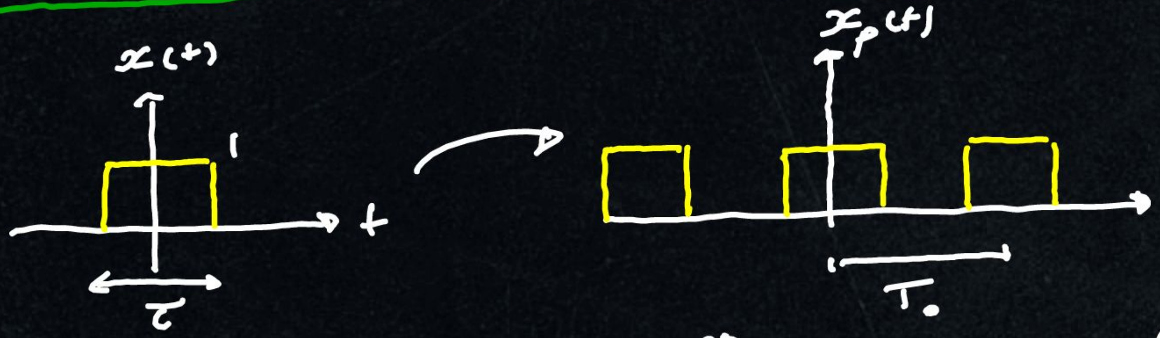
4.5] FT of periodic signals via convolution theorem



$$x_p(t) = x(t) * \sum_{n=-\infty}^{\infty} \delta(t - nT_0) = \sum_{n=-\infty}^{\infty} x(t - nT_0)$$

$x_p(t)$ ↓ periodic

$x(t)$ ↓ non-periodic

$$X_p(f) = X(f) \left[f_0 \sum_{n=-\infty}^{\infty} \delta(f - nf_0) \right]$$

$$X_p(f) = \sum_{n=-\infty}^{\infty} f_0 \, X(nf_0) \, \delta(f - nf_0)$$

$$\Rightarrow \boxed{X_n = f_0 \, X(nf_0)}$$

This can be used to find the FS coefficients

EX :- Determine the FS cofficients

of $x_p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_0)$

$$x_p(t) = x(t) * \sum_{n=-\infty}^{\infty} \delta(t - nT_0) = \sum_{n=-\infty}^{\infty} x(t - nT_0)$$

$$\Rightarrow x(t) = \delta(t) \Rightarrow X(f) = 1$$

$$X_n = f_0 X(nf_0) = \boxed{f_0}$$

This is for $x_p(t) = \sum_{n=-\infty}^{\infty} X_n e^{-j2\pi n f_0 t}$

$$= \sum_{n=-\infty}^{\infty} \delta(t - nT_0)$$

4.6] Energy spectral Density Function

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} \underbrace{|X(f)|^2}_{} df$$

↳ This is called the energy spectral density function

$$G(f) = |X(f)|^2$$

proof :-

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} x(t)\, x^*(t)\, dt$$

$$E = \int_{-\infty}^{\infty} x(t) \left(\int_{-\infty}^{\infty} X^*(f)\, e^{-j2\pi f t} df \right) dt$$

$$E = \int_{-\infty, f}^{\infty} X^*(f) \left[\int_{-\infty, t}^{\infty} x(t)\, e^{-j2\pi f t} dt \right] df = \int_{-\infty}^{\infty} X(f)\, X^*(f)\, df = \int_{-\infty}^{\infty} |X(f)|^2 df$$

EX:- Calculate the energy of the signal $x(t)$ in the frequency band $[-\beta, \beta]$.

$$x(t) = A e^{-\alpha t} u(t)$$

$$E = \int_{-\beta}^{\beta} |X(f)|^2 \, df \quad ; \quad X(f) = \frac{A}{\alpha + j2\pi f}$$

$$|X(f)| = \frac{A}{\sqrt{\alpha^2 + (2\pi f)^2}} \Rightarrow |X(f)|^2 = \frac{A^2}{\alpha^2 + (2\pi f)^2}$$

$$E = \int_{-\beta}^{\beta} \frac{A^2}{\alpha^2 + (2\pi f)^2} \, df = \frac{A^2}{2\pi} \int_{-2\pi\beta}^{2\pi\beta} \frac{1}{\alpha^2 + \omega^2} \, d\omega = \frac{A^2}{2\pi} \left[\frac{1}{\alpha} \tan^{-1}\left(\frac{\omega}{\alpha}\right) \right] \Bigg|_{-2\pi\beta}^{2\pi\beta}$$

$$E = \frac{A^2}{\pi\alpha} \tan^{-1}\left(2\pi \frac{\beta}{\alpha}\right) \rightsquigarrow \text{Total energy} = \lim_{\beta \to \infty} E = \frac{A^2}{2\alpha}$$

4.7] Response of LTI system

$$x(t) \longrightarrow \boxed{LTI} \longrightarrow y(t)$$

$$y(t) = h(t) * x(t) \xrightarrow{FT} Y(f) = H(f)\,X(f)$$

$$G_x(f) = |X(f)|^2 \qquad G_Y(f) = |Y(f)|^2$$

input energy density function

output energy density function

$$G_Y(f) = |Y(f)|^2 = |H(f)|^2\, G_x(f)$$

$$x(t) \longrightarrow \boxed{LTI} \longrightarrow y(t)$$

$$\sum_{i=0}^{M} \alpha_i \frac{d^{(i)}y(t)}{dt^{(i)}} = \sum_{k=0}^{N} \beta_k \frac{d^{(k)}x(t)}{dt^{(k)}}$$

$$\xrightarrow{FT} \quad Y(f)\sum_{i=0}^{M} \alpha_i (j2\pi f)^{(i)} = X(f) \sum_{k=0}^{N} \beta_k (j2\pi f)^{k}$$

$$H(f) = \frac{Y(f)}{X(f)} = \frac{\beta_0 + \beta_1 (j2\pi f) + \beta_2 (j2\pi f)^2 + \cdots}{\alpha_0 + \alpha_1 (j2\pi f) + \alpha_2 (j2\pi f)^2 + \cdots}$$

Ex:- Find the frequency response of the system:-

$$y''(t) + 5y'(t) + 6y(t) = 2x'(t) + 4x(t)$$

$$\left[(j2\pi f)^2 + 5(j2\pi f) + 6\right] Y(f) = \left[2(j2\pi f) + 4\right] X(f)$$

$$H(f) = \frac{Y(f)}{X(f)} = \frac{2(j2\pi f) + 4}{(j2\pi f)^2 + 5(j2\pi f) + 6}$$

$$H(\omega) = \frac{j2\omega + 4}{(j\omega)^2 + j5\omega + 6} = \frac{4 + j2\omega}{(6 - \omega^2) + j5\omega}$$

$\downarrow$
Frequency Response
$(\omega = 2\pi f)$