n-1}} \underbrace{\left[\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{m-n}} \right]}_{\text{Geometric series}}$$

$a_1 \Rightarrow$ First term $= \frac{1}{2}$

Ratio $= r = \frac{1}{2}$.

$$= \frac{1}{2^{n-1}} \left[\frac{a_1(1 - r^{m-n})}{1 - r} \right]$$

$$= \frac{1}{2^{n-1}} \left[\frac{\frac{1}{2}\left(1 - \left(\frac{1}{2}\right)^{m-n}\right)}{1 - \frac{1}{2}} \right]$$

$$\therefore |x_n - x_m| \leq \frac{1}{2^{n-1}} \left(1 - \frac{1}{2^{m-n}}\right), \text{ if } m > n.$$

It follows that $|x_n - x_m| < \frac{1}{2^{n-1}}$, for all integers $m > n \geq 1$. But given $\varepsilon > 0$, we can choose $N \in \mathbb{N}$ so large that

$$n \geq N \Longrightarrow \frac{1}{2^{n-1}} < \varepsilon$$

We have proved that $\{x_n\}$ is Cauchy.

By Thm ①, therefore, it converges to some real number. ∎

Rmk ③. A sequence that satisfies $x_{n+1} - x_n \to 0$ is not necessarily Cauchy.

Proof: Consider the sequence $x_n := \log n$.

$$x_{n+1} - x_n = \log(n+1) - \log n = \log \frac{n+1}{n} \to \log 1 = 0$$

as $n \to \infty$. $\{x_n\}$ cannot be Cauchy because it does not conv. ($\lim_{n\to\infty} x_n = \lim_{n\to\infty} \log n = \infty$). ∎

H.W's Exercises page 60 0, 1, 2, 3, 4, 5.

CH3 Functions on $\mathbb{R}$.

3.1 Two-sided limits

Df ① Let $a \in \mathbb{R}$, let I be an open interval which contains a, and let f be a real function defined on I except possibly at a. Then we say that $f(x)$ converges (approaches) to L as x approaches a, and write

$\lim_{x \to a} f(x) = L$ iff $\forall \varepsilon > 0$, $\exists$ a $\delta > 0$ (which in general depends on ε, f, I, and a) such that

$$\boxed{0 < |x-a| < \delta \Longrightarrow |f(x) - L| < \varepsilon.} \quad (*)$$

Rmk. 1) ε represents the maximal error allowed in the approximation $f(x)$ to L.

2) According to Df ①, to show that a function has a limit, we must begin with a general $\varepsilon > 0$ and describe how to choose a δ which satisfies $(*)$

Ex. ① Let $f(x) = mx + b$, where $m, b \in \mathbb{R}$.
prove that $\lim_{x \to a} f(x) = f(a)$, $\forall a \in \mathbb{R}$.

Proof: If $m=0$, then $|f(x)-f(a)| = |b-b| = 0 < \varepsilon$ for all x. If $m \neq 0$, given $\varepsilon > 0$, set $\delta = \frac{\varepsilon}{|m|}$

If $|x-a| < \delta$, then

$$|f(x)-f(a)| = |mx+b-(ma+b)| = |m|\,|x-a|$$
$$< |m|\,\delta = |m| \frac{\varepsilon}{|m|} = \varepsilon$$

Thus, by Df ①, $\lim_{x \to a} f(x) = f(a)$. ∎

Ex ②. If $f(x) = x \sin \frac{1}{x}$, $x \neq 0$, then

$$\lim_{x \to 0} f(x) = 0$$

Pf. Let $\varepsilon > 0$, set $\delta = \varepsilon$. If $|x| < \delta = \varepsilon$, then $|f(x) - 0| = |x \sin \frac{1}{x} - 0|$
$$\leq |x| < \varepsilon.$$

Ex ③. If $f(x) = x^2 + x - 3$, prove that $\lim_{x \to 1} f(x) = -1$

Let $\varepsilon > 0$. Notice that

$$f(x) - L = x^2 + x - 3 + 1 = x^2 + x - 2 = (x+2)(x-1)$$

If $0 < \delta \leq 1$, then $|x-1| < \delta \implies 0 < x < 2$, so $|x+2| \leq |x| + 2 < 4$. Set $\delta = \min\{1, \frac{\varepsilon}{4}\}$.

It follows that if $|x-1| < \delta$, then

$$|f(x)-L| = |x-1|\,|x+2| < 4|x-1| < 4\delta \leq 4\cdot\frac{\varepsilon}{4} = \varepsilon.$$

Thus, by Df①, $\lim\limits_{x\to 1} f(x) = -1$.

<u>Thm①.</u> If $\lim\limits_{x\to a} f(x)$ exists, then it is unique, i.e.,

if $\lim\limits_{x\to a} f(x) = L_1$ and $\lim\limits_{x\to a} f(x) = L_2$, then

$L_1 = L_2$.

<u>Proof:</u> Spse that $\lim\limits_{x\to a} f(x) = L_1$ & $\lim\limits_{x\to a} f(x) = L_2$

and let $\varepsilon > 0$. From Df①, $\exists$ $\delta_1, \delta_2 > 0$

such that $|f(x) - L_1| < \varepsilon$ if $0 < |x-a| < \delta_1$

and $|f(x) - L_2| < \varepsilon$ if $0 < |x-a| < \delta_2$.

If $\delta = \min\{\delta_1, \delta_2\}$, then

$$\begin{aligned} |L_1 - L_2| &= |L_1 - f(x) + f(x) - L_2| \\ &\leq |f(x) - L_1| + |f(x) - L_2| \\ &< \varepsilon + \varepsilon = 2\varepsilon \quad \text{if } |x-a| < \delta, \end{aligned}$$

i.e., $|L_1 - L_2| < 2\varepsilon$, $\forall \varepsilon > 0$

$\Rightarrow L_1 = L_2$ ∎

The next result shows that even when a function f is defined at a, $\lim_{x\to a} f(x)$, in general, independent of the value of $f(a)$.

<u>Lemma:</u> Let $a \in \mathbb{R}$, let I be an open interval which contains a, and let f, g be real functions defined $\forall x \in I$ except possibly at a.

If $f(x) = g(x)$, $\forall x \in I \setminus \{a\}$ and $\lim_{x\to a} f(x) = L$

then $\lim_{x\to a} g(x)$ exists and $\lim_{x\to a} g(x) = \lim_{x\to a} f(x)$

<u>PF.</u> let $\varepsilon > 0$ and chose $\delta > 0$ small enough so that (*) holds and $|x-a| < \delta \Rightarrow x \in I$.

Suppose that $0 < |x-a| < \delta$. We have $f(x) = g(x)$ by hypothesis and $|f(x) - L| < \varepsilon$ by (*). It follows that $|g(x) - L| < \varepsilon$. that is, $\lim_{x\to a} g(x) = L$. ▮

<u>Ex(4)</u> Prove that $\lim_{x\to 1} g(x)$ exists, if

$$g(x) = \frac{x^3 + x^2 - x - 1}{x^2 - 1}.$$

Pf. Set $f(x) = x+1$ and observe that

$$g(x) = \frac{x^3+x^2-x-1}{x^2-1} = \frac{x^2(x+1)-(x+1)}{x^2-1}$$

$$= \frac{(x^2-1)(x+1)}{x^2-1} = x+1, \quad x \neq \pm 1$$

$$= f(x).$$

and observe that, by Ex1, $\lim_{x \to 1} f(x) = 2$.

It follows from previous lemma that $g(x)$ has a limit at $x=1$ and $\lim_{x\to 1} g(x) = 2$ ∎

Theorem ②. [Sequential characterization of limits].

Let $a \in \mathbb{R}$, let I be an open interval contains a, and let f be a real function defined $\forall x \in I$ except possibly at a. then

$\lim_{x \to a} f(x) = L$ iff $f(x_n) \longrightarrow L$ as $n \longrightarrow \infty$

for every sequence $x_n \in I \setminus \{a\}$ which converges to a as $n \longrightarrow \infty$.

Proof: ($\Rightarrow$) Suppose that $\lim_{x \to a} f(x) = L$. then given $\varepsilon > 0$, $\exists$ a $\delta > 0$ s.t.

$0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon$. ... (I)

Let $x_n \in I \setminus \{a\}$ s.t. $\lim_{n \to \infty} x_n = a$, then $\exists$ an $N \in \mathbb{N}$ s.t $n \geq N \implies |x_n - a| < \delta$.

Since $x_n \neq a$, it follows from (I) that $|f(x_n) - L| < \varepsilon$ for all $n \geq N$. therefore, $f(x_n) \longrightarrow L$ as $n \longrightarrow \infty$.

($\Leftarrow$) Conversely, suppose that $f(x_n) \longrightarrow L$ as $n \to \infty$ for every sequence $x_n \in I \setminus \{a\}$ which converges to a (i.e., $x_n \to a$). Suppose that $\lim_{x \to a} f(x) \neq L$, then there is an $\varepsilon > 0$ (say ε_0) such that the implication "$0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon$" does not hold for any $\delta > 0$. thus, for each $\delta = \frac{1}{n}$, $n \in \mathbb{N}$, $\exists$ a point $x_n \in I$ which satisfies two conditions:

$0 < |x_n - a| < \frac{1}{n}$ and $|f(x_n) - L| \geq \varepsilon_0$.

Now the first condition and the squeeze thm imply that $x_n \neq a$, $x_n \longrightarrow a$ as $n \to \infty$, so by hypothesis, $f(x_n) \longrightarrow L$, as $n \to \infty$.

In particular, $|f(x_n) - L| < \varepsilon_0$ for n large, which contradicts the second condition ▤

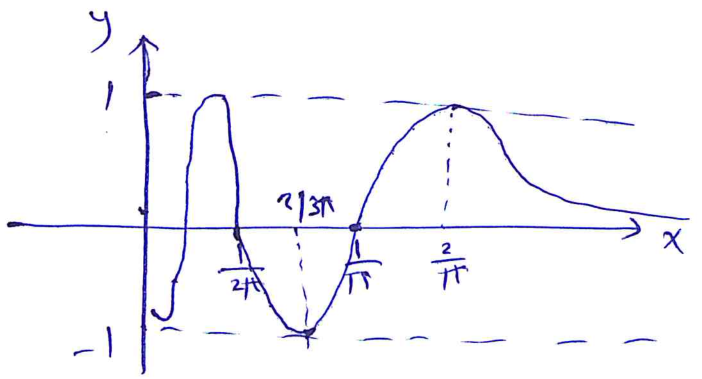
Rmk. To show that the limit of a function f does not exist as $x \longrightarrow a$, using this thm we need to find two sequences converges to a (say $x_n \longrightarrow a$ and $y_n \longrightarrow a$) whose images under f have different limits.

(i.e., $f(x_n) \longrightarrow L_1$ and $f(y_n) \longrightarrow L_2$, where $L_1 \neq L_2$). $\neq \lim\limits_{x \to a} f(x)$

Ex.⑤. Prove that $f(x) = \begin{cases} \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

has no limit as $x \longrightarrow 0$.

Pf.



By examining the graph of $y=f(x)$ (see the above Fig.), we consider

$$x_n := \frac{2}{(4n+1)\pi} \text{ and } y_n := \frac{2}{(4n+3)\pi}, n\in\mathbb{N}$$

Clearly x_n and $y_n \longrightarrow 0$ as $n\longrightarrow\infty$.

But, since $f(x_n)=1$ and $f(y_n)=-1$ for all $n\in\mathbb{N}$, $f(x_n)\longrightarrow 1$

and $f(y_n)\longrightarrow -1$

as $n\to\infty$. Thus by Thm(2), $\lim_{x\to 0} f(x)$ DNE ▭

Rmk. Thm(2) allows us to translate results about limits of sequences to results about limits of functions. Let's see the following theorems.

Thm③. Suppose that $a\in\mathbb{R}$, that I is an open interval which contains a and that f, g are real functions defined $\forall x\in I$ except possibly at a. If $f(x)$ and $g(x)$ converges as $x\to a$ (i.e.,

$\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist), then so do $(f+g)(x) = f(x)+g(x)$, $(fg)(x) = f(x)g(x)$, $(\alpha f)(x) = \alpha f(x)$, and $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ (when $\lim_{x\to a} g(x) \neq 0$). In fact,

(i) $\lim_{x\to a} (f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

(ii) $\lim_{x\to a} (\alpha f)(x) = \alpha \lim_{x\to a} f(x)$

(iii) $\lim_{x\to a} (fg)(x) = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right)$.

and when $\lim_{x\to a} g(x) \neq 0$,

(iv) $\lim_{x\to a} \left(\frac{f}{g}\right)(x) = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$.

<u>proof.</u> Let $\lim_{x\to a} f(x) := L$, and $\lim_{x\to a} g(x) := M$.

(i) If $x_n \in I\setminus\{a\}$ s.t. $x_n \to a$, then by thm ②, $f(x_n) \to L$ and $g(x_n) \to M$ as $n\to\infty$. By Thm (ch 2),

$(f+g)(x_n) = f(x_n) + g(x_n) \longrightarrow M+L$ as $n\to\infty$

Since this holds for any sequence $x_n \in I \setminus \{a\}$ which converges to a, we conclude by Thm 2,

$$\lim_{x \to a} (f+g)(x) = L + M$$
$$= \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

(ii), (iii), + (iv) Exercises. ■

Thm 4: [Squeeze thm for functions]

Suppose that $a \in \mathbb{R}$, that I is an open interval which contains a, and that f, g, h are real functions defined $\forall x \in I$ except possibly at a.

(i) If $g(x) \leq h(x) \leq f(x)$ for all $x \in I \setminus \{a\}$ and $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = L$, then $\lim_{x \to a} h(x)$ exists, and $\lim_{x \to a} h(x) = L$.

(ii) If $|g(x)| \leq M$ for all $x \in I \setminus \{a\}$ (i.e., g is bdd) and $\lim_{x \to a} f(x) = 0$, then $\lim_{x \to a} f(x) g(x) = 0$.

