CH4: Fourier Transform

⇒ It is used to determine the spectral representation of aperiodic signal

4.1] Definition

The Fourier transform of $x(t)$ is determined as follows:

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt$$

Note: If $X(f)$ is given, the $x(t)$ is determined as follows:

$$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi f t}\, df$$

4.2] properties of Fourier Transform

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt$$

① $X(-f) = X^{*}(f)$

② $x(t)$ is even $\Rightarrow X(f)$ is real and even

③ $x(t)$ is odd $\Rightarrow X(f)$ is imaginary and odd

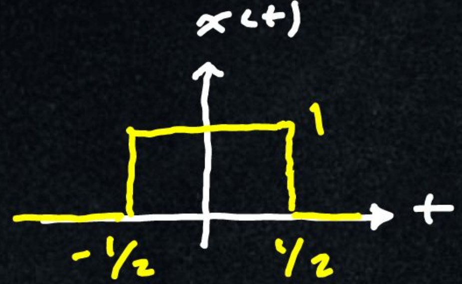
$x(-t) = x(t)$

$x(-t) = -x(t)$

$X(-f) = X(f)$

$X(-f) = -X(f)$

EX :- Determine the FT of $x(t) = \Pi(t)$



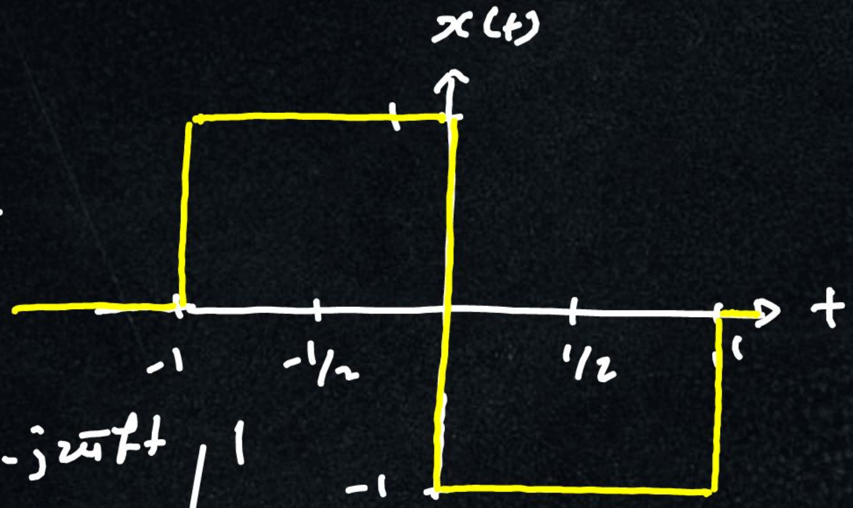
$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt$$

$$X(f) = \int_{-1/2}^{1/2} e^{-j2\pi f t}\, dt = \frac{1}{-j2\pi f}\, e^{-j2\pi f t}\Big|_{-1/2}^{1/2}$$

$$X(f) = \frac{1}{-j2\pi f}\left[e^{-j\pi f} - e^{j\pi f}\right] = \left[\frac{e^{j\pi f} - e^{-j\pi f}}{2j}\right]\frac{1}{\pi f} = \frac{\sin \pi f}{\pi f}$$

$$X(f) = \text{sinc}(f)$$

EX: Determine the FT of $x(t) = \Pi(t+1/2) - \Pi(t-1/2)$



$$X(f) = \int_{-1}^{0} e^{-j2\pi f t}\,dt - \int_{0}^{1} e^{-j2\pi f t}\,dt$$

$$X(f) = \frac{1}{j2\pi f} e^{-j2\pi f t}\Big|_{0}^{-1} + \frac{1}{j2\pi f} e^{-j2\pi f t}\Big|_{0}^{1}$$

$$X(f) = \frac{1}{j2\pi f}\left[\left(e^{j2\pi f} - 1\right) + \left(e^{-j2\pi f} - 1\right)\right] = \frac{1}{j2\pi f}\left[e^{j2\pi f} + e^{-j2\pi f} - 2\right]$$

