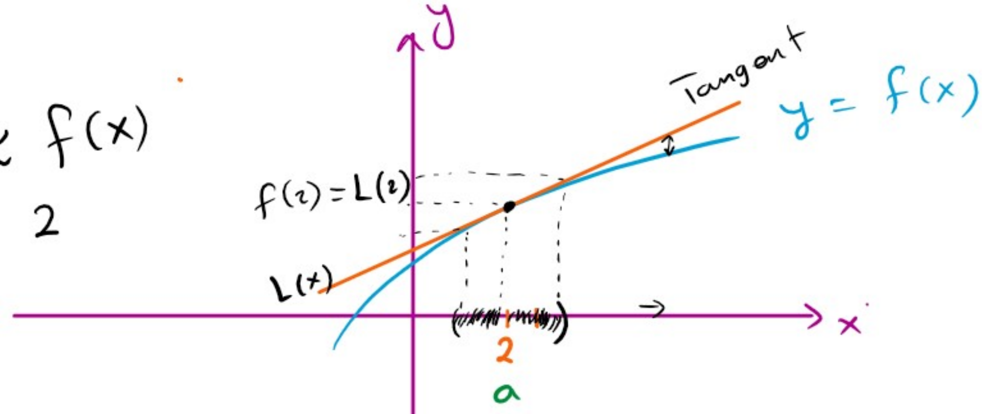
linearization

$f(2)$

$L(x) \approx f(x)$
near 2

$L(x)$



$L(x)$: linearization

$$L(x) = f(a) + f'(a)(x-a)$$

معادلة الخط المستقيم

$y - y_0 = m(x - x_0)$

$(x_0, y_0) = (a, f(a))$

$m = f'(a)$

$y - f(a) = f'(a)(x-a)$

$L(x) \approx f(x)$

$L(x)$ approximates $f(x)$ near a

$\Downarrow$

Standard linear approximation of f at $x = a$

Exp Let $f(x) = \sqrt{x+1}$

1) Find linearization of f at $x = 0$

$a = 0$

$L(x) = f(a) + f'(a)(x-a)$

$= f(0) + f'(0)(x-0)$

$f'(x) = \dfrac{1}{2\sqrt{x+1}}$

$f'(0) = \dfrac{1}{2\sqrt{0+1}}$

$f(a) = f(0) = \sqrt{0+1} = \sqrt{1} = 1$

$\dots \boxed{\tfrac{1}{\ }} x$

$= f(0) + f'(0)\,x$ $\qquad f'(0) = \frac{1}{2\sqrt{0+1}}$ $\qquad = 1$

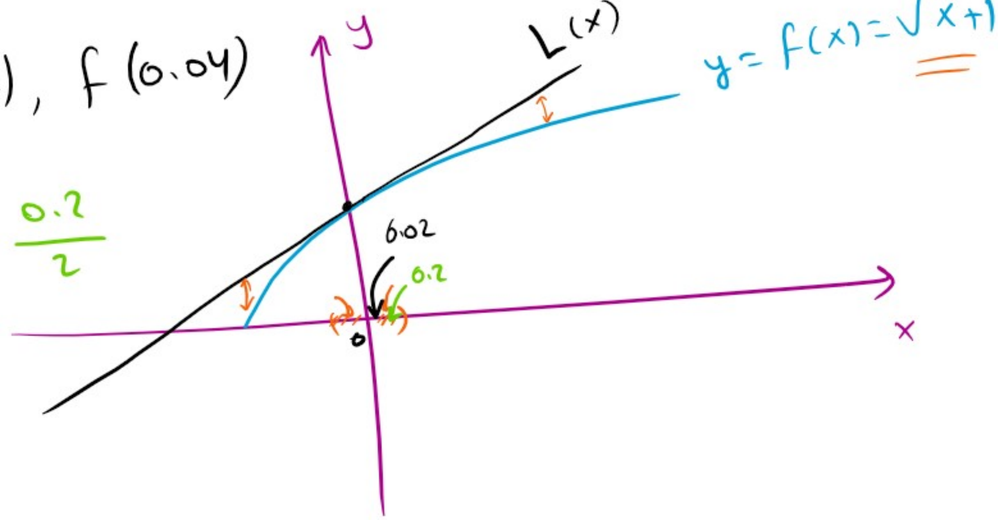
$= 1 + \boxed{\frac{1}{2}}\,x$ $\qquad = \frac{1}{2(1)}$