Thm(5): [Comparison thm for functions]

Suppose that $a \in \mathbb{R}$, that I is an open interval which contains a, and that f, g are real functions defined $\forall x \in I$ except possibly at a. If f and g have limit as $x \to a$ and $f(x) \le g(x)$ $\forall x \in I \setminus \{a\}$, then $\lim_{x\to a} f(x) \le \lim_{x \to a} g(x)$.

Rmk. We shall refer to thm 5 as "taking the limit of an inequality".

Rmk. The limit thms (Thm 3, 4, and 5) allow us to prove that limits exist without using $(\varepsilon - \delta)$ definition (Df ①).

Ex. 6 Prove that $\lim_{x\to 1} \dfrac{x-1}{3x+1} = 0$

Pf. By Example (1), $\lim_{x\to 1}(x-1) = 0$ and $\lim_{x\to 1}(3x+1) = 4$. Hence, by Thm (3)(iv),

$$\lim_{x\to 1} \frac{x-1}{3x+1} = \frac{\lim_{x\to 1}(x-1)}{\lim_{x\to 1}(3x+1)} = \frac{0}{4} = 0.$$

H.W's 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 (All).

3.2 One-sided limits and limits at infinity

Df①. Let $a \in \mathbb{R}$ and f be a real function.

(i) $f(x)$ is said to converge to L as x approaches a from the right iff f is defined on some open interval I with left endpoint a and for every $\varepsilon > 0$ $\exists$ a $\delta > 0$ (which in general depends on ε, I, f and a) such that

$$a+\delta \in I \text{ and } a < x < a+\delta \Longrightarrow |f(x) - L| < \varepsilon.$$

Here, L is called the right-hand limit of f at a, and denote it by

$$f(a^+) := L =: \lim_{x \to a^+} f(x).$$

(ii) $f(x)$ is said to converge to L as x approaches a from the left iff f is defined on some open interval I with right endpoint a and for every $\varepsilon > 0$, $\exists$ a $\delta > 0$ such that

$$a-\delta \in I \text{ and } a - \delta < x < a \Longrightarrow |f(x) - L| < \varepsilon.$$

Here, L is called the left-hand limit of f at and denote by

$$f(a^-) := L =: \lim_{x\to a^-} f(x)$$

Example [1] (i) prove that $f(x) = \begin{cases} x+1, & x \geq 0 \\ x-1, & x < 0 \end{cases}$

has one-sided limits at $x=0$ but $\lim_{x\to 0} f(x)$ DNE.

(ii) prove that $\lim_{x\to 0^+} \sqrt{x} = 0$

proof: (i) Let $\varepsilon > 0$ and set $\delta = \varepsilon$.

If $0 < x < \delta$, then $|f(x) - L| = |x+1-1| = |x| < \delta = \varepsilon$.

Hence $\lim_{x\to 0^+} f(x)$ exists and equals 1.

Similarly, $\lim_{x\to 0^-} f(x)$ exists and equals -1

Indeed, set $\delta = \varepsilon$. If $-\delta < x < 0$, then

$|f(x) - L| = |x-1+1| = |x| < \delta = \varepsilon$.

However, $\lim_{x\to 0} f(x)$ DNE since,

$$x_n = \frac{(-1)^n}{n} \longrightarrow 0 \quad \text{(by squeeze thm)}$$

but $f(x_n) = f\left(\frac{(-1)^n}{n}\right) = (-1)^n\left(1 + \frac{1}{n}\right)$ does not converge as $n \to \infty$

Hence, by the sequential characterization of Limits, $\lim_{x \to 0} f(x)$ DNE.

(ii) Let $\varepsilon > 0$ and set $\delta = \varepsilon^2$. If $0 < x < \delta$, then $|f(x) - L| = |\sqrt{x} - 0| = \sqrt{x} < \sqrt{\delta} = \varepsilon$. ∎

Rmk. Not every function has one-sided limits $\left(\text{ex. } f(x) = \begin{cases} \sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases} \right)$. The last example show that even a function has one-sided limits, it may not have a two-sided limit. The following Thm show that if both one-sided limits at a exist and are EQUAL, then the two-sided limit at a exists.

Thm ① Let f be a real function. Then the limit $\lim_{x \to a} f(x)$ exists and equals L iff

$$L = \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x).$$

Proof. ($\Rightarrow$) Suppose that $\lim_{x \to a} f(x) = L$ exists. Then given $\varepsilon > 0$, $\exists$ a $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon.$$

If $a < x < a+\delta$, then $|x-a| < \delta$ and this implies $|f(x)-L| < \varepsilon$. Hence $\lim_{x \to a^+} f(x) = L$ exists.

Similarly, if $a-\delta < x < a$, then $|x-a| < \delta$ and this implies $|f(x)-L| < \varepsilon$, which means

$\lim_{x \to a^-} f(x) = L$ exists.

($\Leftarrow$) Conversely, suppose that $L = \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$

Then given $\varepsilon > 0$, $\exists$ a $\delta_1 > 0$ s.t.

$$a < x < a+\delta_1 \implies |f(x)-L| < \varepsilon. \quad (1)$$

and $\exists$ a $\delta_2 > 0$ s.t.

$$a-\delta_2 < x < a \implies |f(x)-L| < \varepsilon. \quad (2)$$

Set $\delta = \min\{\delta_1, \delta_2\}$. Then

$|x-a| < \delta \Rightarrow a-\delta < x < a+\delta$ which implies $a < x < a+\delta_1$ or $a-\delta_2 < x < a$ (depending on whether x is to the right or to the left of a). Hence, (1) & (2) give $|f(x)-L| < \varepsilon$ that is $\lim_{x \to a} f(x) = L$ ▤

Df②. (limits at infinity).

Let $a, L \in \mathbb{R}$ & let f be a real function.

(i) $f(x)$ is said to converge to L as $x \to \infty$ iff $\exists$ a $c > 0$ such that $(c, \infty) \subset Dom(f)$ and given $\varepsilon > 0$ there is an $M \in \mathbb{R}$ s.t.,

$x > M \implies |f(x) - L| < \varepsilon$, in which case we shall write $\lim\limits_{x \to \infty} f(x) = L$ or $f(x) \to L$ as $x \to \infty$.

Similarly, $f(x) \longrightarrow L$ as $x \to -\infty$ iff $\exists$ a $c > 0$ s.t., $(-\infty, -c) \subset Dom(f)$ and given $\varepsilon > 0$ there is an $M \in \mathbb{R}$ such that $x < M \implies |f(x) - L| < \varepsilon$, in which case we shall write $\lim\limits_{x \to -\infty} f(x) = L$ or $f(x) \to L$ as $x \to -\infty$.

(ii) $f(x)$ is said to converge to ∞ as $x \to a$ (i.e., $\lim\limits_{x \to a} f(x) = \infty$) iff there is an open interval I containing a such that $I \setminus \{a\} \subset Dom(f)$ and given $M \in \mathbb{R}$

there is a $\delta > 0$ such that

$$0 < |x-a| < \delta \implies f(x) > M,$$ in which case we write $\lim_{x \to a} f(x) = \infty$ or $f(x) \to \infty$ as $x \to a$

Similarly, $f(x)$ is said to converge to $-\infty$ as $x \to a$ (i.e., $\lim_{x \to a} f(x) = -\infty$) iff $\exists$ an open interval I containing a such that

$I \setminus \{a\} \subset \mathrm{Dom}(f)$ and given $M \in \mathbb{R}$

there is a $\delta > 0$ such that

$$0 < |x-a| < \delta \implies f(x) < M.$$

Rmk. Obvious modifications of this Df, we define $f(x) \to \pm\infty$ as $x \to a^+$ & $x \to a^-$, and $f(x) \to \pm\infty$ as $x \to \pm\infty$.

Ex. 2 (i) Prove that $\lim_{x \to \infty} \frac{1}{x} = 0$.

(ii) prove that $\lim_{x \to 1^-} \frac{x+2}{2x^2-3x+1} = -\infty$

Proof: (i) Given $\varepsilon > 0$, set $M = \frac{1}{\varepsilon}$. If $x > M$, then $|f(x) - L| = |\frac{1}{x} - 0| = \frac{1}{x} < \frac{1}{M} = \varepsilon$.

Thus, $\frac{1}{x} \to 0$ as $x \to \infty$.

(ii) Let $M \in \mathbb{R}$. We need to find $\delta > 0$ s.t. $1-\delta < x < 1 \Rightarrow f(x) < M$, where $f(x) = \dfrac{x+2}{2x^2-3x+1}$. Without loss of generality, assume that $M < 0$.

As $x \to 1^-$, $2x^2-3x+1$ is negative and $2x^2-3x+1 \to 0$ (observe that $2x^2-3x+1$ is a parabola opening upward with roots $\frac{1}{2}$ and 1).

Therefore, choose $\delta \in (0,1)$ such that

$$1-\delta < x < 1 \Rightarrow \frac{2}{M} < 2x^2-3x+1 < 0, \text{ i.e.,}$$

$$\frac{-1}{2x^2-3x+1} > -\frac{M}{2} > 0.$$ Since $0 < x < 1$ also implies $2 < x+2 < 3$, it follows that

$$-\frac{x+2}{2x^2-3x+1} > -M, \text{ i.e.,}$$

$$f(x) = \frac{x+2}{2x^2-3x+1} < M, \text{ for all } 1-\delta < x < 1$$

Notation $\lim_{\substack{x \to a \\ x \in I}} f(x)$ (*) [a is an extended real number]

- (*) will denote $\lim_{x \to a} f(x)$ (when it exists)
- If a is a finite left endpoint of I, then (*) will denote $\lim_{x \to a^+} f(x)$ (when it exists).
- If a is a finite right endpoint of I, then (*) will denote $\lim_{x \to a^-} f(x)$ (when it exists)
- If $a = \pm\infty$ is an endpoint of I, then (*) will denote $\lim_{x \to \pm\infty} f(x)$ (when each exists).

Using this notation, we can state a Sequential characterization of Limits valid for two-sided, one-sided, and infinite limits.

Thm ② Let a be an extended real number, and let I be a nondegenerate open interval ~~I~~ which either contains a or has a as one of its endpoints. Suppose further that f is a real function defined on I except possibly at a. Then

$\lim_{\substack{x \to a \\ x \in I}} f(x)$ exists and equals L

if and only if $f(x_n) \to L$ for all sequences $x_n \in I$ which satisfy $x_n \neq a$ and $x_n \to a$ as $n \to \infty$.

Proof. Since we have proved this for two-sided limits, we must show it for the remaining 8 cases which notation (*) represents. Since the proofs are similar, we shall give the details for only one case, namely, $\boxed{\lim_{x\to a} f(x) = \infty}$. Thus, we must prove that

$\boxed{\lim_{x \to a} f(x) = \infty \text{ iff } f(x_n) \to \infty \text{ for any sequence } x_n \in I \text{ which converges to } a \text{ and satisfies } x_n \neq a \text{ for } n \in \mathbb{N}.}$

$\Rightarrow$) Suppose that $\lim_{x\to a} f(x) = \infty$. If $x_n \in I$, $x_n \to a$ as $n \to \infty$, and $x_n \neq a$, then given $M \in \mathbb{R}$, $\exists$ a $\delta > 0$ such that

$0 < |x-a| < \delta \Rightarrow f(x) > M.$

and $\exists$ an $N \in \mathbb{N}$ such that

$n \geq N \Rightarrow |x_n - a| < \delta.$

Consequently, $n \geq N \Rightarrow f(x_n) > M$, i.e.,

$f(x_n) \to \infty$ as $n \to \infty$ as required.

($\Leftarrow$) Conversely, Suppose to the contrary that $f(x_n) \to \infty$ for any sequence $x_n \in I$ which converges to a and satisfies $x_n \neq a$ but $\lim_{x \to a} f(x) \neq \infty$. By the definition of "convergence" to ∞ there are numbers $M_0 \in \mathbb{R}$ and $x_n \in I$ s.t., $|x_n - a| < \frac{1}{n}$ and $f(x_n) \leq M_0 \quad \forall n \geq N.$

Now, $|x_n - a| < \frac{1}{n} \Rightarrow a - \frac{1}{n} < x_n < a + \frac{1}{n}$

this implies, using squeeze thm, $x_n \to a$ but the condition $f(x_n) \leq M_0$, $\forall n \geq N$ implies that $f(x_n) \not\to \infty$ as $n \to \infty$ which is a contradiction & this proves thm 2 in the case $\lim_{x \to a} f(x) = \infty$, $a \in I$.

Rmk: Using Thm 2, we can prove limit theorems represented in sec. 3.1. These limits thms can be used to evaluate infinite limits and limits at $\pm\infty$.

Ex. ③ prove that $\lim_{x\to\infty} \frac{2x^2-1}{1-x^2} = -2$.

Proof: Since the limit of a product is the product of the limits, we have by Ex②(a) that $\lim_{x\to\infty} \frac{1}{x^m} = 0$, for any $m \in \mathbb{N}$.

Multiplying numerator and denominator of the expression above by $\frac{1}{x^2}$, we obtain

$$\lim_{x\to\infty} \frac{2x^2-1}{1-x^2} = \lim_{x\to\infty} \frac{2-\frac{1}{x^2}}{-1+\frac{1}{x^2}}$$

$$= \frac{\lim_{x\to\infty}\left(2-\frac{1}{x^2}\right)}{\lim_{x\to\infty}\left(-1+\frac{1}{x^2}\right)} = \frac{2}{-1} = -2$$

$\blacksquare$

H.W's Exercises p. 81

0, 1, 2, 3, 4, 5, 6, 7, 8 (i.e., All)

Also, Give such an example on Q8.

3.3 Continuity

Df① Let $\phi \neq E \subseteq \mathbb{R}$ and $f: E \longrightarrow \mathbb{R}$.

(i) f is said to be continuous at a point $a \in E$ iff $\forall \varepsilon > 0, \exists$ a $\delta > 0$ (depends on $\varepsilon, f,$ and a) s.t.

$|x-a| < \delta$ and $x \in E \Rightarrow |f(x) - f(a)| < \varepsilon$ (*)

(ii) f is said to be continuous on E iff f is continuous at every $x \in E$.

Rmk. Let I be an open interval which contains a point a and $f: I \longrightarrow \mathbb{R}$. Then f is continuous at $a \in I$ iff $f(a) = \lim_{x \to a} f(x)$.

proof: see the book.

