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Birzeit University
Mathematics Department
Math3331
Quiz I&II

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Second Semester 2019/2020
Name:..

Time: 20 minutes
Date: 05/03/2020
Number:..............

Exercise#1 [5 points].

a) Find all real numbers x that satisfy the inequality $x^2 > \frac{1}{x}$.

b) Show that

$$\max\{\alpha,\ \beta\} = \frac{\alpha+\beta+|\alpha-\beta|}{2},\ \ \forall\, \alpha,\beta \in \mathbb{R}.$$

Exercise#2 [5 points]. Let

$$E = \Big\{ r \big|\ r \text{ is a rational and } r^2 < 2 \Big\}.$$

Show that E has no rational supremum.

Exercise#3 [5+5 points]. Let A and B be bounded nonempty subsets of $\mathbb{R}$.

a) Show that $A \cup B$ is bounded.

b) Prove that $\sup\big(A \cup B\big) = \max\big\{\sup A,\ \sup B\big\}$.

Sol. Ex. #1 a) It is clear that if $x<0$, then $\frac{1}{x}<0$ and $x^2>0$. thus, $\forall x<0$: $\frac{1}{x}<x^2$

If $x>0$, $\frac{1}{x}<x^2 \iff x^3>1 \iff x>1$

then $\left\{x : \frac{1}{x} < x^2\right\} = (-\infty, 0) \cup (1, \infty)$.

b) Let $\alpha, \beta \in \mathbb{R}$: $\alpha < \beta$ or $\alpha \geq \beta$

Good Luck

If $\alpha < \beta$: $\max\{\alpha, \beta\} = \beta$ and

$\frac{\alpha+\beta+|\alpha-\beta|}{2} = \frac{\alpha+\beta+\beta-\alpha}{2} = \beta$

If $\alpha \geq \beta$: $\max\{\alpha, \beta\} = \alpha$ and

$$\frac{\alpha+\beta+|\alpha-\beta|}{2} = \frac{\alpha+\beta+\alpha-\beta}{2} = \alpha.$$

Then in both cases we have equality.

Ex#2. Suppose that E has a rational sup. Say $\sup E = q$ where q is rational. Since $\sup E$ is an upper bound for E, it follows that $\sqrt{2} < q$. Density then implies that $\exists\, q_2 \in \mathbb{Q}$ such that

$\sqrt{2} < q_2 < q$. This contradicts the assumption that $q = \sup E$. ∎

Ex#3. a) Since A is bdd, then $\exists M > 0$ such that $|a| \leq M$, $\forall a \in A$. Similarly, since B is bdd, then $\exists N > 0$ such that $|b| \leq M$, $\forall b \in B$.

For $c \in A \cup B \iff c \in A$ or $c \in B$

$\Rightarrow |c| \leq M$ or $|c| \leq N$

$\Rightarrow |c| \leq \max\{M, N\}, \forall c \in A \cup B$.

$\Rightarrow A \cup B$ is bdd.

b) We know that $A \subset A \cup B \Rightarrow \sup A \leq \sup(A \cup B)$

$B \subset A \cup B \Rightarrow \sup B \leq \sup(A \cup B)$

$$\Rightarrow \boxed{\max\{\sup A, \sup B\} \leq \sup(A \cup B) \quad \cdots (1)}$$

$\forall \alpha \in A \cup B$: $\alpha \in A$ or $\alpha \in B$

$\Rightarrow \alpha \leq \sup A$ or $\alpha \leq \sup B$

$\Rightarrow \alpha \leq \max\{\sup A, \sup B\}$

$\Rightarrow \max\{\sup A, \sup B\}$ is an upper bound of $A \cup B$. this implies that

$$\boxed{\sup(A \cup B) \leq \max\{\sup A, \sup B\}} \; (2)$$

because $\sup(A \cup B)$ is the smallest upper bound

Hence, (1) and (2) give the equality.