$$X(f) = \frac{1}{j\pi f}\left[\frac{e^{j2\pi f} + e^{-j2\pi f}}{2} - 1\right] = \frac{-1}{j\pi f}\left[1 - \cos(2\pi f)\right] = \frac{-2\sin^2 \pi f}{j\pi f}$$

$$\boxed{X(f) = j2\pi f\,\mathrm{sinc}^2(\pi f)}$$

4.3] Fourier Transform Theorems

① Superposition

$$x_1(t) \xrightarrow{FT} X_1(f) \qquad x_2(t) \xrightarrow{FT} X_2(f)$$

$$x(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t) \xrightarrow{FT} X(f) = \alpha_1 X_1(f) + \alpha_2 X_2(f)$$

where α_1 & α_2 are constants

proof

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt = \int_{-\infty}^{\infty} (\alpha_1 x_1(t) + \alpha_2 x_2(t))\, e^{-j2\pi f t}\, dt$$

$$X(f) = \alpha_1 \underbrace{\int_{-\infty}^{\infty} x_1(t)\, e^{-j2\pi f t}\, dt}_{X_1(f)} + \alpha_2 \underbrace{\int_{-\infty}^{\infty} x_2(t)\, e^{-j2\pi f t}\, dt}_{X_2(f)} = \alpha_1 X_1(f) + \alpha_2 X_2(f)$$

② Delay

$$x(t) \xrightarrow{FT} X(f) \Rightarrow x(t-\tau) = e^{-j2\pi f\tau} X(f)$$

proof :-

$$F[x(t-\tau)] = \int_{-\infty}^{\infty} x(t-\tau)\, e^{-j2\pi f t}\, dt$$

let $t-\tau = \lambda \Rightarrow dt = d\lambda$

$$F[x(t-\tau)] = \int_{-\infty}^{\infty} x(\lambda)\, e^{-j2\pi f(\tau+\lambda)}\, d\lambda = e^{-j2\pi f\tau} \underbrace{\int_{-\infty}^{\infty} x(\lambda)\, e^{-j2\pi f\lambda}\, d\lambda}_{X(f)}$$

$$\boxed{F[x(t-\tau)] = e^{-j2\pi f\tau} X(f)}$$

Ex:- If $F[\Pi(t)] = \text{sinc}(f)$, find the FT of $x(t)$

such that $x(t) = \Pi(t+1/2) - \Pi(t-1/2)$

$$X(f) = e^{j2\pi f(1/2)}\,\text{sinc}(f) - e^{-j2\pi f(1/2)}\,\text{sinc}(f)$$

$$X(f) = \text{sinc}(f)\left[\frac{e^{j\pi f} - e^{-j\pi f}}{2j}\right] 2j$$

$$X(f) = \text{sinc}(f)\,\sin(\pi f)\,(2j)\,\frac{\pi f}{\pi f}$$

$$\boxed{X(f) = 2j\pi f\,\text{sinc}^2(f)}$$

③ Scaling

$$x(t) \xrightarrow{FT} X(f) \Rightarrow x(\alpha t) = \frac{1}{|\alpha|} X(f/\alpha)$$

where α is constant

proof :-

$$F[x(\alpha t)] = \int_{-\infty}^{\infty} x(\alpha t)\, e^{-j2\pi f t}\, dt$$

let $\alpha t = \lambda \Rightarrow dt = \frac{1}{\alpha} d\lambda$

$$F[x(\alpha t)] = \mp \frac{1}{\alpha} \underbrace{\int_{-\infty}^{\infty} x(\lambda)\, e^{-j2\pi (f/\alpha)\lambda}\, d\lambda}_{X(f/\alpha)} \quad ; \begin{cases} + & \alpha > 0 \\ - & \alpha < 0 \end{cases}$$

$$\Rightarrow F[x(\alpha t)] = \frac{1}{|\alpha|} X(f/\alpha)$$

EX:- If the FT of $x(t) = \Pi(t)$ is $X(f) = \text{sinc}(f)$, determine the FT of $x(0.5t)$ & $x(-2t)$