Thm(1) [sequential characterization of Continuity]

Spse that E is a nonempty subset of $\mathbb{R}$, that $a \in E$, and that $f: E \longrightarrow \mathbb{R}$. Then the following statements are equivalent:

(i) f is cont. at $a \in E$.

(ii) If $x_n \longrightarrow a$ and $x_n \in E$, then $f(x_n) \longrightarrow f(a)$ as $n \to \infty$

Thm ②. Let E be a nonempty subset of $\mathbb{R}$ and $f, g : E \to \mathbb{R}$. If f, g are continuous at a point $a \in E$ (resp. continuous on the set E), then so are $f+g$, fg, and αf (for any $\alpha \in \mathbb{R}$).

Moreover, f/g is cont. at $a \in E$ when $g(a) \neq 0$ (resp., on E when $g(x) \neq 0$ for all $x \in E$).

Df ②. Spse that A and B are subsets of $\mathbb{R}$, that $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$. If $f(A) \subseteq B$ for every $x \in A$, then the composition of g with f is the function $g \circ f : A \to \mathbb{R}$ defined by $(g \circ f)(x) := g(f(x))$, $x \in A$.

Thm ③. Spse that A and B are subsets of $\mathbb{R}$, that $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$, and that $f(x) \in B$, $\forall x \in A$.

(i) If $A := I \setminus \{a\}$, where I is a nondegenerate interval which either contains a or has a as one of its endpoints, if

$$L := \lim_{\substack{x \to a \\ x \in I}} f(x)$$

exists and belongs to B, and if g is cont. at $L \in B$, then

$$\lim_{\substack{x \to a \\ x \in I}} (g \circ f)(x) = g\left(\lim_{\substack{x \to a \\ x \in I}} f(x)\right).$$

(ii) If f is cont. at $a \in A$ and g is cont. at $f(a) \in B$, then $g \circ f$ is cont at $a \in A$.

Proof (i) Suppose that $x_n \in I \setminus \{a\}$ and that $x_n \to a$ as $n \to \infty$. Since $f(A) \subseteq B$. $f(x_n) \in B$. Also, by the sequential characterization of limits, $f(x_n) \to L$ as $n \to \infty$. Since g is cont. at $L \in B$, it follows from thm (1) that $(g \circ f)(x_n) := g(f(x_n)) \to g(L)$ as $n \to \infty$. Hence, by the sequential characterization of limits again, $(g \circ f)(x) \to g(L)$ as $x \to a$ in I.

(ii) Exercise

<u>Df ③</u>. Let $\phi \neq E \subseteq \mathbb{R}$. A function $f: E \to \mathbb{R}$ is said to be <u>bounded</u> on E iff $\exists$ an $M \in \mathbb{R}$ s.t., $|f(x)| \leq M$, $\forall x \in E$. (f is dominated by M on E)

<u>Rmk:</u> Notice that whether a function f is bounded or not on a set E depends on E as well as on f. For ex, $f(x) = \frac{1}{x}$ is bounded on $[1, \infty)$ but unbounded on $(0, 2)$. Again, $f(x) = x^2$ is bounded on $(-2, 2)$ (dominated by 4) but unbounded on $[0, \infty)$.

<u>thm ④.</u> [<u>Extreme value thm</u>]

If I is a closed, bounded interval and $f: I \to \mathbb{R}$ is continuous on I, then f is <u>bounded</u> on I. Moreover, if $M = \sup_{x \in I} f(x)$ and $m = \inf_{x \in I} f(x)$, then $\exists$ points $x_m, x_M \in I$ s.t.

$$f(x_M) = M \text{ and } f(x_m) = m.$$

Proof: Spse first that f is not bounded on I. Then $\exists$ $x_n \in I$ s.t.

$$|f(x_n)| > n, \quad n \in \mathbb{N}. \qquad (**)$$

Since I is bounded, by the Bolzano-Weierstrass Thm, $\{x_n\}$ has a convergent subsequence, say $x_{n_k} \longrightarrow a$ as $k \to \infty$. Since I is closed, by the Comparison Thm, $a \in I$. In particular, $f(a) \in \mathbb{R}$. On the other hand, substituting n_k for n in $(**)$ and taking the limit as $k \to \infty$, we have $|f(a)| = \infty$, a contradiction. Hence, f is bounded on I.

We have proved that both M and m are finite real numbers. To show that $\exists$ an $x_M \in I$ s.t. $f(x_M) = M$, suppose to the contrary, that $f(x) < M$ for all $x \in I$.

Then $g(x) = \dfrac{1}{M - f(x)}$ is cont; hence bounded on I. In particular, $\exists$ a $C > 0$ such that $|g(x)| = g(x) \leq C$.

It follows that $f(x) \leq M - \frac{1}{C}$, $\forall x \in I$.

It follows that $\sup_{x\in I} f(x) \leq \sup_{x \in I} \left(M - \frac{1}{C}\right)$.

This implies $M \leq M - \frac{1}{C} < M$,

a contradiction. Hence, $\exists$ an $x_M \in I$ such that $f(x_M) = M$. Similarly, you can prove that $\exists$ an $x_m \in I$ s.t., $f(x_m) = m$ (please do it). ∎

Rmk. ① We also call the value M (resp., m) the maximum (resp., the minimum) of f on I.

② The Extreme value Thm (Thm ④) is false if either "closed" or "bounded" is dropped from the hypothesis.

Counterexamples (2) $(0,1)$ is bounded interval but not closed and $f(x) = \frac{1}{x}$ is continuous and unbounded on $(0,1)$.

$[0,\infty)$ is closed but not bounded, and the function $f(x) = x$ is cont. and unbounded on $[0,\infty)$.

<u>Lemma</u>. Spse that $a < b$ and that $f : [a,b) \to \mathbb{R}$
If f is continuous at $x_0 \in [a,b)$ and $f(x_0) > 0$, then $\exists$ an $\varepsilon > 0$ and a point $x_1 \in [a,b)$ such that $x_1 > x_0$ and $f(x) > \varepsilon$, $\forall x \in [x_0, x_1]$.

<u>Proof</u>. Strategy: If $f(x_0) > 0$, then $f(x) > \frac{f(x_0)}{2}$ for x near x_0. the following are the details.

let $\varepsilon = \frac{f(x_0)}{2}$. Since $x_0 < b$, then

$$\delta_0 := \frac{b - x_0}{2} > 0 \text{ and } x \in [a, x_0 + \delta) \Longrightarrow x \in [a,b).$$

Since f is cont. at x_0, then we can choose $0 < \delta < \delta_0$ s.t. $x \in [a,b)$ and

$$|x - x_0| < \delta \Longrightarrow |f(x) - f(x_0)| < \varepsilon.$$

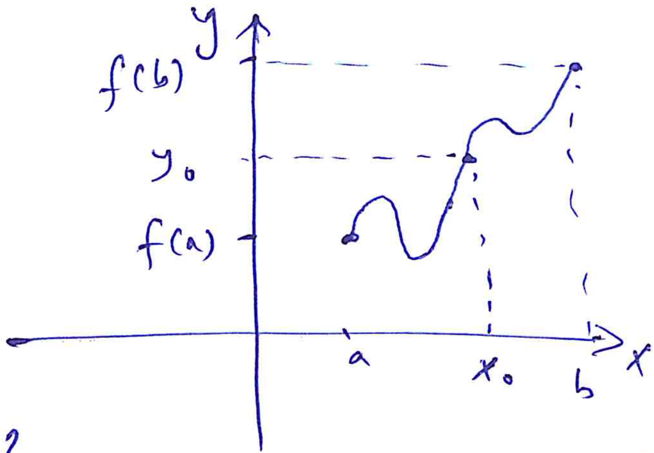
Fix $x_1 \in (x_0, x_0 + \delta)$ and spse that $x \in [x_0, x_1]$. By the choice of ε & δ, it is clear that

$$-\frac{f(x_0)}{2} < f(x) - f(x_0) < \frac{f(x_0)}{2}.$$

Solving the left-hand ineq. we conclude that
$f(x) > \frac{f(x_0)}{2} = \varepsilon$, as required ∎

· A real number y_0 is said to lie between two numbers c and d iff $c < y_0 < d$ or $d < y_0 < c$.

Thm ⑤ [Intermediate Value Theorem]
Spse that $a < b$ and that $f : [a,b] \longrightarrow \mathbb{R}$ is continuous. If y_0 lies between $f(a)$ and $f(b)$, then $\exists$ an $x_0 \in (a,b)$, such that $f(x_0) = y_0$.



Proof. We may suppose that $f(a) < y_0 < f(b)$.
Consider the set

$E = \{ x \in [a,b] : f(x) < y_0 \}$. Since $a \in E$ and $E \subseteq [a,b]$, then E is a nonempty, bounded subset of $\mathbb{R}$. Hence by the Completeness Axiom, $x_0 := \sup E$ is a finite real number. It remains to prove that $x_0 \in (a,b)$ and $f(x_0) = y_0$. Since E has a finite sup x_0 by Thm, $\exists$ a sequence $x_n \in E$ such that

$x_n \longrightarrow x_0$ as $n \longrightarrow \infty$.

Since $E \subseteq [a,b]$, it follows from the Comparison thm,

$x_0 \in [a,b]$. Moreover, by the continuity of f and the def'n of E, we have $f(x_0) = \lim_{n\to\infty} f(x_n) \leq y_0$

To show that $f(x_0) = y_0$, suppose to the contrary that $f(x_0) < y_0$. Then $y_0 - f(x)$ is a const. function on $[a,b)$ whose value at $x = x_0$ is positive. Hence, by previous lemma, we can choose an ε and an $x_1 > x_0$ such that

$y_0 - f(x_1) > \varepsilon > 0$. In particular, $x_1 \in E$ and $x_1 > \sup E$, a contradiction. We have proved that $x_0 \in [a,b]$ and $y_0 = f(x_0)$. Since we assumed that $f(a) < y_0 < f(b)$, it follows that x_0 cannot equal a or b. We conclude that $x_0 \in (a,b)$. $\blacksquare$

Ex ①. prove that $f(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 1 & x = 0 \end{cases}$ is

Continuous on $(-\infty, 0)$ and $[0,\infty)$, discont. at $x=0$, and both $f(0^+)$ and $f(0^-)$ exist.

Pf: Since $f(x) = 1$ for $x \geq 0$, it is clear that $f(0^+) = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 1 = 1$ exists and $f(x) \longrightarrow f(a)$ as $x \longrightarrow a$ for any $a > 0$. Inparticular, f is cont. on $[0, \infty)$. Similarly, $f(0^-) = \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (-1) = -1$ exists and f is continuous on $(-\infty, 0)$. Finally, since $f(0^+) \neq f(0^-)$, then $\lim_{x \to 0} f(x)$ DNE. therefore, f is not cont. at $x = 0$.

Ex. 2 Assume that $\sin x$ is cont. on $(-\infty, \infty)$, prove that $f(x) = \begin{cases} \sin(\frac{1}{x}), & x \neq 0 \\ 1, & x = 0 \end{cases}$ is cont. on $(-\infty, 0)$ and $(0, \infty)$, dis cont. at 0, and neither $f(0^+)$ nor $f(0^-)$ exists.

proof: The function $g(x) = \frac{1}{x}$ is cont. for $x \neq 0$. Hence by thm 3, $f(x) = (\sin \circ g)(x) = \sin(\frac{1}{x})$ is cont. on $(-\infty, 0) \cup (0, \infty)$. To prove that

$f(0^+)$ DNE, Let $x_n = \frac{2}{(2n+1)\pi}$, and observe that $\sin\left(\frac{1}{x_n}\right) = (-1)^n$, $n \in \mathbb{N}$.

Since $x_n \downarrow 0$ but $(-1)^n$ does not converge, it follows from (the sequential characterization of continuity Thm) that $f(0^+)$ DNE.

A similar way proves that $f(0^-)$ DNE (please do it as Exercise).

Ex③. The Dirichlet function is defined by

$$f(x) := \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q}. \end{cases}$$

prove that every point $x \in \mathbb{R}$ is a point of discontinuity of f (i.e. f is nowhere continuous)

proof: By Density of Rationals and Irrationals given any $a \in \mathbb{R}$ and $\delta > 0$ we can choose $x_1 \in \mathbb{Q}$ and $x_2 \in \mathbb{Q}^c$ such that $|x_1 - a| < \delta$ and $|x_2 - a| < \delta$.

Since $f(x_1) = 1$ and $f(x_2) = 0$, then f cannot be continuous at a.

Ex(4). Prove that $f(x) = \begin{cases} \frac{1}{q}, & x = \frac{p}{q} \in \mathbb{Q} \text{ (in reduced form)} \\ 0, & x \notin \mathbb{Q}. \end{cases}$

is continuous at every irrational in $(0,1)$ but discontinuous at every rational in $(0,1)$.

Proof: First, we shall prove that f is discont. at every rational in $(0,1)$. Let a be a rational in $(0,1)$ and spse that f is cont. at a. If x_n is a sequence of irrationals s.t. $x_n \longrightarrow a$, then $f(x_n) \longrightarrow f(a)$ as $n \to \infty$, i.e, $f(a) = 0$. But $f(a) \neq 0$ by defn.

Hence, f is discontinuous at every rational in $(0,1)$.

Next, we want to prove that f is cont. at every irrational in $(0,1)$. Indeed, Let a be an irrational in $(0,1)$. We must show that $f(x_n) \longrightarrow f(a)$ for every sequence $x_n \in (0,1)$ which satisfies $x_n \to a$ as $n \to \infty$. We may suppose that $x_n \in \mathbb{Q}$. $\forall n \in \mathbb{N}$, write $x_n = \frac{p_n}{q_n}$ in reduced form. Since $f(a)=0$, it suffices to show that $q_n \to \infty$ as $n \to \infty$ (since We need to prove that $f(x_n) = \frac{1}{q_n} \longrightarrow f(a) = 0$).