$\qquad = \frac{1}{2}$

$$\boxed{L(x) = 1 + \frac{x}{2}}$$

$L(x)$ approximates $\sqrt{x+1}$ near $x = 0$

2) Estimate $f(0.2)$, $f(0.04)$



$f(0.2) \approx L(0.2) = 1 + \frac{0.2}{2}$

$= 1 + 0.1$

$= 1.1$

$f(0.2) \approx 1.1$

$f(0.04) \approx L(0.04) = 1 + \frac{0.04}{2} = 1 + 0.02 = 1.02$

$f(0.04) \approx 1.02$

3) Find true values of $f(0.2)$, $f(0.04)$

$f(0.2) = \sqrt{0.2 + 1} = \sqrt{1.2} = 1.095445115$

$f(0.04) = \sqrt{0.04 + 1} = \sqrt{1.04} = 1.019803902$

4) Find Error

|value|

(4) Find Error

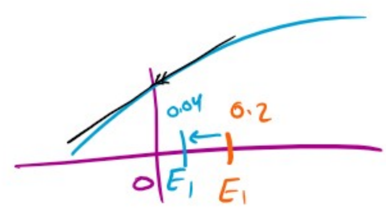
$$\text{Error} = E = \left| \text{True value} - \text{Estimated value} \right|$$

$$E_1 = \left| f(0.2) - L(0.2) \right| = \left| 1.095445115 - 1.1 \right| \simeq 0.00455$$

$$E_2 = \left| f(0.04) - L(0.04) \right| = \left| 1.0198039027 - 1.02 \right| \simeq 0.000196$$

$$E_2 < E_1$$

السبب



as $x \to 0 \Rightarrow E \downarrow \Rightarrow L(x)$ is better approximation

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$$L(x) = f(a) + f'(a)(x-a)$$

Tangent line

$(\overset{x_0}{a}, \overset{y_0}{f(a)})$, $\overset{m}{f'(a)}$

$$y - y_0 = m(x - x_0)$$

$$y - f(a) = f'(a)(x-a)$$

$$y = f(a) + f'(a)(x-a)$$

Exp $f(x) = \sqrt{x+1}$ $\quad$ $x = 0$

$$f(x) = (1+x)^{\frac{1}{2}}$$

=

$$L(x) = 1 + \frac{x}{2}$$

$$f(x) = (1+x)^{\frac{1}{2}}$$

$$L(x) = 1 + \frac{1}{2}x$$

Remark $f(x) = (x+1)^m$, $x = 0$

The linearization about $x=0$ is

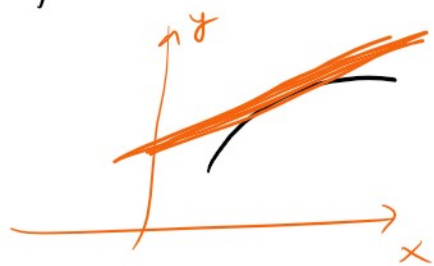
$$L(x) = 1 + mx$$

$\approx$ approximation

تقريب

Exp Find $L(x)$ at $x=0$ for

① $f(x) = (1-x)^{\frac{3}{2}}$



$$\boxed{L(x) = 1 - \frac{3}{2}x}$$

$f(0) = \ldots$

$f'(0) = $ ———

② $g(x) = \dfrac{1}{\sqrt[3]{1+x}}$

$$= \frac{1}{(1+x)^{\frac{1}{3}}}$$

$-\frac{1}{3}$

$$= (1+x)^{-\frac{1}{3}}$$

$$L(x) = 1 + \left(-\frac{1}{3}\right)x$$

$$= 1 - \frac{x}{3}$$

Exp Find linearization of $f(x) = \cos x + 1$ at $x = \frac{\pi}{3}$

$$a = \frac{\pi}{3} \Rightarrow f(a) = f\left(\frac{\pi}{3}\right) = \cos\frac{\pi}{3} + 1$$