$$F[x(0.5t)] = \frac{1}{0.5} X(f/0.5) = 2X(2f)$$

$$\Rightarrow F[x(0.5t)] = 2\,\text{sinc}(2f)$$

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even

$$F[x(-2t)] = \frac{1}{2} X(f/(-2)) = \frac{1}{2}\,\text{sinc}\left(\frac{f}{-2}\right)$$

$$\Rightarrow F[x(-2t)] = \frac{1}{2}\,\text{sinc}(f/2)$$

④ Inversion

$$F[x(t)] = X(f) \Rightarrow F[x(-t)] = X(f)$$

proof:-

$$F[x(-t)] = \int_{-\infty}^{\infty} x(-t)\, e^{-j2\pi f t}\, dt$$

let $\lambda = -t \Rightarrow dt = -d\lambda$

$$F[x(-t)] = -\int_{\infty}^{-\infty} x(\lambda)\, e^{-j2\pi(-f)\lambda}\, d\lambda = \underbrace{\int_{-\infty}^{\infty} x(\lambda)\, e^{-j2\pi(-f)\lambda}\, d\lambda}_{X(-f)}$$

$$\Rightarrow F[x(-t)] = X(-f)$$

⑤ Duality

$$F[x(t)] = X(f) \Rightarrow F[X(t)] = x(-f)$$

proof :-

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt$$

$$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi f t}\, df$$

let $f = t \Rightarrow$ $$x(f) = \int_{-\infty}^{\infty} X(t)\, e^{-j2\pi(-f)t}\, dt$$

$$x(-f) = \int_{-\infty}^{\infty} X(t)\, e^{-j2\pi f t}\, dt = F[X(t)]$$

Ex :- $x(t) = \Pi(t + 1/2) - \Pi(t - 1/2)$

$$X(f) = j2\pi f \,\text{sinc}^2(f)$$

Determine the FT of $x(-t)$

$$\underbrace{F[x(-t)] = X(-f) = -j2\pi f \,\text{sinc}^2(f)}_{\text{inversion theorem}}$$

↳ Exercise: plot $x(-t)$ and find $F[x(-t)]$ using the definition of FT

Ex :- Find the FT of $x(t) = \text{sinc}(t)$

$$F[\Pi(t)] = \text{sinc}(f)$$

$$\Rightarrow F[\text{sinc}(t)] = \Pi(-f) = \Pi(f)$$

even

Duality theorem

⑥ Frequency - shift

$$F[x(t)] = X(f) \Rightarrow F[x(t)\,e^{j2\pi f_0 t}] = X(f-f_0)$$

proof :-

$$F[x(t)\,e^{j2\pi f_0 t}] = \int_{-\infty}^{\infty} x(t)\,e^{j2\pi f_0 t}\,e^{-j2\pi f t}\,dt$$

$$= \int_{-\infty}^{\infty} x(t)\,e^{-j2\pi(f-f_0)t}\,dt = X(f-f_0)$$

⑦ Convolution

$$F[x_1(t)] = X_1(f) \quad \& \quad F[x_2(t)] = X_2(f)$$

$$\Rightarrow F[x_1(t) * x_2(t)] = X_1(f)\,X_2(f)$$

proof :-

$$F[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x_1(\tau)\,x_2(t-\tau)\,d\tau\right] e^{-j2\pi f \overset{\lambda+\tau}{t}}\,dt$$

let $\lambda = t - \tau$

$$\Rightarrow \; = \underbrace{\int_{-\infty,\,d\tau}^{\infty} x_1(\tau)\,e^{-j2\pi f\tau}\,d\tau}_{X_1(f)}\;\underbrace{\int_{-\infty,\,d\lambda}^{\infty} x_2(\lambda)\,e^{-j2\pi f\lambda}\,d\lambda}_{X_2(f)}$$

$$\Rightarrow F[x_1(t) * x_2(t)] = X_1(f)\,X_2(f)$$

EX:- Find $F[\Pi(t)\, e^{j\pi t}]$

let $x(t) = \Pi(t)$

$$X(f) = \operatorname{sinc}(f)$$

$$\left.\begin{aligned} F[x(t)\, e^{j\pi t}] &= X(f - f_0) \\ f_0 &= \frac{1}{2} \end{aligned}\right\} \Rightarrow F[\Pi(t)\, e^{j\pi t}] = \operatorname{sinc}(f - 1/2)$$