Suppose to the contrary that there exist integers $n_1 < n_2 < \cdots$ such that $|q_{n_k}| \le M < \infty$ for $k \in \mathbb{N}$. Since $x_{n_k} \in (0,1)$, it follows that the set $E := \left\{ x_{n_k} = \frac{p_{n_k}}{q_{n_k}} : k \in \mathbb{N} \right\}$

contains only a finite number of pts. Hence, the limit of any sequence in E must belong to E, a contradiction since a is such a limit and is irrational.

Rmk. the composition of two functions $g \circ f$ can be nowhere continuous, even though f is discont. only on $\mathbb{Q}$ and g is discont. at only one point.

proof. Let $f(x) = \begin{cases} \frac{1}{q}, & x = \frac{p}{q} \in \mathbb{Q} \text{ (in reduced form)} \\ 0, & x \notin \mathbb{Q}. \end{cases}$

$$g(x) = \begin{cases} 1, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

clearly, $(g \circ f)(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q}. \end{cases}$

Hence, $g \circ f$ is the Dirichlet function, nowhere continuous by Ex ③.

H.w's (Exercises p. 90; 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 i.e., All).

Good Luck.

3.4 Uniform Continuity

Df①. Let E be a nonempty subset of $\mathbb{R}$ and $f: E \longrightarrow \mathbb{R}$. Then f is said to be uniformly continuous on E iff $\forall \varepsilon > 0$, $\exists$ a $\delta > 0$ s.t

$$|x-a| < \delta \text{ and } x, a \in E \Longrightarrow |f(x)-f(a)| < \varepsilon .$$

Notice that δ here depends on ε and f, but not on a and x.

Ex① Prove that $f(x) = x^2$ is uniformly continuous on $(0,1)$.

Proof. Given $\varepsilon > 0$, set $\delta = \frac{\varepsilon}{2}$. If $x, a \in (0,1)$, then $|x+a| \leq |x| + |a| \leq 2$

Therefore, if $x, a \in (0,1)$ and $|x-a| < \delta$, then

$$|f(x) - f(a)| = |x^2 - a^2| = |x+a|\,|x-a| \leq 2|x-a| < 2\delta = \varepsilon$$

$\blacksquare$

Rmk. ① The difference between the def'ns of continuity and uniform continuity is that for a continuous function, δ may depend on a whereas for a uniformly continuous function,

δ must be chosen <u>independently</u> of a.

② Every uniformly continuous function on E is also continuous on E. But the converse is not true. For ex. $f(x) = x^2$ is cont. on $(-\infty, \infty)$ but it is not uniformly cont.

Ex. ② Show that $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$.

proof. Spse to the contrary that f is uniformly cont. on $\mathbb{R}$. Then $\exists$ a $\delta > 0$ such that

$$|x-a| < \delta \implies |f(x) - f(a)| < 1, \ \forall x, a \in \mathbb{R}$$

By the <u>Archimedean Principle</u>, choose $n \in \mathbb{N}$ so large that $n\delta > 1$. Set $a = n$ and $x = n + \frac{\delta}{2}$. Then $|x-a| = |n + \frac{\delta}{2} - n| = \frac{\delta}{2} < \delta$

but
$$1 > |f(x) - f(a)| = |x^2 - a^2| = |x-a||x+a|$$
$$= \frac{\delta}{2} \cdot \left(2n + \frac{\delta}{2}\right)$$
$$= n\delta + \frac{\delta^2}{4} > n\delta > 1$$

this implies $1 > 1$ which is a contradiction. Hence f is not uniformly cont. on $\mathbb{R}$ ▤

<u>lemma</u> Suppose that $E \subseteq \mathbb{R}$ and that $f: E \longrightarrow \mathbb{R}$ is uniformly continuous. If $x_n \in E$ is Cauchy, then $\{f(x_n)\}$ is Cauchy.

<u>Proof:</u> Let $\varepsilon > 0$ and choose $\delta > 0$ such that $|x-a| < \delta$, $x, a \in E \Longrightarrow |f(x) - f(a)| < \varepsilon$.

Since $\{x_n\}$ is Cauchy. choose $N \in \mathbb{N}$ such that $n, m \geq N \Longrightarrow |x_n - x_m| < \delta$.

Then $n, m \geq N \Longrightarrow |f(x_n) - f(x_m)| < \varepsilon$.

This means $\{f(x_n)\}$ is Cauchy $\blacksquare$.

<u>Thm</u> ① Suppose that I is a closed, bounded interval. If $f: I \longrightarrow \mathbb{R}$ is continuous on I, then f is uniformly continuous on I.

<u>Proof:</u> Spse to the contrary that f is continuous but not uniformly continuous on I.

Then $\exists$ a $\varepsilon_0 > 0$ and $x_n, y_n \in I$ such that

$|x_n - y_n| < \frac{1}{n}$ and $|f(x_n) - f(y_n)| \geq \varepsilon_0$, $n \in \mathbb{N}$

By the Bolzano–Weierstrass thm and the Comparison Thm, $\{x_n\}$ has a convergent subseq. Say $x_{n_k} \longrightarrow x \in I$ as $k \to \infty$.

Similarly, the sequence $\{y_{n_k}\}_{k=1}^{\infty}$ has a convergent subsequence Say $y_{n_{k_j}} \longrightarrow y \in I$ as $j \to \infty$. Since $x_{n_{k_j}} \longrightarrow x$ as $j \to \infty$ and f is continuous, it follows that

$$|f(x) - f(y)| \geq \varepsilon_0, \text{ i.e } f(x) \neq f(y).$$

But $|x_n - y_n| < \frac{1}{n}$ for all $n \in \mathbb{N}$ so by the Squeeze Thm implies $x = y$ Therefore, $f(x) = f(y)$, a contradiction. ∎

Rmk. Thm(1) might not hold if "closed" replaced by "open".

Ex. $f(x) = \frac{1}{x}$ is continuous on $(0,1)$ but not uniformly cont. on $(0,1)$. (see the book p. 93)

Thm (2). Suppose that $a<b$ and that $f:(a,b)\to\mathbb{R}$. Then f is uniformly continuous on (a,b) iff f can be continuously extended to $[a,b]$, i.e., iff there is a continuous function $g:[a,b]\to\mathbb{R}$ which satisfies

$$f(x)=g(x),\quad x\in(a,b).$$

Proof. See the book p.94.

Ex. Prove that $f(x)=\dfrac{x-1}{\ln x}$ is uniformly continuous on $(0,1)$.

Proof. $\displaystyle\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}\frac{x-1}{\ln x}=0.$

$\displaystyle\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}\frac{x-1}{\ln x}=\lim_{x\to 1^-}\frac{1}{1/x}=1$

Let $g(x)=\begin{cases}\dfrac{x-1}{\ln x}, & x\in(0,1)\\ 0, & x=0\\ 1, & x=1\end{cases}$

Notice that $g:[0,1] \to \mathbb{R}$ is a continuous function on $[0,1]$ and $g(x) = f(x), \forall x \in (0,1)$

Hence f is continuously extendable to $[0,1]$, So by thm ②, f is uniformly continuous on $(0,1)$.

<u>Rmk.</u> Let f be cont. on a bounded, open, nondegenerate interval (a,b). Notice that f is continuously extendable to $[a,b]$ iff $\lim\limits_{x\to a^+} f(x)$ and $\lim\limits_{x\to b^-} f(x)$ exist.

Indeed, when they exist, we define g at $x=a$ and $x=b$ as

$$g(a) = \lim_{x\to a^+} f(x), \quad g(b) = \lim_{x\to b^-} f(x)$$

<u>H.W's</u> <u>Exercises p.95</u> [0, 1, 2, 3, 4, 5, 6, ~~[illegible]~~]

~~[illegible]~~

<u>Good Luck</u>.

CH4 Differentiability on $\mathbb{R}$

4.1 the Derivative

Df ①. A real function f is said to be differentiable at a point $a \in \mathbb{R}$ iff f is defined on some open interval I containing a and

$$\boxed{f'(a) := \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \text{ exists}} \quad (*)$$

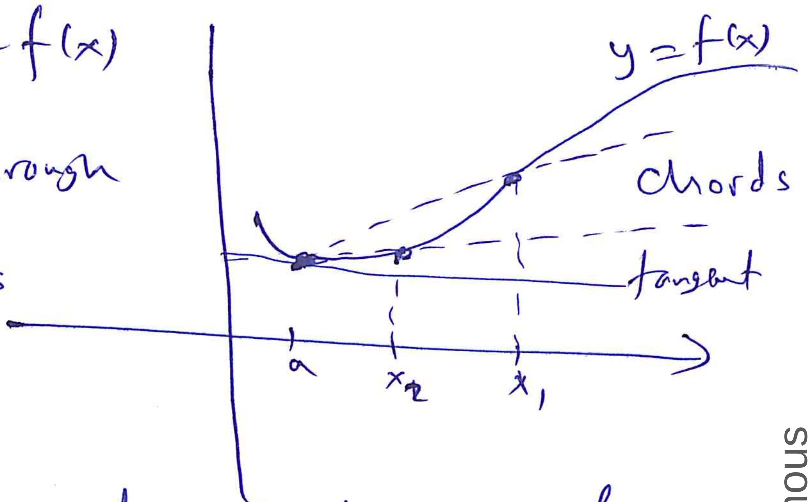
In this case $f'(a)$ is called the derivative of f at a.

Rmk. ① the assumption that f be defined on an open interval containing a is made so that the quotients in $(*)$ are defined for all $h \neq 0$ sufficiently small.

② the graph of $y = f(x)$ has a non-vertical tangent line at $(a, f(a))$ iff $f'(a)$ exists, in this case the slope of the tangent line is $f'(a)$.

Let us Consider a geometric interpretation of $(*)$

Spse that f is diffble at a. A secant line of the graph $y=f(x)$ is a line passing through at least two points on the graph, and



a chord is a line segment which runs from one point on the graph to another.

Let $x=a+h$, the slope of the chord passing through $(x, f(x))$, $(a, f(a))$ is

$$\frac{f(x)-f(a)}{x-a}$$

Since $x=a+h$, (*) becomes

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

Hence, as $x \longrightarrow a$, the slopes of the chords through $(x, f(x))$ and $(a, f(a))$ approximate the slope of the tangent line of $y=f(x)$ at $x=a$

Thus, the slope of the tangent line to $y=f(x)$ at $x=a$ is $f'(a)$.

- $y=f(x)$ has a unique tangent line at $(a, f(a))$ iff $f'(a)$ exists.

- If f is diffble at each point in E, then f' is a function on E.

Notations: $D_x f = \frac{df}{dx} = f^{(1)}(x) = f'(x) = y' = \frac{dy}{dx}$ when $y=f(x)$.

- Higher order derivatives are defined as $f^{(n+1)}(a) := \left(f^{(n)}\right)'(a)$, $n \in \mathbb{N}$ provided these derivatives exist.

Notation $D_x^n f$, $\frac{d^n f}{dx^n}$, $f^{(n)}$, and $\frac{d^n y}{dx^n}$, $y^{(n)}$ when $y=f(x)$

Thm ①. A real function f is diff'ble at $x=a\in\mathbb{R}$ iff $\exists$ an open interval I and a function $F: I\longrightarrow\mathbb{R}$ such that $a\in I$, f is defined on I, F is continuous at a, and

$$f(x) = F(x)(x-a) + f(a).$$

holds $\forall x\in I$, in which case $F(a)=f'(a)$

Proof. ($\Longrightarrow$) suppose that f is diff'ble at a then f is defined on some open interval I containing a, and the limit

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} \text{ exists.}$$

Define F on I by

$$F(x) := \begin{cases} \dfrac{f(x)-f(a)}{x-a}, & x\neq a \\ f'(a), & x=a \end{cases}$$

Then $f(x) = F(x)(x-a)+f(a)$, $\forall x\in I$, and F is continuous at a. since $f'(a)$ exists.

($\Leftarrow$) Conversely, suppose that $\exists$ an open interval I and $F: I \longrightarrow \mathbb{R}$ s.t $a \in I$, f is defined on I, F is continuous at a and $f(x) = F(x)(x-a) + f(a)$, $\forall x \in I$. Then

$$F(x) = \frac{f(x) - f(a)}{x-a}, \quad x \neq a.$$

The continuity of F implies that

$$F(a) = \lim_{x \to a} F(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x-a} \text{ exists}$$

therefore, f is diff'ble at a and

$$f'(a) = F(a).$$

$\blacksquare$

<u>Thm 2.</u> A real function f is diff'ble at $x=a$ iff $\exists$ a function T of the form $T(x) := mx$ such that

$$\lim_{h \to 0} \frac{f(a+h) - f(a) - T(h)}{h} = 0$$

Proof: ($\Rightarrow$) suppose that f is diff'ble at a, and define T as $T(x) = mx$, where $m = f'(a)$. Then by (*),

$$\frac{f(a+h)-f(a)-T(h)}{h} = \frac{f(a+h)-f(a)}{h} - f'(a)$$

$$\longrightarrow 0 \quad \text{as } h \to 0 .$$

Conversely, suppose that $\exists$ a function T of the form $T(x) = mx$ s.t

$$\lim_{h\to 0} \frac{f(a+h)-f(a)-T(h)}{h} = 0,$$

then for $h \neq 0$,

$$\frac{f(a+h)-f(a)}{h} = m + \frac{f(a+h)-f(a)-mh}{h}$$

$$= m + \underbrace{\frac{f(a+h)-f(a)-T(h)}{h}}_{\searrow 0}$$

$$\Rightarrow \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = m + 0 \quad \text{(assumption)}$$

$$= m$$

that is, $f'(a)$ exists and equals m

therefore, f is diff'ble at $x = a$ ▤

Thm ③. If f is diff'ble at a, then f is continuous at a.

Proof: Suppose that f is diff'ble at a. By Thm ①, $\exists$ an open interval I and a function F, continuous at a, such that

$$f(x) = F(x)(x-a) + f(a), \quad \forall x \in I.$$

Taking the limit as $x \to a$, we see that

$$\lim_{x \to a} f(x) = F(a) \cdot 0 + f(a) = f(a).$$

In particular, $f(x) \to f(a)$ as $x \to a$; i.e., f is continuous at a ▤

Rmk. the converse of Thm ③ is false

example. Show that $f(x) = |x|$ is continuous at 0 but not diff'ble there.

proof: Since $x \to 0 \Rightarrow |x| \to 0$, f is continuous at 0. On the other hand

$$f'_+(0) = \lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^+} \frac{|h| - 0}{h}$$

$= \lim_{h \to 0^+} \frac{h}{h} = 1$

$f'_-(0) = \lim_{h \to 0^-} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^-} \frac{-h}{h} = -1$

Since $f'_+(0) \neq f'_-(0)$, it follows that $f'(0)$ doesnot exist. Therefore, f is not diffble at 0. ⊟

Df ②. Let I be a nondegenerate interval.

