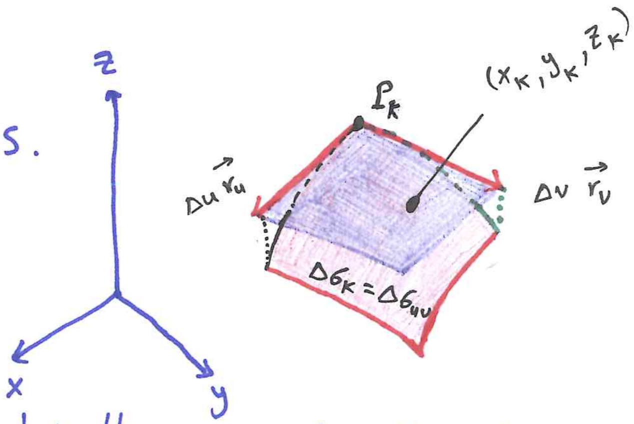
$\boxed{16.6}$ Surface Integrals

- Recall that the line integral is computed over a curve (16.1).
- Now we extend line integral to surface integral over surface.
- This helps to compute flow of liquid across membrance (غشاء)
  or upward force on a falling parachute (مظلة)

- Suppose there is an electrical charge over a surface S.
- Suppose $G(x,y,z)$ is the function that gives charge density (per unit area) at each point on S.
- we need to calculate the total charge on S.



- Assume, as in section 16.5, that the surface S is defined parametrically on region R in the uv-plane by:

$$\vec{r}(u,v) = f(u,v)\,\vec{i} + g(u,v)\,\vec{j} + h(u,v)\,\vec{k} \quad , \quad (u,v) \in R$$

- The subdivisions of R divids S into surface elements with area

$$\Delta 6_K = \Delta 6_{uv} = |\vec{r}_u \times \vec{r}_v| \, du \, dv$$

- Choose a point (x_K, y_K, z_K) in the K^{th} surface element (not plane).
- Multiply the value of the function G at (x_K, y_K, z_K) by the area $\Delta 6_K$ and form Riemann sum over S:

$$\sum_{K=1}^{n} G(x_K, y_K, z_K)\, \Delta 6_K$$

- Take limit as $n \longrightarrow \infty$ ($\Delta u \to 0$ and $\Delta v \to 0$). This limit, if it exists, define the surface integral of G over the surface S as

$$\iint_S G(x,y,z)\, d6 = \lim_{n \to \infty} \sum_{K=1}^{n} G(x_K, y_K, z_K)\, \Delta 6_K$$

Note The formula for evaluating the surface integral depends on whether S is parametrically, implicitly or explicitly described.

1) For a smoth surface S defined parametrically as $\vec{r}(u,v) = f(u,v)\vec{i} + g(u,v)\vec{j} + h(u,v)\vec{k}$, $(u,v) \in R$ and a continuous function $G(x,y,z)$ defined on S, the surface integral of G over S is the double integral over R

$$\iint_S G(x,y,z)\,d\sigma = \iint_R G\big(f(u,v), g(u,v), h(u,v)\big)\,|\vec{r}_u \times \vec{r}_v|\,du\,dv$$

2) For a surface S given implicitly by $F(x,y,z)=c$, where F is a continuously diff function and S lying above its closed and bounded shadow region R, the surface integral of the continuous function G over S is the double integral over R

$$\iint_S G(x,y,z)\,d\sigma = \iint_R G(x,y,z)\,\frac{|\nabla F|}{|\nabla f \cdot \vec{p}|}\,dA,$$

where $\vec{p}$ is a unit vector normal to R and $\nabla F \cdot \vec{p} \neq 0$.

3) For a surface given explicitly as $z = f(x,y)$, where f is a continuously diff function over a region R in the xy-plane, the surface integral of the continuous function G over S is the double integral over R

$$\iint_S G(x,y,z)\,d\sigma = \iint_R G(x,y,f(x,y))\sqrt{f_x^2 + f_y^2 + 1}\,dx\,dy$$

Remark: If S is partioned by smooth curves $S_1, S_2, \dots, S_n$ nonoverlapping, then

$$\iint_S G\,d\sigma = \iint_{S_1} G\,d\sigma + \iint_{S_2} G\,d\sigma + \cdots + \iint_{S_n} G\,d\sigma$$

Exp Integrate the given function over the given surface: $\boxed{17.8}$

$F(x,y,z) = z$ over the portion of the plane $x+y+z=4$ that lies above the square $0 \le x \le 1$ and $0 \le y \le 1$ in the xy-plane.