$$= \frac{1}{2} + 1$$

$$= \frac{3}{2}$$

$$f'(x) = -\sin x$$

$$f'\left(\frac{\pi}{3}\right) = -\sin\frac{\pi}{3}$$

$$= -\frac{\sqrt{3}}{2}$$

$$f'(a) = f'\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

$$L(x) = f\left(\frac{\pi}{3}\right) + f'\left(\frac{\pi}{3}\right)\left(x - \frac{\pi}{3}\right)$$

$$= \frac{3}{2} + -\frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right)$$

$$= \frac{3}{2} - \frac{\sqrt{3}}{2}x + \frac{\sqrt{3}\pi}{6}$$

$$= \left(\frac{3}{2} + \frac{\sqrt{3}\pi}{6}\right) - \frac{\sqrt{3}}{2}x$$

Differential

$y = f(x) \rightarrow$ differentiable

$y' = f'(x)$

$\frac{dy}{dx} = f'(x)$

$\frac{dy}{dx}\Big|_{x=a} = f'(a)$

$$\boxed{dy = f'(a)\, dx}$$

differential

(Estimated Value)

$dy \approx \Delta y \rightarrow \Delta f$

True Value

$\Delta y = y_2 - y_1$

$= f(x_2) - f(x_1) = \Delta f$

$\Delta x = x_2 - x_1$

$= dx$

Error $= E = |$ True Value $-$ Estimated value $|$

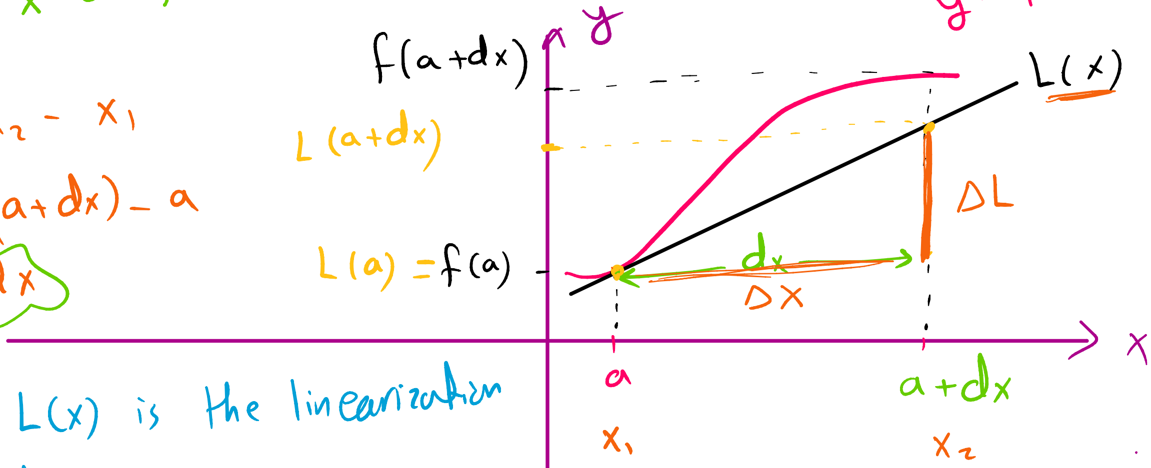
$$= |\Delta y - dy|$$

$\Delta L \approx f(x)$??
$dy \approx f(x)$

Exp show that $dy = \Delta L$

Assume x changes from a to $a + dx$

$\Delta X = x_2 - x_1$
$= (a + dx) - a$
$= dx$



Assume $L(x)$ is the linearization of f at $x = a$

True change is $\Delta f = \underbrace{f(x_2)}_{y_2} - \underbrace{f(x_1)}_{y_1} = f(a+dx) - f(a) = \Delta y$

Estimated change "differential" $dy = f'(a)\,dx$

$$= \frac{\Delta L}{\cancel{\Delta x}} \cancel{dx}$$

$$= \Delta L$$

Hence, we can use differential dy to estimate the true change Δy

$dy \approx \Delta y$ | $\Delta L \approx \Delta y$

$dy \approx \Delta f$ | $\Delta L \approx \Delta f$

Exp ① Find differential of

$y = x^3 + 2x$ at $x = 1$ (x_1, a), $dx = \frac{1}{5}$

$dy = f'(a)\,dx$ (estimated change)