(i) A function $f : I \to \mathbb{R}$ is said to be diffble on I if and only if

$$f'_I(a) := \lim_{\substack{x \to a \\ x \in I}} \frac{f(x) - f(a)}{x - a} \text{ exists}$$

and is finite $\forall a \in I$.

(ii) f is said to be Continuously diffble on I if and only if f'_I exists and is continuous on I.

Rmk. when a is not an endpoint of I, $f'_I(a)$ is the same as $f'(a)$.

If is diffble on $[a,b]$. Then

$$f'(a) = \lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h}$$ and

$$f'(b) = \lim_{h\to 0^-} \frac{f(b+h)-f(b)}{h}.$$

Ex. Show that $f(x) = x^{3/2}$ is diffble on $[0,\infty)$ and $f'(x) = \frac{3\sqrt{x}}{2}$, $\forall x\in[0,\infty)$.

Pf. By the power Rule, $f'(x) = \frac{3}{2}x^{\frac{1}{2}} = \frac{3\sqrt{x}}{2}$ $\forall x\in(0,\infty)$. And by def'n,

$$f'(0) = \lim_{h\to 0^+} \frac{h^{3/2}-0}{h} = \lim_{h\to 0^+} \sqrt{h} = 0.$$

$\therefore f'(x) = \frac{3\sqrt{x}}{2}$, $\forall x\in[0,\infty)$. ☒

Notation $C^{n}(I)$.

Let I be a nondegenerate interval. For $n\in\mathbb{N}$, we define the collection of functions $C^n(I)$ by

$C^n(I) := \{ f : f : I \longrightarrow \mathbb{R}$ and $f^{(n)}$ exists and is continuous on $I \}$

- When $f \in C^n(I)$, $\forall n \in \mathbb{N}$, we shall denote it by $f \in C^\infty(I)$.

- Notice that $C^1(I)$ is precisely the collection of real functions which are continuously diff'ble on I.

- $C^n([a,b]) = C^n[a,b]$.

- $C^\infty(I) \subset C^m(I) \subset C^n(I)$, for all integers $m > n > 0$

- Not every function which is diff'ble on $\mathbb{R}$ belongs to $C^1(\mathbb{R})$.

Ex. $f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & , x \neq 0 \\ 0 & , x = 0 \end{cases}$

is diff'ble on $\mathbb{R}$ but not continuously diff'ble on any interval contains the origin.

Proof. By def'n.

$$f'(0) = \lim_{h\to 0} \frac{f(h)-f(0)}{h} = \lim_{h\to 0} h\sin\left(\frac{1}{h}\right) = 0$$

and $f'(x) = 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$, $x \neq 0$.

Thus, f is diffble on $\mathbb{R}$ but

$\lim_{x\to 0} f'(x)$ does not exist. In particular,

f' is not continuous on any interval which contains the orgin. $\blacksquare$

Rmk. A function which is diffble on two sets is not necessarily diffble on their union.

example $f(x) = |x|$ is diffble on $[0,1]$ and on $[-1,0]$ but not on $[-1,1]$.

Proof. Since $f(x) = \begin{cases} x & \text{when } x > 0 \\ -x & \text{when } x < 0 \end{cases}$, it is

clear that f is diffble on $[-1,0)\cup(0,1]$

(with $f'(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$)

we proved that f is not diffble at $x=0$.

However, $f'_{[0,1]}(0) = \lim_{h\to 0^+} \frac{|h|}{h} = 1$

$$f'_{[-1,0]}(0) = \lim_{h\to 0^-} \frac{|h|}{h} = -1.$$

Therefore, f is diffble on $[0,1]$ and on $[-1,0]$ but not on $[-1,1]$ ∎

<u>Exercises</u> (H.W's) 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 (All).

4.2. Differentiability Theorems

Thm ④. Let $I \subseteq \mathbb{R}$ be an interval, let $a \in I$, $\alpha \in \mathbb{R}$ and let $f: I \to \mathbb{R}$, $g: I \to \mathbb{R}$ be functions that diff'ble at a,

then $f+g$, αf, $f \cdot g$ and [when $g(a) \neq 0$] $\frac{f}{g}$ are all diff'ble at a. In fact,

(i) $(f+g)'(a) = f'(a) + g'(a)$

(ii) $(\alpha f)'(a) = \alpha f'(a)$.

(iii) $(f \cdot g)'(a) = g(a) f'(a) + f(a) g'(a)$

(iv) $\left(\frac{f}{g}\right)'(a) = \frac{g(a) f'(a) - f(a) g'(a)}{g^2(a)}$

Proof: We shall prove (iii) and (iv), leaving (i), (ii) as exercises.

(iii) Let $p := fg$, then for $x \in I$, $x \neq a$ we have

$$\frac{p(x) - p(a)}{x-a} = \frac{f(x) g(x) - f(a) g(a)}{x-a}$$

$$= \frac{f(x) g(x) - f(a) g(x) + f(a) g(x) - f(a) g(a)}{x-a}$$

$$= \frac{f(x)-f(a)}{x-a} \cdot g(x) + f(a) \cdot \frac{g(x)-g(a)}{x-a},$$

Since g is continuous at a, by thm 3, then $\lim_{x\to a} g(x) = g(a)$. Since f and g are diff'ble at $x=a$, we deduce that

$$\lim_{x\to a} \frac{p(x)-p(a)}{x-a} = f'(a) \cdot g(x) + f(a)\, g'(a)$$

Hence $p := fg$ is diff'ble at a and (iii) holds.

(iv) Let $q := \frac{f}{g}$. Since g is diff'ble at a, it is continuous at that point (Thm). Therefore, since $g(a) \neq 0$, then (by thm), $\exists$ an interval $J \subseteq I$ with $a \in J$ s.t. $g(x) \neq 0$, $\forall x \in J$.

For $x \in J$, $x \neq a$, we have

$$\frac{q(x)-q(a)}{x-a} = \frac{\frac{f(x)}{g(x)} - \frac{f(a)}{g(a)}}{x-a}$$

$$= \frac{f(x)\,g(a) - f(a)\,g(x)}{(x-a)\,g(x)\,g(a)}$$

$$= \frac{f(x)g(a) - f(a)g(a) + f(a)g(a) - f(a)\,g(x)}{(x-a)\;g(x)\,g(a)}$$

$$= \frac{1}{g(x)g(a)}\left[\frac{f(x)-f(a)}{x-a}\cdot g(a) - f(a)\cdot\frac{g(x)-g(a)}{x-a}\right]$$

Using the continuity of g at a and the differentiability of f and g at a, then we get

$$q'(a) = \lim_{x\to a}\frac{q(x)-q(a)}{x-a} = \frac{f'(a)\,g(a) - g'(a)\,f(a)}{g^2(a)}$$

Thus, $q = \frac{f}{g}$ is diff'ble at a and (iv) holds. ∎

Rmk. Formula in (i) is Called the Sum Rule, in (iii) is Called the Product Rule, in (ii) the homogeneous Rule and in (iv) is Called the Quotient Rule.

Corollary. If $f_1, f_2, ---, f_n$ are functions on an interval I to $\mathbb{R}$ that are diffble at $a \in I$, then:

(i) the function $f_1 + f_2 + --- + f_n$ is diffble at a and $(f_1 + f_2 + -- + f_n)'_{(a)} = f_1'(a) + f_2'(a) + --- + f_n'(a)$

(ii) the function $f_1 f_2 --- f_n$ is diffble at a, and $(f_1 f_2 -- f_n)'(a) = f_1'(a) f_2(a) --- f_n(a)$

$$+ f_1(a) f_2'(a) --- f_n(a) + ----- + f_1(a) f_2(a) --- f_n'(a).$$

Proof: Use Mathematical Induction.

Thm ⑤ [Chain Rule]

Let f and g be real functions. If f is diffble at a and g is diffble at $f(a)$, then $g \circ f$ is diffble at a with

$$(g \circ f)'(a) = g'(f(a)) \, f'(a).$$

Proof: By thm ①, $\exists$ open intervals I and J, and functions $F: I \longrightarrow \mathbb{R}$, continuous at a, and $G: J \longrightarrow \mathbb{R}$, continuous at $f(a)$,

such that $F(a) = f'(a)$, $G(f(a)) = g'(f(a))$,

$$\boxed{f(x) = F(x)(x-a) + f(a), \quad x \in I} \quad (A)$$

and $$\boxed{g(y) = G(y)\left(y - f(a)\right) + g(f(a)), \; y \in J.} \quad (B)$$

Since f is continuous at a, we may assume that $f(x) \in J$, $\forall x \in I$.

Fix $x \in I$. Apply (B) to $y = f(x)$ and (A) to to write

$$(g \circ f)(x) = g(f(x))$$

$$= G(f(x))\left(f(x) - f(a)\right) + g(f(a)) \quad [\text{using (B)}]$$

$$= G(f(x))\,F(x)(x-a) + (g \circ f)(a) \quad [\text{using (A)}]$$

Set $H(x) = G(f(x))F(x)$ for $x \in I$.

Since F is continuous at a and G is continuous at $f(a)$, it is clear that H is continuous at a. Moreover,

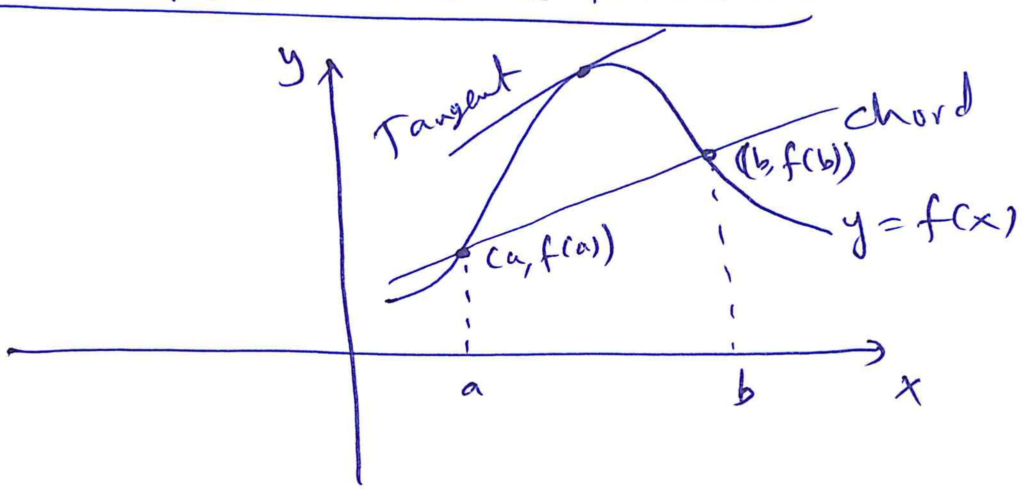
$$H(a) = G(f(a))\,F(a) = g'(f(a))\,f'(a)$$

It follows from Thm ①, $(g\circ f)'(a) = H(a)$, i.e.,

$$(g\circ f)'(a) = g'(f(a))\,f'(a).$$

<u>Exercises p.106</u> (H.w's) 0, 1, 2, 3, 4, 5, 8, 9.

4.3 The Mean value Theorem



lemma. [Rolle's thm].

Suppose that $a, b \in \mathbb{R}$ with $a < b$. If f is continuous on $[a,b]$, diffble on (a,b), and if $f(a) = f(b)$, then $f'(c) = 0$ for some $c \in (a,b)$.

Proof. By the Extreme value thm, f has a finite maximum M and a finite minimum m on $[a,b]$

If $M = m$, then f is constant on (a,b) and $f'(x) = 0$, $\forall x \in (a,b)$.

Suppose that $M \neq m$. Since $f(a) = f(b)$, f must assume one of the values M or m at some $c \in (a,b)$. We may suppose that $f(c) = M$ (similar proof when $f(c) = m$)

Since M is the max. of f on $[a,b]$, we have $f(c+h)-f(c)\le 0$ for all h satisfy $c+h\in(a,b)$. In the case $h>0$ this implies

$$f'(c)=\lim_{h\to 0^+}\frac{f(c+h)-f(c)}{h}\le 0$$

and the case $h<0$,

$$f'(c)=\lim_{h\to 0^-}\frac{f(c+h)-f(c)}{h}\ge 0.$$

It follows that $f'(c)=0$ $\blacksquare$

Rmk. ① the continuity hypothesis in Rolle's thm cannot be relaxed at even one point in $[a,b]$.

proof. $f(x)=\begin{cases} x, & x\in[0,1) \\ 0, & x=1 \end{cases}$ is continuous on $[0,1)$, diff'ble on $(0,1)$, and $f(0)=f(1)=0$ but $f'(x)$ is never zero.

② the differentiability hypothesis in Rolle's thm cannot be relaxed at even one point in (a,b).

proof: $f(x)=|x|$ is cont. on $[-1,1]$, diffble on $(-1,1)\setminus\{0\}$, and $f(-1)=f(1)=1$ but $f'(x)$ is never zero.

Thm 6 Suppose that $a,b\in\mathbb{R}$ with $a<b$.

(i) [Generalized Mean value Thm]. If f, g are continuous on $[a,b]$ and diffble on (a,b), then there is a $c\in(a,b)$ such that

$$g'(c)\big(f(b)-f(a)\big)=f'(c)\big(g(b)-g(a)\big).$$

(ii) [Mean value Thm] If f is continuous on $[a,b]$, and diffble on (a,b), then there is a $c\in(a,b)$ such that

$$f(b)-f(a)=f'(c)(b-a).$$

proof. (i) set

$$h(x)=f(x)\big(g(b)-g(a)\big)-g(x)\big(f(b)-f(a)\big).$$

Since $h'(x) = f'(x)\big(g(b)-g(a)\big) - g'(x)\big(f(b)-f(a)\big)$

it is clear that h is cont. on $[a,b]$, diff'ble on (a,b), and $h(a) = h(b) = 0$.