EX:- let $x_1(t) = x_2(t) = \Pi(t)$, Find $F[x_1(t) * x_2(t)]$

$$F[x_1(t) * x_2(t)] = X_1(f)\, X_2(f) = \operatorname{sinc}^2(f)$$

⑧ Multiplication

$F[x_1(t)] = X_1(f)$ & $F[x_2(t)] = X_2(f)$

$\Rightarrow F[x_1(t)\,x_2(t)] = X_1(f) * X_2(f)$

proof :-

$$F[x_1(t)\,x_2(t)] = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} X_1(f')\,e^{j2\pi f' t}\,df'\right] x_2(t)\,e^{-j2\pi f t}\,dt$$

$$= \int_{-\infty}^{\infty} X_1(f') \int_{-\infty}^{\infty} x_2(t)\,e^{-j2\pi(f-f')t}\,dt\,df'$$

$$= \int_{-\infty}^{\infty} X_1(f')\,X_2(f-f')\,df' = X_1(f) * X_2(f)$$

9 Differentiation

$$F[x(t)] = X(f)$$

$$\Rightarrow F\left[\frac{d^{(n)}x(t)}{dt^{(n)}}\right] = (j2\pi f)^n X(f)$$

Proof :-

$\Rightarrow$ It is done using $\underbrace{\text{induction method}}_{\text{exercise}}$

$$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi f t}\, df$$

$$\frac{dx(t)}{dt} = \int_{-\infty}^{\infty} j2\pi f\, X(f)\, e^{j2\pi f t}\, df$$

$$\frac{dx(t)}{dt} = F^{-1}\left[(j2\pi f)\, X(f)\right]$$

$$\Rightarrow\; F\left[\frac{dx(t)}{dt}\right] = (j2\pi f)\, X(f)$$

$\Rightarrow$ Assume that $F\left[\frac{d^{(k)}x(t)}{dt^{(k)}}\right] = (j2\pi f)^{k} X(f)$

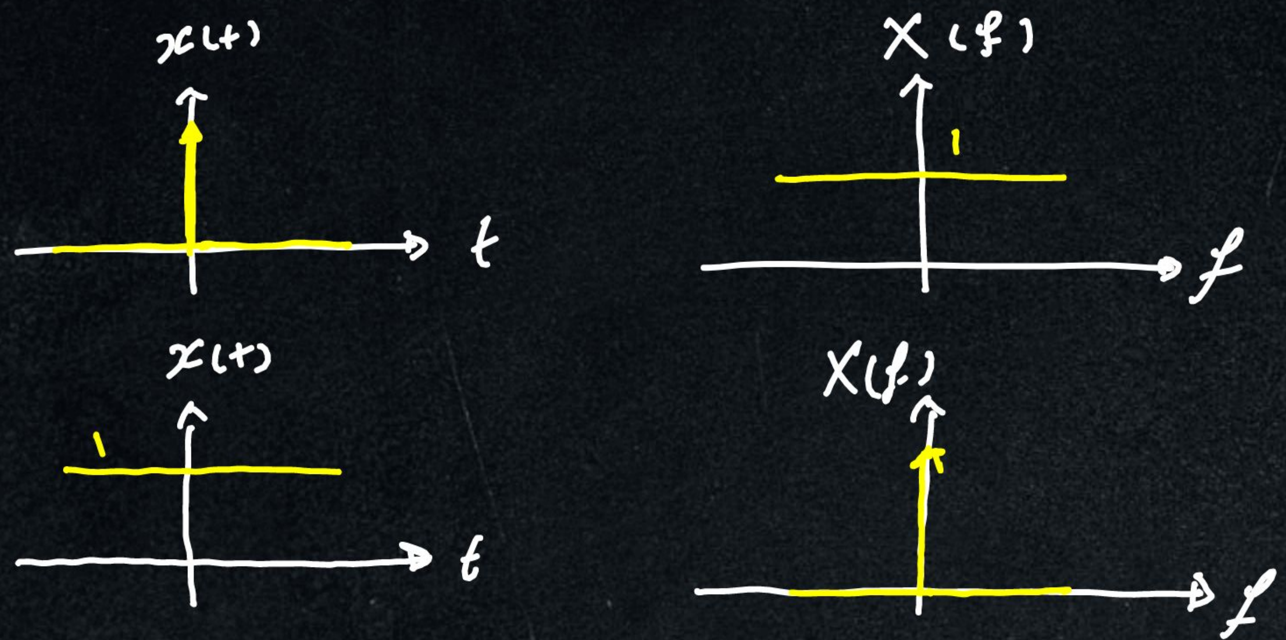
Show that $F\left[\frac{d^{(k+1)}x(t)}{dt^{(k+1)}}\right] = (j2\pi f)^{k+1} X(f)$