$= f'(1)\,dx$

$= \boxed{5}\,\boxed{\frac{1}{5}}$

$= 1$ ✓

$y' = 3x^2 + 2$

$y'(1) = 3(1)^2 + 2$

$= 3 + 2$

$= 5$

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② Find True change

$\Delta f = f(x_2) - f(x_1)$

$= f\left(\frac{6}{5}\right) - f(1)$

$= f(1.2) - f(1)$

$y = x^3 + 2x$

$f(1) = 1^3 + 2$

$= 1 + 2 = 3$

$f(1.2) = (1.2)^3 + 2(1.2)$

$= 4.128$

$\Delta x = dx = x_2 - x_1$

$\frac{1}{5} = x_2 - 1$

$x_2 = 1 + \frac{1}{5}$

$x_2 = \frac{6}{5}$

$= f(1.2) - f(1)$ | $= 4.128$ | $x_2 = \frac{6}{5}$

$= 4.128 - 3$

$= 1.128$ True change

$dy = 1$ estimated change

3

Find error $\Rightarrow E = |TV - EV|$

$= |1.128 - 1| = \underline{\underline{0.128}}$

Exp Find differential for $y = \tan(2x)$ at $x = 0$ if $dx = \frac{1}{4}$

$f'(x) = 2\sec^2(2x)$

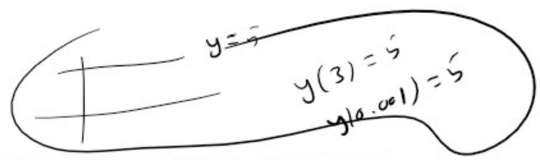
$f'(0) = 2\sec^2 0$
$= 2(1)^2$
$= 2$

$dy = f'(a)\,dx$

$= f'(0)\,dx$

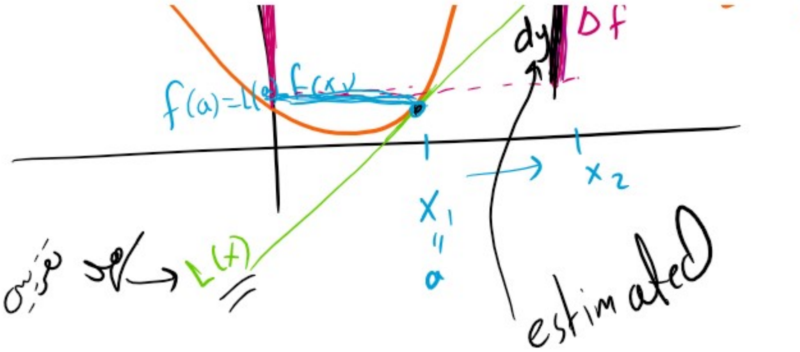
$= \boxed{2}\left(\frac{1}{4}\right)$

$= \frac{1}{2}$



$f(x)$

$dy \approx \Delta f$
↓
differential



$\boxed{L(x) = f(x) \text{ only at } a} \longrightarrow E = 0$

$E = |\Delta f - dy| = 0$

Exp r : radius of circle

Assum r changes from 10 cm to 10.1 cm

$r_1 = 10$
$r_2 = 10.1$
$\Delta r = dr = r_2 - r_1$
$= 10.1 - 10$
$= 0.1$

① Estimate the change in the circle's area

estimated change in area of circle → dA

$A = r^2 \pi$

$dA = 2r\pi \, dr$

$= 2(10)\pi (0.1)$

$dA = 2\pi$ EV

$dA = dy$
$dx = dr$

② Find true change in the area

$\Delta A = A(r_2) - A(r_1)$