Thus, by Rolle's Thm, $h'(c) = 0$ for some $c \in (a,b)$. That is, there is a $c \in (a,b)$ s.t.

$$g'(c)\big(f(b)-f(a)\big) = f'(c)\big(g(b)-g(a)\big).$$

(ii) Set $g(x) = x$ and apply part (i). Then $\exists$ a $c \in (a,b)$ s.t

$$f(b) - f(a) = f'(c)(b-a). \quad \blacksquare$$

<u>Rmk.</u> ① The Generalized Mean Value Thm is also called <u>Cauchy's Mean Value Thm</u>

② For a geometric interpretation of (ii), see the opening graph p.19 (Note).

③ The Mean Value Thm is most often used to extract information about f from f' as follows.

Df: let E be a nonempty subset of $\mathbb{R}$ and $f: E \to \mathbb{R}$.

(i) f is said to be <u>increasing</u> (resp., strictly increasing) on E iff

$$x_1, x_2 \in E \text{ and } x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2) \; [\text{resp. } f(x_1) < f(x_2)]$$

(ii) f is said to be <u>decreasing</u> (resp., strictly decreasing) on E iff

$$x_1, x_2 \in E \text{ and } x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2) \; [\text{resp., } f(x_1) > f(x_2)]$$

(iii) f is said to be <u>monotone</u> (resp., strictly monotone) on E iff f is either decreasing or increasing (resp., either strictly decreasing or strictly increasing) on E.

<u>ex.</u> $f(x) = x^2$ is <u>strictly monotone</u> on $[0,1]$, and on $[-1,0]$, it is not monotone on $[-1,1]$.

<u>Thm 7.</u> Suppose that $a, b \in \mathbb{R}$, with $a < b$, that f is continuous on $[a,b]$, and that f is diffble on (a,b).

i) If $f'(x)>0$ [resp., $f'(x)<0$], $\forall x\in(a,b)$, then f is strictly increasing [resp, strictly decreasing) on $[a,b]$.

ii) If $f'(x)=0$, $\forall x\in(a,b)$, then f is constant on $[a,b]$.

iii) If g is continuous on $[a,b]$ and diffble on (a,b), and if $f'(x)=g'(x)$, $\forall x\in(a,b)$, then $f-g$ is constant on $[a,b]$.

Proof: (i) Let $a\le x_1<x_2\le b$. By the Mean Value thm, $\exists$ a $c\in(a,b)$ such that $f(x_2)-f(x_1)=f'(c)(x_2-x_1)$. Thus, $f(x_2)>f(x_1)$ when $f'(c)>0$ and $f(x_2)<f(x_1)$, when $f'(c)<0$.

(ii) If $f'(x)=0$, then by the proof of part (i), f is both increasing and decreasing, and hence constant on $[a,b]$.

(iii) Follows from part (ii) applied to $h=f-g$. ($h'(x)=f'(x)-g'(x)=0 \Rightarrow h(x)=f(x)-g(x)$ is constant on $[a,b]$)

Thm ⑧. Suppose that f is increasing on $[a,b]$.

(i) If $c \in [a,b)$, then $f(c^+)$ exists and
$$f(c) \le f(c^+) = \lim_{x\to c^+} f(x)$$

(ii) If $c \in (a,b]$, then $f(c^-)$ exists and
$$f(c^-) = \lim_{x \to c^-} f(x) \le f(c)$$

Proof. See the textbook p.112.

Thm ⑨. If f is monotone on an interval I, then f has at most countably many points of discontinuity on I.

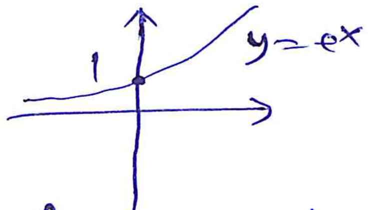
proof. See the textbook p.113.

Application (Thm ⑦ (i)).

ex. Prove that $1+x < e^x$, $\forall x>0$.

Proof. Let $f(x) = e^x - x$, and observe that

$f'(x) = e^x - 1 > 0$, $\forall x > 0$



It follows from Thm ⑦ (i), that $f(x)$ is strictly increasing on $(0,\infty)$. Thus, As $x>0$, then $e^x - x = f(x) > f(0) = 1$. In particular, $e^x > x+1$, $\forall x>0$ ∎

Thm (10). [Bernoulli's Inequality].

Let α be a positive real number.

If $0 < \alpha \leq 1$, then $(1+x)^\alpha \leq 1+\alpha x$, $\forall x \geq -1$,

and if $\alpha \geq 1$, then $(1+x)^\alpha \geq 1+\alpha x$, $\forall x \geq -1$.

Proof. Case 1 $0 < \alpha \leq 1$.

Fix $x \geq -1$ and let $f(t) = t^\alpha$, $t \in [0, \infty)$.

Since $f'(t) = \alpha t^{\alpha-1}$, it follows from the Mean value thm (applied to $a=1$, and $b=1+x$) that $f(1+x) - f(1) = f'(c)(1+x-1)$

(*) $\boxed{f(1+x) - f(1) = \alpha x c^{\alpha-1}}$, for some c between 1 and $1+x$.

Sub Case 1.1 $x > 0$. Then $c > 1$. Since $0 < \alpha \leq 1$ implies $\alpha - 1 \leq 0$, it follows that $c^{\alpha-1} \leq 1$, hence $x c^{\alpha-1} \leq x$. therefore, we have by (*) that $(1+x)^\alpha = f(1+x) = f(1) + \alpha x c^{\alpha-1}$

$$\leq f(1) + \alpha x = 1 + \alpha x$$

as required.

Sub Case 1.2. $-1 \le x \le 0$. Then $c \le 1$ so $c^{\alpha-1} \ge 1$. But since $x \le 0$, it follows that $x c^{\alpha-1} \le x$ as before, we have by (*) that

$$(1+x)^\alpha = f(1+x) = f(1) + \alpha x c^{\alpha-1}$$
$$\le f(1) + \alpha x = 1 + \alpha x.$$

Case 2. $\alpha \ge 1$. (Exercise) $\boxminus$

Ex. prove that the sequence $x_n = \left(1+\frac{1}{n}\right)^n$ is increasing and $\lim\limits_{n\to\infty} x_n = L$ satisfies $2 < L \le 3$ [$\lim\limits_{n\to\infty} x_n = e$ as you know $= 2.718281828459\ldots$]

proof. $x_n = \left(1+\frac{1}{n}\right)^n$ is increasing, since by Bernoulli's Inequality,

$$x_n = \left(1+\frac{1}{n}\right)^n = \left[\left(1+\frac{1}{n}\right)^{\frac{n}{n+1}}\right]^{n+1}$$
$$\le \left[1 + \left(\frac{1}{n}\right)\left(\frac{n}{n+1}\right)\right]^{n+1}$$
$$= \left(1+\frac{1}{n+1}\right)^{n+1} = x_{n+1}.$$

- To prove that this sequence is bounded above, observe that by the Binomial Formula

$$\left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k}\left(\frac{1}{n}\right)^k \cdot (1)^{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k}\left(\frac{1}{n}\right)^k$$

Now, $$\binom{n}{k}\left(\frac{1}{n}\right)^k = \frac{n!}{k!\,(n-k)!\cdot n^k}$$

$$= \frac{n(n-1)(n-2)\cdots(n-(k-1))\cancel{(n-k)!}}{k!\,\cancel{(n-k)!}\,n^k}$$

$$= \frac{n(n-1)(n-2)\cdots(n-k+1)}{n^k}\cdot\frac{1}{k!}$$

$$\leq 1\cdot\frac{1}{k!} \leq \frac{1}{2^{k-1}},$$

for all $k \in \mathbb{N}$. Next,

$$2 = \left(1+\frac{1}{1}\right) < \left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n}\binom{n}{k}\left(\frac{1}{n}\right)^k$$

$$= 1+\sum_{k=1}^{n}\binom{n}{k}\left(\frac{1}{n}\right)^k$$

$$\leq 1+\sum_{k=1}^{n}\frac{1}{2^{k-1}} = 1+\frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}} = 3-\frac{1}{2^{n-1}}$$

$$< 3$$

for $n>1$. Hence, by the Monotone Convergence thm, the limit L exists and satisfies

$$2 < L \leq 3.$$

<u>Thm (11)</u> [Intermediate value theorem for derivatives]. [or Darboux's Thm].
Suppose that f is diffble on $[a,b]$ with $f'(a) \neq f'(b)$. If y_0 is a real number which lies between $f'(a)$ and $f'(b)$, then there is an $x_0 \in (a,b)$ such that $f'(x_0) = y_0$.

<u>Proof</u>. Suppose that y_0 lies between $f'(a)$ and $f'(b)$. By symmetry, we may assume that $f'(a) < y_0 < f'(b)$. Set $F(x) = f(x) - y_0 x$, for $x \in [a,b]$, and observe that F is diffble on $[a,b]$. Hence, by the Extreme Value Thm, F has an absolute minimum, say $F(x_0)$, on $[a,b]$. Now, $F'(a) = f'(a) - y_0 < 0$, so $F(a+h) - F(a) < 0$ for $h>0$ sufficiently small. Hence, $F(a)$ is NOT the absolute minimum of F on $[a,b]$. Similarly, $F(b)$ is not the absolute minimum of F on $[a,b]$. Hence, the absolute minimum $F(x_0)$

must occur on (a,b), i.e., $x_0 \in (a,b)$ and $0 = F'(x_0) = y_0 - f'(x_0)$. Hence, $f'(x_0) = y_0$.

Ex. The function $g: [-1,1] \longrightarrow \mathbb{R}$ defined by

$$g(x) := \begin{cases} 1, & 0 < x \leq 1 \\ 0, & x = 0 \\ -1, & -1 \leq x < 0, \end{cases}$$

clearly fails to satisfy the intermediate value property on $[-1,1]$. Therefore, by Darboux's Thm, there does not exist a function f s.t. $f'(x) = g(x)$, $\forall x \in [-1,1]$. In otherwords, g is NOT the derivative on $[-1,1]$ of any function.

H.W's Exercises 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 (i.e, All).

Good Luck

Chapter 5 Integrability on $\mathbb{R}$

5.1 The Riemann Integral.

Df ①. Let $a, b \in \mathbb{R}$, with $a < b$.

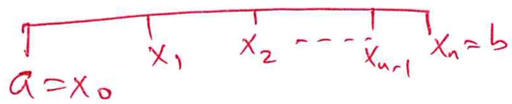
(i) A *partition* of $[a,b]$ is a set of subintervals

$$[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n],$$

where $a = x_0 < x_1 < \cdots < x_n = b$. $(*)$

Thus any set of $(n+1)$ points satisfying $(*)$ defines a partion P of $[a,b]$, which we denote by

$$P = \{x_0, x_1, \ldots, x_n\}.$$



(ii) the *norm* of a partition $P = \{x_0, x_1, \ldots, x_n\}$ is the number

$$\|P\| := \max\{x_1 - x_0, x_2 - x_1, \ldots, x_n - x_{n-1}\}$$
$$= \max_{1 \le j \le n} |x_j - x_{j-1}|$$

(iii) A *refinement* of a partition $P = \{x_0, x_1, \ldots, x_n\}$ is a partition Q of $[a,b]$ which satisfies $Q \supseteq P$. And we say that Q is finer than P.

<u>Ex.</u> Prove that for each $n \in \mathbb{N}$,

$P_n = \{ \frac{j}{2^n} : j = 0, 1, ---, 2^n \}$ is a partition of $[0,1]$ and P_m is finer than P_n when $m > n$.

<u>Proof:</u> Fix $n \in \mathbb{N}$. If $x_j = \frac{j}{2^n}$, then

$0 = x_0 < x_1 < x_2 < ---- < x_{2^n} = 1$. Thus,

P_n is a partition of $[0,1]$.

Next, we need to show that $P_m \supseteq P_n$ when $m > n$.

Let $m > n$ and set $p = m - n$. If $0 \le j \le 2^n$, then $\frac{j}{2^n} = \frac{j \cdot 2^p}{2^m}$ and $0 \le j\, 2^p \le 2^m$.

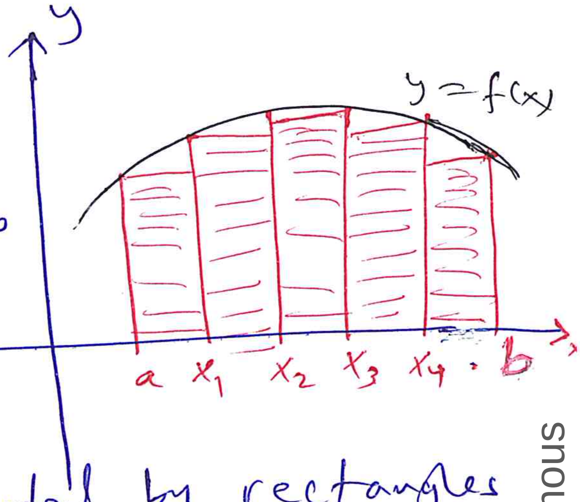
thus P_m is finer than P_n. ∎

<u>Rmk</u> ① If P and Q are partitions of $[a,b]$, then $P \cup Q$ is finer than both P and Q.

② If Q is a refinement of P (i.e., $Q \supseteq P$), then $\|Q\| \le \|P\|$

Recall, let f be nonnegative on $[a,b]$,

$\int_a^b f(x)dx$ = Area of the region bounded by $y=f(x)$, $y=0$, $x=a$, $x=b$ ~~if~~ when this integral exists.



this Area can be approximated by rectangles whose base lie in $[a,b]$ and whose heights approximate f. If the tops of these rectangles lie above $y=f(x)$, then $A_{approximate} > A_{exact}$. If the tops of these rectangles lie below $y=f(x)$, then $A_{approximate} < A_{exact}$.

Df ②. let $a,b \in \mathbb{R}$ with $a<b$. Let $P=\{x_0, x_1, ---, x_n\}$ be a partition of $[a,b]$, set $\Delta x_j := x_j - x_{j-1}$, for $j = 1, 2, ---, n$ and suppose that $f:[a,b] \longrightarrow \mathbb{R}$ is bounded.