- Parametrization $\vec{r}(x,y) = x\,\vec{i} + y\,\vec{j} + (4-x-y)\,\vec{k}$

- $\vec{r}_x = \vec{i} - \vec{k}$

  $\vec{r}_y = \vec{j} - \vec{k}$

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \vec{i} + \vec{j} + \vec{k}$$

$$|\vec{r}_x \times \vec{r}_y| = \sqrt{1+1+1} = \sqrt{3}$$

- $$\iint_S F(x,y,z)\,d\sigma = \iint_R F(x,y,z)\,|\vec{r}_x \times \vec{r}_y|\,dx\,dy$$

$$= \sqrt{3}\int_0^1\int_0^1 (4-x-y)\,dx\,dy = \int_0^1 \sqrt{3}\left(\frac{7}{2}-y\right)dy = 3\sqrt{3}$$

Exp Integrate $G(x,y,z) = x+y+z$ over the portion of the plane $2x+2y+z=2$ that lies in the first octant.

- $f(x,y,z) = 2x+2y+z \Rightarrow \nabla f = 2\vec{i} + 2\vec{j} + \vec{k}$

- $z = 2-2x-2y \Rightarrow G(x,y,z) = x+y+(2-2x-2y)$

  $= 2-x-y \qquad \Rightarrow \vec{p} = \vec{k}$

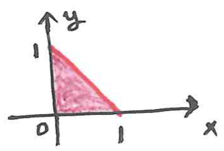
- $|\nabla f| = \sqrt{4+4+1} = 3$ and $|\nabla f \cdot \vec{k}| = |1| = 1$

- $$\iint_S G(x,y,z)\,d\sigma = \iint_R (2-x-y)\frac{|\nabla f|}{|\nabla f \cdot \vec{p}|}\,dA$$

when $z=0 \Rightarrow$

$x+y=1 \Rightarrow$

$y = 1-x$

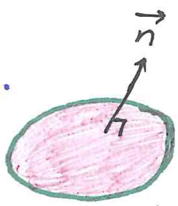


$$= \int_0^1\int_0^{1-x} (2-x-y)\,3\,dy\,dx$$

$$= 3\int_0^1 \left(\frac{3}{2} - 2x + \frac{x^2}{2}\right)dx = 2$$

Orientation

- A smooth surface S is orientable (or two-sided) if it is possible to define a field $\vec{n}$ of unit normal vectors on S that varies continuously with position.

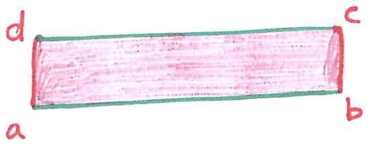


Positive direction

- Any patch or subportion of an orientable surface is orientable.
- Spheres and smooth closed surfaces (enclose solids) like cones are orientable.
- The surface together with its normal field is called an oriented surface.
- The vector $\vec{n}$ at any point is called the positive direction at that point.

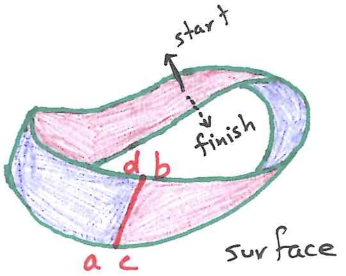
<u>Exp</u> Give an example of nonorientable (or one-sided) surface.

- The Möbius band is not orientable



Paper

$\Rightarrow$ Match a with c and b with d



- No matter where we start to construct a continuous unit normal field, moving the vector continuously around the surface will return the vector to the starting point with a direction apposit to the one we start with.
- The vector at that point can not point both ways
- Hence, no such field exists.

Surface Integral for Flux

- Recall that, from section 16.2, the flux of a two-dimensional field $\vec{F}$ across a plane curve C is $\int_C \vec{F}\cdot\vec{n}\,ds$.