$= A(10.1) - A(10)$

$= (10.1)^2 \pi - 10^2 \pi$

$102.01\pi - 100\pi$

③ Find error

$\Rightarrow 102.01\pi - 100\pi$

TV $= 2.01\pi$

③ Find error

$E = |TV - EV|$

$= |2.01\pi - 2\pi| = \underline{\underline{0.01\pi}}$

④ Estimate the area of the large circle

$A(r_1) + dA$

$= 100\pi + 2\pi$

$= 102\pi$

True area of large Circle is

$A(r_2) = A(10.1)$

$= 102.01\pi$

Exp ① $f(x) = c \Rightarrow f'(x) = 0$

$f(x) = \sqrt{5} \Rightarrow f'(x) = 0$

② $f(x) = a x^n \Rightarrow f'(x) = an x^{n-1}$

$f(x) = 3x^2 - 5x^3 + 1$

$f'(x) = 6x - 15x^2$

③ $f(x) = (x^3 + 4x)^5$ Find $f'(1)$

$f'(x) = 5(x^3 + 4x)^4 (3x^2 + 4)$

$f'(1) = 5(1 + 4)^4 (3 + 4)$

$= 5(5)^4 (7)$

$= (5)^5 (7)$ ....

Exp show that if $f(x) = \cot x$ then $f'(x) = -\csc^2 x$

$$f(x) = \cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$$

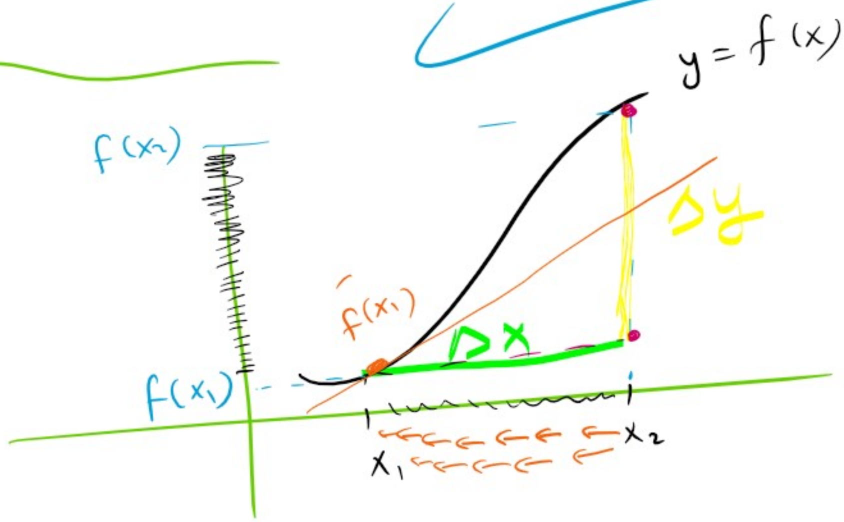
$$f'(x) = \frac{(\sin x)(-\sin x) - (\cos x)(\cos x)}{\sin^2 x}$$

$$= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x}$$

$$= \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x}$$

$$= \frac{-1}{\sin^2 x} = -\csc^2 x$$

$y = f(x)$



$\Delta x = x_2 - x_1$

$\Delta y = y_2 - y_1$

$= f(x_2) - f(x_1)$

$= f(x_2) - f(x_1)$

average change $= \dfrac{\Delta y}{\Delta x} =$

$\dfrac{\text{طبي } 50}{\text{طول } 50}$

معدل طول طلاب الجامعة $\leftarrow$ $1.72 = \dfrac{1.7 + 1.6 + 1.75 + \cdots}{50}$

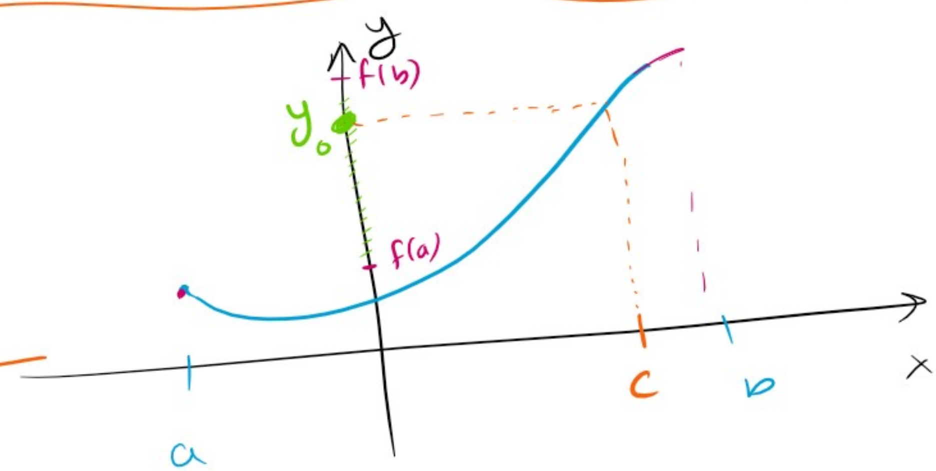
$$f'(x_1) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