(i) the upper Riemann sum of f over P is

$$\boxed{U(f,P) := \sum_{j=1}^{n} M_j(f)\Delta x_j},$$ where

$$\boxed{M_j(f) = \sup_{x_{j-1} \leq t \leq x_j} f(t)}$$

(ii) the lower Riemann sum of f over P is $\boxed{L(f,P) = \sum_{j=1}^{n} m_j(f)\,\Delta x_j}$,

where $\boxed{m_j(f) = \inf_{x_{j-1} \le t \le x_j} f(t)}$

Note: Since f is bounded, then $M_j(f)$ and $m_j(f)$ exist and are finite.

Rmk. If $g: \mathbb{N} \longrightarrow \mathbb{R}$, then

$$\sum_{k=m}^{n} \big(g(k+1) - g(k)\big) = g(n+1) - g(m).$$

Proof. Use induction on n. (see the textbook). ■

by this Rmk., If $P = \{x_0, x_1, \dots, x_n\}$ is a partition of $[a,b]$, then

$$\sum_{j=1}^{n} \Delta x_j = \sum_{j=1}^{n} x_j - x_{j-1}$$

$$= x_n - x_0 = b - a.$$

Rmk. If $f(x)=\alpha$ is constant on $[a,b]$, then $U(f,P)=L(f,P)=\alpha(b-a)$, for all partitions P of $[a,b]$.

Proof:

Since $M_j(f)=\sup\limits_{t\in[x_{j-1},x_j]} f(t) = \sup\limits_{t\in[x_{j-1},x_j]}(\alpha) = \alpha$,

then
$$U(f,P)=\sum_{j=1}^{n} M_j(f)\,\Delta x_j$$
$$=\sum_{j=1}^{n}\alpha\,\Delta x_j$$
$$=\alpha\sum_{j=1}^{n}\Delta x_j = \alpha(b-a).$$

Similarly,
$$L(f,P)=\sum_{j=1}^{n} m_j(f)\,\Delta x_j = \sum_{j=1}^{n}\alpha\,\Delta x_j$$
$$=\alpha\sum_{j=1}^{n}\Delta x_j = \alpha(b-a) \quad \blacksquare$$

Rmk. $L(f,P)\le U(f,P)$ for all partitions P and all bounded functions f.

Proof: By definition, $m_j(f)\le M_j(f)$ for all j, then $L(f,P)\le U(f,P)$ $\blacksquare$

Rmk. If P is any partition of $[a,b]$. and Q is a refinement of P, then

$$L(f,P) \leq L(f,Q) \leq U(f,Q) \leq U(f,P).$$

Pf (see textbook).

Rmk. If P and Q are any partitions of $[a,b]$, then $L(f,P) \leq U(f,Q)$.

proof. Since $P \cup Q$ is a refinement of P and Q, it follows from last remark that

$$L(f,P) \leq L(f, P\cup Q) \leq U(f, P\cup Q) \leq U(f,Q)$$

for any pair of partitions P, Q whether Q is a refinement of P or not $\blacksquare$

Df (3) Let $a,b \in \mathbb{R}$ with $a<b$. A function $f: [a,b] \to \mathbb{R}$ is said to be (Riemann) integrable on $[a,b]$ iff f is bounded on $[a,b]$ and $\forall\, \varepsilon > 0$, $\exists$ a partion P of $[a,b]$ such that $U(f,P) - L(f,P) < \varepsilon$.

Thm ①. Suppose that $a, b \in \mathbb{R}$ with $a < b$. If f is continuous on $[a,b]$, then f is integrable on $[a,b]$.

Proof. Let $\varepsilon > 0$. Since f is uniformly continuous on $[a,b]$, $\exists$ a $\delta > 0$ such that

$$\boxed{|x-y| < \delta \Longrightarrow |f(x) - f(y)| < \frac{\varepsilon}{b-a} \qquad (1)}$$

Let $P = \{x_0, x_1, \ldots, x_n\}$ be any partition of $[a,b]$ which satisfies $\|P\| < \delta$. Fix an index j. Then, by Extreme value thm, there are points x_m and x_M in $[x_{j-1}, x_j]$ such that $f(x_m) = m_j(f)$ and $f(x_M) = M_j(f)$.

Since $\|P\| < \delta$, we also have $|x_M - x_m| < \delta$.

Hence by (1),

$$M_j(f) - m_j(f) = |M_j(f) - m_j(f)| = |f(x_M) - f(x_m)| < \frac{\varepsilon}{b-a}.$$

In particular,

$$U(f,P) - L(f,P) = \sum_{j=1}^{n} [M_j(f) - m_j(f)] \Delta x_j < \frac{\varepsilon}{b-a} \sum_{j=1}^{n} \Delta x_j = \varepsilon$$

<u>Ex.</u> the Dirichlet function

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$$ is not Riemann integrable on $[0,1]$.

<u>proof.</u> clearly f is bounded on $[0,1]$. the supremum of f over any nondegenrate interval is 1 and $\inf = 0$ (أشرح تعلل).

therefore $U(f,P) - L(f,P) = 1 - 0 = 1$ for any partition P of $[0,1]$ (i.e., $\exists\, \varepsilon_0 = 1 > 0$ s.t for any partion P of $[0,1]$,

$$U(f,P) - L(f,P) = \varepsilon_0 = 1$$), that is, f is not integrable on $[0,1]$ ∎

<u>Ex.</u> show that the function

$$f(x) = \begin{cases} 0, & 0 \le x < \frac{1}{2} \\ 1, & \frac{1}{2} \le x \le 1 \end{cases}$$ is <u>integrable</u> on $[0,1]$

<u>proof.</u> let $\varepsilon > 0$, choose $0 < x_1 < \frac{1}{2} < x_2 < 1$ such that $x_2 - x_1 < \varepsilon$.

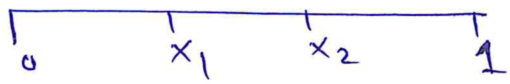
The set $P := \{0, x_1, x_2, 1\}$ is a partition of $[0,1]$. Since $m_1(f) = 0 = M_1(f)$, $m_2(f) = 0 < 1 = M_2(f)$ and $m_3(f) = M_3(f) = 1$,

then $U(f,P) - L(f,P)$

$$= \sum_{j=1}^{3} M_j(f)\,\Delta x_j - \sum_{j=1}^{3} m_j(f)\,\Delta x_j$$

$$= \left(\cancelto{0}{M_1(f)}\,\Delta x_1 + \cancelto{1}{M_2(f)}\,\Delta x_2 + \cancelto{1}{M_3(f)}\,\Delta x_3\right)$$

$$- \left(\cancelto{0}{m_1(f)}\,\Delta x_1 + \cancelto{0}{m_2(f)}\,\Delta x_2 + \cancelto{1}{m_3(f)}\,\Delta x_3\right)$$



$$= \Delta x_2 + \Delta x_3 - \Delta x_3 = \Delta x_2 = x_2 - x_1 < \varepsilon$$

$\Rightarrow U(f,P) - L(f,P) = x_2 - x_1 < \varepsilon$.

therefore, f is integrable on $[0,1]$ $\blacksquare$

Df ④. Let $a, b \in \mathbb{R}$ with $a < b$, and $f : [a,b] \to \mathbb{R}$ be bounded.

(i) the upper integral of f on $[a,b]$ is

$$(U)\int_a^b f(x)\,dx := \inf\{U(f,P) : P \text{ is a partition of } [a,b]\}$$

(ii) the lower integral of f on $[a,b]$ is

$$(L)\int_a^b f(x)\,dx := \sup\{L(f,P) : P \text{ is a partition of } [a,b]\}$$

(iii) If $(U)\int_a^b f(x)\,dx = (L)\int_a^b f(x)\,dx$, then

$$\int_a^b f(x)\,dx := (U)\int_a^b f(x)\,dx = (L)\int_a^b f(x)\,dx.$$

Rmk. ① we define the integral of any bounded function f on $[a,a]$ to be zero, i.e.

$$\int_a^a f(x)\,dx := 0.$$

② A bounded function might not be integrable

Ex. $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$ (Dirichlet function).

<u>Rmk.</u> If $f : [a,b] \longrightarrow \mathbb{R}$ is bounded, then $(U)\int_a^b f(x)\,dx$ and $(L)\int_a^b f(x)\,dx$ exist and are finite, and satisfy

$$(L)\int_a^b f(x)\,dx \leq (U)\int_a^b f(x)\,dx.$$

<u>Proof.</u> we know $L(f,P) \leq U(f,Q)$ for all partitions P and Q of $[a,b]$. Taking the Sup over all partitions P of $[a,b]$, we have

$$\boxed{(L)\int_a^b f(x)\,dx \leq U(f,Q)}, \text{ i.e.,}$$

$(L)\int_a^b f(x)\,dx$ exists and is finite.

Taking the inf over all partitions ~~P~~ Q of $[a,b]$,

we ~~conclude~~ $\cancel{L(f,P) \leq (U)\int_a^b f(x)\,dx}$

we conclude that $(U)\int_a^b f(x)\,dx$ is also finite and $(U)\int_a^b f(x)\,dx \geq (L)\int_a^b f(x)\,dx$. $\boxtimes$

Thm ② Let $a, b \in \mathbb{R}$ with $a < b$, and $f : [a,b] \to \mathbb{R}$ be bounded. Then f is integrable on $[a,b]$ iff

$$(L)\int_a^b f(x)\,dx = (U)\int_a^b f(x)\,dx$$

Proof: suppose that f is integrable. Let $\varepsilon > 0$ and choose a partion P of $[a,b]$ such that

$$U(f,P) - L(f,P) < \varepsilon.$$

By def'n, $(U)\int_a^b f(x)\,dx \leq U(f,P)$ and

$$(L)\int_a^b f(x)\,dx \geq L(f,P).$$ therefore,

$$\left|(U)\int_a^b f(x)\,dx - (L)\int_a^b f(x)\,dx\right|$$

$$= (U)\int_a^b f(x)\,dx - (L)\int_a^b f(x)\,dx \quad \text{since } (L)\int_a^b f \leq (U)\int_a^b f$$

$$\leq U(f,P) - L(f,P) < \varepsilon.$$

Since $\left| (U)\int_a^b f(x)\,dx - (L)\int_a^b f(x)\,dx \right| < \varepsilon, \ \forall \varepsilon > 0,$

this implies $(L)\int_a^b f(x)\,dx = (U)\int_a^b f(x)\,dx$

Conversely, suppose that $(L)\int_a^b f(x)\,dx = (U)\int_a^b f(x)\,dx$.

Let $\varepsilon > 0$ and choose, by the Approximation Property, partitions P_1 and P_2 of $[a,b]$ s.t.

$$U(f,P_1) < (U)\int_a^b f(x)\,dx + \frac{\varepsilon}{2}$$

and $$L(f,P_2) > (L)\int_a^b f(x)\,dx - \frac{\varepsilon}{2}.$$

Set $P = P_1 \cup P_2$. Since P is a refinement of P_1 and P_2, it follows that

$$U(f,P) - L(f,P) \le U(f,P_1) - L(f,P_2)$$

Given

$$\le \cancel{(U)\int_a^b f(x)\,dx} + \frac{\varepsilon}{2} - \cancel{(L)\int_a^b f(x)\,dx} + \frac{\varepsilon}{2}$$

$$= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \quad \blacksquare$$

Thm ③. If $f(x) = \alpha$ is constant on $[a,b]$, then

$$\int_a^b f(x)\,dx = \alpha(b-a).$$

Proof. By thm 1, f is integrable on $[a,b]$ since it is continuous on $[a,b]$. Hence, it follows from Thm ② and Rmk ($U(f,P) = L(f,P) = \alpha(b-a)$) that

$$\int_a^b f(x)\,dx = (U)\int_a^b f(x)\,dx$$

$$= \inf\{U(f,P) : P \text{ is a partion of } [a,b]\}$$

$$= \inf\{\alpha(b-a) : \text{ " " " }\}$$

$$= \alpha(b-a)$$

H.w's (Exercise P.138)

0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 (All).

5.2 Riemann Sums

Df ①. Let $f:[a,b] \to \mathbb{R}$.

(i) A Riemann sum of f with respect to a partition $P=\{x_0,\dots,x_n\}$ of $[a,b]$ generated by samples $t_j \in [x_{j-1}, x_j]$ is

$$\boxed{S(f,P,t_j) := \sum_{j=1}^{n} f(t_j)\,\Delta x_j}.$$

(ii) The Riemann sums of f are said to ~~be~~ converge to $I(f)$ as $\|P\| \to 0$ iff $\forall \varepsilon>0$, $\exists$ a partition P_ε of $[a,b]$ such that

$$P=\{x_0,\dots,x_n\} \supseteq P_\varepsilon \Rightarrow |S(f,P,t_j) - I(f)| < \varepsilon$$

for all choices of $t_j \in [x_{j-1}, x_j]$, $j=1,\dots,n$.

In this case we use the notation

$$I(f) = \lim_{\|P\|\to 0} S(f,P,t_j)$$

$$:= \lim_{\|P\|\to 0} \sum_{j=1}^{n} f(t_j)\,\Delta x_j .$$

Thm(1) Let $a,b\in\mathbb{R}$ with $a<b$ and Spse that $f:[a,b]\to\mathbb{R}$. Then

f is Riemann integrable on $[a,b]$ iff

$$I(f)=\lim_{\|P\|\to 0}\sum_{j=1}^{n} f(t_j)\Delta x_j \quad \text{exists.}$$

in which case $I(f)=\int_a^b f(x)\,dx$.