Def The flux of a three-dimensional vector field $\vec{F}$ across an oriented surface S in the direction of $\vec{n}$ is
$$\text{Flux} = \iint_S \vec{F}\cdot\vec{n}\,d\sigma$$

- If $\vec{F}$ is the velocity field of a three-dimensional fluid flow, then the flux of F across S is the net rate at which fluid is crossing S in the chosen positive direction

Exp* Find the flux $\vec{F} = z^2\,\vec{i} + x\,\vec{j} - 3z\,\vec{k}$ outward (normal away from the x-axis) through the surface cut from the parabolic cylinder $z = 4 - y^2$ by the planes $x=0$, $x=1$, $z=0$.

S_1 **Parametrization:** $\vec{r}(x,y) = x\,\vec{i} + y\,\vec{j} + (4-y^2)\,\vec{k}$, $0 \le x \le 1$

- when $z=0 \Rightarrow 0 = 4-y^2 \Rightarrow y = \pm 2 \Rightarrow -2 \le y \le 2$

- $\vec{r}_x = \vec{i}$ and $\vec{r}_y = \vec{j} - 2y\,\vec{k}$

$$\vec{n} = \frac{\vec{r}_x \times \vec{r}_y}{|\vec{r}_x \times \vec{r}_y|}$$

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & 1 & -2y \end{vmatrix} = 2y\,\vec{j} + \vec{k}$$

- $\text{Flux} = \iint_S \vec{F}\cdot\vec{n}\,d\sigma = \int_0^1\int_{-2}^{2} \vec{F}\cdot\frac{\vec{r}_x\times\vec{r}_y}{|\vec{r}_x\times\vec{r}_y|}\,|\vec{r}_x\times\vec{r}_y|\,dy\,dx$

$$= \int_0^1\int_{-2}^{2}\left[2xy - 3(4-y^2)\right]dy\,dx = \int_0^1 -32\,dx = -32$$

$\boxed{S_2}$ • The gradient of $g(x,y,z) = y^2 + z = 4$ is

$$\nabla g = 2y\vec{j} + \vec{k} \quad \text{with} \quad |\nabla g| = \sqrt{4y^2+1}$$

• The outward normal field on S is

$$\vec{n} = \frac{\nabla g}{|\nabla g|} = \frac{2y\vec{j} + \vec{k}}{\sqrt{4y^2+1}}$$

• with $\vec{p} = \vec{K}$, we have $|\nabla g \cdot \vec{p}| = |1| = 1$

• $$\text{Flux} = \iint_S \vec{F}\cdot\vec{n}\, d6 = \iint_R \vec{F} \cdot \frac{\nabla g}{|\nabla g|} \frac{|\nabla g|\, dA}{|\nabla g \cdot \vec{p}|}$$

$$= \iint_R (2xy - 3z)\, dA$$

$$= \int_0^1 \int_{-2}^{2} (2xy - 12 + 3y^2)\, dy\, dx$$

$$= \int_0^1 -32\, dx$$

$$= -32$$

• $\vec{F} = z^2\vec{i} + x\vec{j} - 3z\vec{k}$

$= (4-y^2)^2\vec{i} + x\vec{j} - 3(4-y^2)\vec{k}$

• $\vec{F}\cdot\nabla g = 2xy - 3(4-y^2)$

• when $z = 0 \Rightarrow$

$y^2 = 4 \Rightarrow y = \pm 2$

Moments and Masses of Thin shells (similar to section 16.1)

Thin shells like bowl (كرة مضببة), drums (طبول), dome (قبة)....

- Mass $M = \iint_S \delta \, d\sigma$ where $\delta = \delta(x,y,z)$ is density at (x,y,z) as mass per unit area.

- First moments about the coordinate planes

$$M_{yz} = \iint_S x\,\delta\,d\sigma, \quad M_{xz} = \iint_S y\,\delta\,d\sigma, \quad M_{xy} = \iint_S z\,\delta\,d\sigma$$

- Coordinates of center of mass

$$\bar{x} = \frac{M_{yz}}{M}, \quad \bar{y} = \frac{M_{xz}}{M}, \quad \bar{z} = \frac{M_{xy}}{M}$$

- Moments of inertia about coordinate axes

$$I_x = \iint_S (y^2+z^2)\,\delta\,d\sigma, \quad I_y = \iint_S (x^2+z^2)\,\delta\,d\sigma$$

$$I_z = \iint_S (x^2+y^2)\,\delta\,d\sigma, \quad I_L = \iint_S r^2\,\delta\,d\sigma$$ where

$r(x,y,z)$ is the distance from point (x,y,z) to line L.

