Key

Quiz#5solutions.

Exercise#1 [6 marks].

(a) Use the **definition** to show that $f(x) = x \sin x$ is everywhere continuous function.

Hint. You may use the identity: $\sin A - \sin B = 2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)$.

Pf. Let $a \in \mathbb{R}$ and $\varepsilon > 0$. Set 1 $\delta = \frac{\varepsilon}{1+|a|}$. Then

if $|x-a| < \delta$, then

$$|x\sin x - a\sin a| = |x\sin x - a\sin x + a\sin x - a\sin a|$$

$$\leq |x-a|\,|\sin x| + |a|\,|\sin x - \sin a|$$

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$$\leq |x-a|\cdot 1 + |a|\cdot 2\left|\sin\left(\tfrac{x-a}{2}\right)\right|\left|\cos\left(\tfrac{x+a}{2}\right)\right|$$

$$\leq |x-a| + |a|\cdot 2\left|\tfrac{x-a}{2}\right|\cdot 1$$

$$= (1+|a|)\,|x-a|$$

$$< (1+|a|)\,\delta = \varepsilon. \quad \square$$

(b) Use the **nonuniform continuity criterion** to show that $f(x) = x\sin x$ is not uniformly continuous on $\mathbb{R}$.

1 1

Let $x_n = 2n\pi$, $y_n = 2n\pi + \frac{1}{n}$. Then $|x_n - y_n| = \frac{1}{n} \to 0$ as $n \to \infty$.

But $|f(x_n) - f(y_n)| = \left|2n\pi \underbrace{\sin(2n\pi)}_{\to 0} - \left(2n\pi + \tfrac{1}{n}\right)\sin\left(2n\pi + \tfrac{1}{n}\right)\right|$

1

$$= \left(2n\pi + \tfrac{1}{n}\right)\sin\left(\tfrac{1}{n}\right)$$

$$= 2n\pi \sin\left(\tfrac{1}{n}\right) + \tfrac{1}{n}\sin\left(\tfrac{1}{n}\right)$$

$$\longrightarrow 2\pi \text{ as } n \to \infty.$$

Exercise#2 [9 marks]. Prove or disprove.

(a) Let $f : E \to \mathbb{R}$ be continuous. If $\{x_n\}$ is a Cauchy sequence in E, then $\{f(x_n)\}$ is also a Cauchy sequence.

1 False. Let $x_n = \frac{2}{n\pi}$, and $f(x) = \sin(\frac{1}{x})$ on $(0,1)$.

2 Since $x_n \to 0$ as $n \to \infty$ (conv.), it is clearly Cauchy.

But $f(x_n) = \begin{cases} 0, & n \text{ is even} \\ (-1)^{\frac{n-1}{2}}, & n \text{ is odd} \end{cases}$ and hence the seq. $\{f(x_n)\}_{n=1}^{\infty}$ is NOT Cauchy.

(b) If f is uniformly continuous on $E \subseteq \mathbb{R}$ and $|f(x)| \geq \alpha > 0$ for all $x \in E$, then $\frac{1}{f}$ is uniformly continuous on E.

1 True. pf. Since f is uniformly cont. on E, then given $\varepsilon > 0$, $\exists$ a $\delta > 0$ s.t if $|x-a| < \delta$, $x, a \in E$, then $|f(x) - f(a)| < \alpha^2 \varepsilon$.

2 Thus, if $|x-a| < \delta$, $x, a \in E$,

$$\left| \frac{1}{f(x)} - \frac{1}{f(a)} \right| = \frac{1}{|f(x)||f(a)|} |f(x) - f(a)|$$

$$< \frac{1}{\alpha^2} |f(x) - f(a)|$$

$$< \frac{1}{\alpha^2} \cdot \alpha^2 \varepsilon = \varepsilon \qquad \blacksquare$$

(c) If $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and $f(q) = 0$, for every rational number q, then $f(x) = 0$ for all $x \in \mathbb{R}$.

1 True.

pf. If x is irrational, then by density of rationals, $\exists$

2 a sequence (r_n) of rational numbers s.t $r_n \to x$.

Since f is continuous, then $f(x) = \lim_{n \to \infty} f(r_n) = 0$

$\therefore f(x) = 0, \forall x \in \mathbb{R}$. $\square$