Prof. Spse that f is integrable on $[a,b]$ and $\varepsilon>0$. By the Approximation Property, there is a partition P_ε of $[a,b]$ such that

$$\boxed{L(f,P_\varepsilon)>\int_a^b f(x)\,dx-\varepsilon \quad \text{and} \quad U(f,P_\varepsilon)<\int_a^b f(x)\,dx+\varepsilon} \quad (1)$$

Let $P=\{x_0,x_1,\dots,x_n\}\supseteq P_\varepsilon$. Then (1) holds with P in place of P_ε. But $m_j(f)\le f(t_j)\le M_j(f)$ for any choice of $t_j\in[x_{j-1},x_j]$. Hence,

$$\int_a^b f(x)\,dx-\varepsilon<L(f,P)\le\sum_{j=1}^{n} f(t_j)\Delta x_j\le U(f,P)<\int_a^b f(x)\,dx+\varepsilon$$

i.e, $-\varepsilon<\sum_{j=1}^{n} f(t_j)\Delta x_j-\int_a^b f(x)\,dx<\varepsilon$, We conclude that

$$\left| \sum_{j=1}^{n} f(t_j)\, \Delta x_j - \int_a^b f(x)\,dx \right| < \varepsilon.$$

for all partitions $P \supseteq P_\varepsilon$ and all choices of $t_j \in [x_{j-1}, x_j]$, $j = 1, 2, \ldots, n$.

Conversely, spse that the Riemann sums of f converge to $I(f)$. Let $\varepsilon > 0$ and choose a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a,b]$ such that

$$\left| \sum_{j=1}^{n} f(t_j)\, \Delta x_j - I(f) \right| < \frac{\varepsilon}{3} \qquad (2)$$

for all choices of $t_j \in [x_{j-1}, x_j]$. Since f is bounded on $[a,b]$ (Exercise 11), use the Approximation Property to choose t_j, $u_j \in [x_{j-1}, x_j]$ such that

$$f(t_j) - f(u_j) > M_j(f) - m_j(f) - \frac{\varepsilon}{3(b-a)}.$$

By (2) and telescoping, we have

$$U(f,P)-L(f,P)=\sum_{j=1}^{n}\left(M_j(f)-m_j(f)\right)\Delta x_j$$

$$< \sum_{j=1}^{n}\left(f(t_j)-f(u_j)\right)\Delta x_j+\frac{\varepsilon}{3(b-a)}\sum_{j=1}^{n}\Delta x_j$$

$$\leq \left|\sum_{j=1}^{n} f(t_j)\Delta x_j - I(f)\right|$$

$$+\left|I(f)-\sum_{j=1}^{n} f(u_j)\Delta x_j\right|$$

$$+\frac{\varepsilon}{3(b-a)}\sum_{j=1}^{n}\Delta x_j$$

$$< \frac{2\varepsilon}{3}+\frac{\varepsilon}{3(b-a)}(b-a)=\varepsilon.$$

$\blacksquare$

Therefore, f is integrable on $[a,b]$.

Thm 2. If f, g are integrable on $[a,b]$ and $\alpha\in\mathbb{R}$, then $f+g$ and αf are integrable on $[a,b]$. In fact,

$$\int_a^b \left(f(x)+g(x)\right)dx=\int_a^b f(x)\,dx+\int_a^b g(x)\,dx \qquad (3)$$

and $$\int_a^b (\alpha f(x))\,dx=\alpha\int_a^b f(x)\,dx \qquad (4)$$

<u>Proof</u>: Let $\varepsilon > 0$ and choose P_ε such that for any partition $P = \{x_0, x_1, \ldots, x_n\} \supseteq P_\varepsilon$ of $[a,b]$ and any choice of $t_j \in [x_{j-1}, x_j]$, we have

$$\left| \sum_{j=1}^{n} f(t_j)\Delta x_j - \int_a^b f(x)\,dx \right| < \frac{\varepsilon}{2}$$

and $$\left| \sum_{j=1}^{n} g(t_j)\Delta x_j - \int_a^b g(x)\,dx \right| < \frac{\varepsilon}{2}.$$

By the Triangle inequality,

$$\left| \sum_{j=1}^{n} f(t_j)\Delta x_j + \sum_{j=1}^{n} g(t_j)\Delta x_j - \int_a^b f(x)\,dx - \int_a^b g(x)\,dx \right|$$

$$< \left| \sum_{j=1}^{n} f(t_j)\Delta x_j - \int_a^b f(x)\,dx \right| + \left| \sum_{j=1}^{n} g(t_j)\Delta x_j - \int_a^b g(x)\,dx \right|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

for any choice of $t_j \in [x_{j-1}, x_j]$.

Hence (3) follows directly from thm ①.

To prove (4), we may assume that $\alpha \neq 0$. choose P_ε such that if $P = \{x_0, \dots, x_n\}$ is finer than P_ε, then

$$\left| \sum_{j=1}^{n} f(t_j)\,\Delta x_j - \int_a^b f(x)\,dx \right| < \frac{\varepsilon}{|\alpha|},$$

for any choice of $t_j \in [x_{j-1}, x_j]$.

Multiply this inequality by $|\alpha|$, we obtain

$$\left| \sum_{j=1}^{n} \alpha f(t_j)\,\Delta x_j - \alpha \int_a^b f(x)\,dx \right| < |\alpha| \frac{\varepsilon}{|\alpha|} = \varepsilon,$$

for any choice of $t_j \in [x_{j-1}, x_j]$. We conclude by thm (1) that (4) holds. ▤

Thm ③. If f is integrable on $[a,b]$, then f is integrable on each subintervals $[c,d]$ of $[a,b]$. Moreover,

$$\boxed{\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx} \qquad (5)$$

proof. See the textbook p. 144.

Thm (4). If f, g are integrable on $[a,b]$ and $f(x) \le g(x)$, $\forall x \in [a,b]$, then

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$$

In particular, if $m \le f(x) \le M$, $\forall x \in [a,b]$, then

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a).$$

proof. Let P be a partition of $[a,b]$. By hypothesis $M_j(f) \le M_j(g)$ whence $U(f,P) \le U(g,P)$.

It follows that

$$\int_a^b f(x)\,dx = (U)\int_a^b f(x)\,dx \le U(f,P) \le U(g,P).$$

for all partitions P of $[a,b]$. Taking the infimum of this inequality over all partition P of $[a,b]$, we obtain

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$$

If $m \le f(x) \le M$ then (by what we proved)

$$m(b-a) = \int_a^b m\,dx \le \int_a^b f(x)\,dx \le \int_a^b M\,dx = M(b-a)$$ ∎

Thm 5. If f is integrable on $[a,b]$, then $|f|$ is integrable on $[a,b]$ and

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx.$$

Proof. let $P=\{x_0,\dots,x_n\}$ be a partition of $[a,b]$.

claim $M_j(|f|)-m_j(|f|) \leq M_j(f)-m_j(f)$ (6)

holds for $j=1,2,\dots,n$.

pf (claim). let $x,y \in [x_{j-1},x_j]$. If $f(x), f(y)$ have the same sign, say both are nonnegative, then $|f(x)|-|f(y)| = f(x)-f(y) \leq M_j(f)-m_j(f)$.

If $f(x), f(y)$ have opposite signs, say, $f(x) \geq 0 \geq f(y)$, then $m_j(f) \leq 0$ and

hence $|f(x)|-|f(y)| = f(x)+f(y) \leq M_j(f)+0$

$\leq M_j(f)-m_j(f)$.

Thus, in either case,

$$\boxed{|f(x)| \leq M_j(f)-m_j(f)+|f(y)|}$$

Taking the sup of this inequality for $x \in [x_{j-1}, x_j]$

$$\sup_{x \in [x_{j-1}, x_j]} |f(x)| \leq M_j(f) - m_j(f) + |f(y)|$$

$$\Rightarrow \quad M_j(|f|) \leq M_j(f) - m_j(f) + |f(y)|$$

Next, taking the inf as $y \in [x_{j-1}, x_j]$,

$$M_j(|f|) \leq M_j(f) - m_j(f) + m_j(|f|)$$

$$\Rightarrow \quad M_j(|f|) - m_j(|f|) \leq M_j(f) - m_j(f).$$

We see that (6) holds, as promised.

Let $\varepsilon > 0$ and choose a partition P of $[a,b]$ such that $U(f,P) - L(f,P) < \varepsilon$. Since (6) implies $U(|f|,P) - L(|f|,P) \leq U(f,P) - L(f,P)$.

it follows that

$$U(|f|,P) - L(|f|,P) < \varepsilon.$$

Thus, $|f|$ is integrable on $[a,b]$.

Since $-|f(x)| \leq f(x) \leq |f(x)|$ holds $\forall x \in [a,b]$

we conclude by Thm (4) that

$$-\int_a^b |f(x)|\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b |f(x)|\,dx. \quad \blacksquare$$

<u>Corollary</u>. If f and g are (Riemann) integrable on $[a,b]$, then so is fg.

<u>Proof</u>: <u>Claim</u> the square of any integrable function is integrable.

<u>pf (claim)</u>. We need to prove that f^2 is integrable on $[a,b]$. Let P be a partition of $[a,b]$.

Since $M_j(f^2) = \left(M_j(|f|)\right)^2$ and $m_j(f^2) = \left(m_j(|f|)\right)^2$

it is clear that

$$M_j(f^2) - m_j(f^2) = \left(M_j(|f|)\right)^2 - \left(m_j(|f|)\right)^2$$
$$= \left(M_j(|f|) + m_j(|f|)\right)\left(M_j(|f|) - m_j(|f|)\right)$$
$$\leq 2M\left(M_j(|f|) - m_j(|f|)\right)$$

where $M = \sup\limits_{x\in[a,b]} |f(x)|$, i.e, $|f(x)| \leq M$, $\forall x \in [a,b]$.

$$\Rightarrow \sum_{j=1}^{n}\left(M_j(f^2) - m_j(f^2)\right)\Delta x_j \leq 2M \sum_{j=1}^{n}\left(M_j(|f|) - m_j(|f|)\right)\Delta x_j$$

$$\Rightarrow U(f^2,P) - L(f^2,P) \leq 2M\left(U(|f|,P) - L(|f|,P)\right)$$
$$< 2M \cdot \frac{\varepsilon}{2M} \quad \text{since } f \text{ is integrable}$$
$$= \varepsilon. \Rightarrow f^2 \text{ is integrable on } [a,b]$$

Then, by claim, f^2, g^2, and $(f+g)^2$ are integrable on $[a,b]$. Since

$$fg = \frac{1}{2}(f+g)^2 - \frac{1}{2}f^2 - \frac{1}{2}g^2$$

it follows from (thm ② p.4) that fg is integrable on $[a,b]$. ∎

<u>Thm ⑥.</u> [<u>First Mean value Thm for integrals</u>]
Spse that f and g are integrable on $[a,b]$ with $g(x) \geq 0$, $\forall x \in [a,b]$. If

$$m = \inf_{x \in [a,b]} f(x), \quad M = \sup_{x \in [a,b]} f(x),$$ then

there is a number $c \in [m, M]$ such that

$$\int_a^b f(x) g(x)\, dx = c \int_a^b g(x)\, dx.$$

In particular, if f is continuous on $[a,b]$, then there is an $x_0 \in [a,b]$ which satisfies

$$\int_a^b f(x) g(x)\, dx = f(x_0) \int_a^b g(x)\, dx.$$

Proof. Since $g \geq 0$ on $[a,b]$, then

$m\, g(x) \leq f(x)\, g(x) \leq M\, g(x)$.

Thm (4) implies

$$m \int_a^b g(x)\,dx \leq \int_a^b f(x) g(x)\,dx \leq M \int_a^b g(x)\,dx .$$

If $\int_a^b g(x)\,dx = 0$, then $\int_a^b f(x) g(x)\,dx = 0$.
and there is nothing to prove.

If $\int_a^b g(x)\,dx \neq 0$, then we have

$$m \leq \frac{\int_a^b f(x) g(x)\,dx}{\int_a^b g(x)\,dx} \leq M$$

Set $c = \dfrac{\int_a^b f(x) g(x)\,dx}{\int_a^b g(x)\,dx}$ and note that $c \in [m, M]$

If f is continuous then by the intermediate value thm, $\exists\, x_0 \in [a,b]$ such that

$f(x_0) = c$

Thm 7. If f is (Riemann) integrable on $[a,b]$, then $F(x)=\int_a^x f(t)\,dt$ exists and is continuous on $[a,b]$.

Proof. (Exercise).

Thm 8 [Second mean value thm for integrals]

Spse that f, g are integrable on $[a,b]$, that $g \geq 0$ on $[a,b]$, and that $m, M \in \mathbb{R}$ which satisfy $m \leq \inf_{x\in[a,b]} f(x)$ and $M \geq \sup_{x\in[a,b]} f(x)$.

Then $\exists$ an $c\in[a,b]$ such that

$$\int_a^b f(x)g(x)\,dx = m\int_a^c g(x)\,dx + M\int_c^b g(x)\,dx.$$

In particular, if f is also nonnegative on $[a,b]$, then $\exists$ an $c\in[a,b]$ which satisfies

$$\int_a^b f(x)g(x)\,dx = M\int_a^b g(x)\,dx.$$

Proof. To prove the first statement, set

$$F(x) = m\int_a^x g(t)\,dt + M\int_x^b g(t)\,dt, \quad \forall x\in[a,b]$$

Observe that by Thm 7 that F is continuous on $[a,b]$. Since $g \geq 0$, we also have

$$m g(t) \leq f(t) g(t) \leq M g(t), \ \forall t \in [a,b].$$

Hence it follows from thm 4 that

$$F(b) = m\int_a^b g(t)\,dt \leq \int_a^b f(t)g(t)\,dt \leq M\int_a^b g(t)\,dt = F(a).$$

Since F is continuous and $\int_a^b f(t)g(t)\,dt$ lies between $F(b)$ and $F(a)$, we conclude by the intermediate value thm that $\exists$ an $c \in [a,b]$ such that

$$F(c) = \int_a^b f(t)g(t)\,dt.$$

$$\left(\text{i.e., } m\int_a^c g(x)\,dx + M\int_c^b g(x)\,dx = \int_a^b f(t)g(t)\,dt\right).$$

The second statement follows from the first statement since we may use $m=0$ when $f \geq 0$. That is $\exists$ an $c \in [a,b]$ s.t

$$\int_a^b f(x)g(x)\,dx = M\int_a^b g(x)\,dx$$

$\boxed{}$

H.W's (Exercises p. 150 (0, 1, 2, 4, 5, 6, 7, 8, 9, 10, 11).