Notes [1] These formulas are like those for line integrals in section 16.1

[2] Their derivations are similar to those in section 6.6

Exp Find the centroid $(\bar{x}, \bar{y}, \bar{z})$ of the portion of the sphere $x^2+y^2+z^2=a^2$ that lies in the first octant. 183

- $f(x,y,z) = x^2+y^2+z^2 = a^2 \Rightarrow \nabla f = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$

$$|\nabla f| = \sqrt{4x^2+4y^2+4z^2} = 2a$$

- To find the integral for M_{xy}, we take $\vec{p} = \vec{k}$

$$\Rightarrow |\nabla f \cdot \vec{p}| = |2z| = 2z$$

- Since $z \geq 0 \Rightarrow d\sigma = \dfrac{|\nabla f|}{|\nabla f \cdot \vec{p}|} dA = \dfrac{2a}{2z} dA = \dfrac{a}{z} dA$

- Mass $M = \iint_S \delta \, d\sigma = \delta \iint_S d\sigma = a\delta \iint_R \frac{1}{z} dA$

$$= a\delta \int_0^{\frac{\pi}{2}} \int_0^a \frac{1}{\sqrt{a^2-r^2}} r \, dr \, d\theta = a\delta \int_0^{\frac{\pi}{2}} a \, d\theta = \frac{\delta \pi a^2}{2}$$

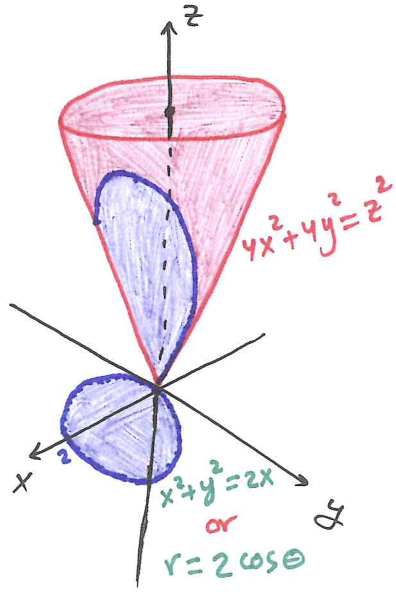
- $M_{xy} = \iint_S z \delta \, d\sigma = \delta \iint_R z \frac{a}{z} dA = a\delta \iint_R dA$

$$= a\delta \int_0^{\frac{\pi}{2}} \int_0^a r \, dr \, d\theta = \frac{\delta \pi a^3}{4}$$

- $\bar{z} = \dfrac{M_{xy}}{M} = \dfrac{\delta \pi a^3}{4} \dfrac{2}{\delta \pi a^2} = \dfrac{a}{2}$

- Because of symmetry $\bar{x} = \bar{y} = \bar{z} = \dfrac{a}{2} \Rightarrow$ centroid $= \left(\frac{a}{2}, \frac{a}{2}, \frac{a}{2}\right)$

Exp Find the moment of inertia about z-axis of a thin shell of constant density δ cut from the cone $4x^2+4y^2-z^2=0$, $z \geq 0$ by the circular cylinder $x^2+y^2=2x$ $\boxed{184}$



- $f(x,y,z)=4x^2+4y^2-z^2=0$

$$\nabla f = 8x\vec{i}+8y\vec{j}-2z\vec{k}$$

$$|\nabla f| = \sqrt{64x^2+64y^2+4z^2}$$

$$= 2\sqrt{16x^2+16y^2+z^2}$$

$$= 2\sqrt{4z^2+z^2} = 2\sqrt{5}\,z$$

- Take $\vec{p}=\vec{k}$ $\Rightarrow |\nabla f\cdot\vec{p}| = |-2z| = 2z$

- $d\sigma = \dfrac{|\nabla f|}{|\nabla f\cdot\vec{p}|}\,dA = \dfrac{2\sqrt{5}\,z}{2z}\,dA = \sqrt{5}\,dA$

- $I_z = \iint_S (x^2+y^2)\,\delta\,d\sigma = \sqrt{5}\,\delta \iint_R (x^2+y^2)\,dx\,dy$

$$= \sqrt{5}\,\delta \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} r^2\, r\,dr\,d\theta$$

$$= \frac{3\sqrt{5}\,\pi\,\delta}{2}$$