Fourier Transform of Useful Functions

① Unit impulse function

$$x(t) = \delta(t)$$

$$X(f) = \int_{-\infty}^{\infty} \delta(t)\, e^{-j2\pi f t}\, dt = \int_{-\infty}^{\infty} \delta(t)\, dt = 1$$



③ Exponential function

$$x(t) = e^{-at}\, u(t)$$

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt = \int_{0}^{\infty} e^{-at}\, e^{-j2\pi f t}\, dt$$

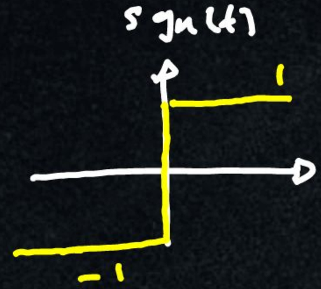
$$X(f) = \int_{0}^{\infty} e^{-(a+j2\pi f)t}\, dt = \left[\frac{1}{a+j2\pi f}\right] e^{-(a+j2\pi f)t} \Big|_{\infty}^{0}$$

$$X(f) = \frac{1}{a+j2\pi f} = F\left[e^{-at}\, u(t)\right]$$

Note: $F\left[x(-t)\right] = F\left[e^{at}\, u(-t)\right] = X(-f) = \dfrac{1}{a-j2\pi f}$

③ Signum Function

$$x(t) = sgn(t) = \begin{cases} 1 & t > 0 \\ -1 & t < 0 \end{cases}$$



$$x(t) = \lim_{a \to 0} \left[e^{-at} u(t) - e^{at} u(-t) \right]$$

$$F[x(t)] = \lim_{a \to 0} \left[F[e^{-at} u(t)] - F[e^{at} u(-t)] \right]$$

$$= \lim_{a \to 0} \left[\frac{1}{a + j2\pi f} - \frac{1}{a - j2\pi f} \right] = \boxed{\frac{1}{j\pi f}}$$

④ Unit step Function

$$x(t) = u(t) = \frac{1}{2}\left[\operatorname{sgn}(t) + 1\right]$$

$$F[x(t)] = \frac{1}{2} F[\operatorname{sgn}(t)] + \frac{1}{2} F[1]$$

$$\boxed{F[x(t)] = \frac{1}{j2\pi f} + \frac{1}{2}\delta(f)}$$

⑤ sinusoidal signal

$$x(t) = \cos(2\pi f_0 t)$$

$$X(f) = F[\cos(2\pi f_0 t)] = F\left[\frac{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2}\right]$$

$$X(f) = \frac{1}{2} F[e^{j2\pi f_0 t} \cdot 1] + \frac{1}{2} F[e^{-j2\pi f_0 t} \cdot 1]$$

$$X(f) = \frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0)$$