IVT

f cont $[a, b]$ ✓

$y_0 \in [f(a), f(b)]$

Then $\exists$ at least number $c \in (a, b)$ s.t $f(c) = y_0$



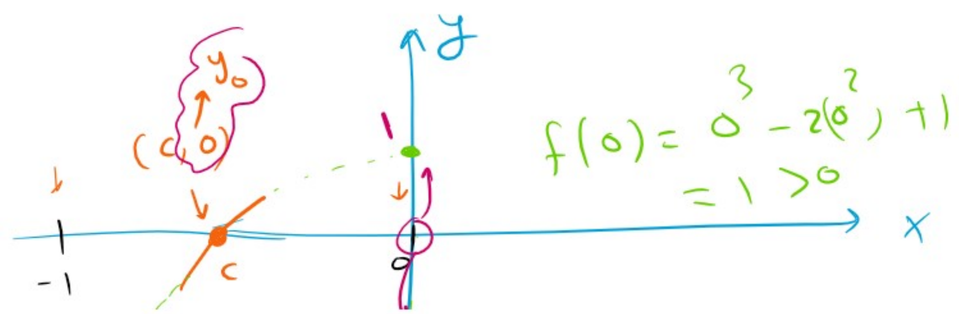
Exp show that $f(x) = x^3 - 2x^2 + 1$ has root on $[-1, 0]$

S1

✓ $f(x)$ poly $\Rightarrow f$ cont on $\mathbb{R} \Rightarrow f$ cont on $[-1, 0]$

$f(c) = 0$

$f(-1) = (-1)^3 - 2(-1)^2 + 1$

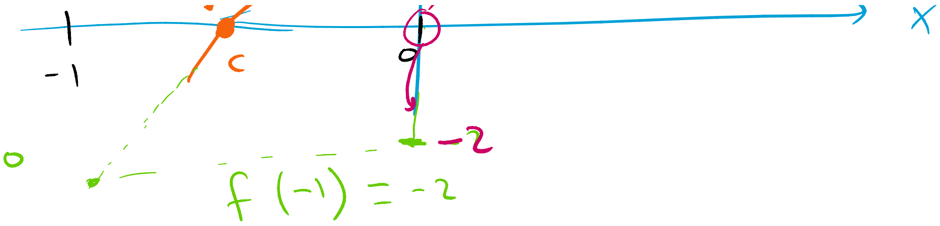


$f(0) = 0^3 - 2(0^2) + 1$
$= 1 > 0$

$f(-1) = (-1)^3 - 2(-1) + 1$

$= -1 - 2 + 1$

$= -3 + 1 = -2 < 0$



Since $y_0 = 0 \in [f(-2), f(0)] = [-2, 1]$

Then by IVT $\Rightarrow \exists\, c \in (-1, 0)$ st $f(c) = y_0$

$f(c) = 0$

$\downarrow$

c is root

Bolzano Th : Special case of IVT when $y_0 = 0$

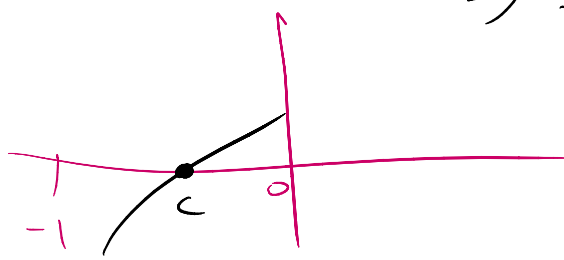
S2 f cont. on $[-1, 0]$

$f(-1)\, f(0) \overset{??}{<} 0$

$(-2)(1) = -2 < 0$

$\Longrightarrow f$ crosses x-axis on $[-1, 0]$

$\Rightarrow \exists\, c \in (-1, 0)$



$\Rightarrow \exists \; c \in (-1, 0)$

s.t $f(c) = 0$

y_0

