Exp $f(x) = \dfrac{x}{x^2+1}$ and $f'(x) = \dfrac{1-x^2}{(x^2+1)^2}$, $f''(x) = \dfrac{2x(x^2-3)}{(x^2+1)^3}$

Find

① $D(f) = (-\infty, \infty) = \mathbb{R}$

② Asy. $f(x) = \dfrac{x^{①}}{x^{②}+1} \Rightarrow \lim\limits_{x\to\infty} f(x) = 0$

$\lim\limits_{x\to-\infty} f(x) = 0$

$\Rightarrow y = 0$ is H. Asy.

$$\lim_{x\to\infty} \frac{x}{x^2+1} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1+\frac{1}{x^2}} = \frac{0}{1+0} = \frac{0}{1} = 0$$

$\nexists$ O. Asy

V. Asy $\quad f(x) = \dfrac{x}{x^2+1} \qquad x^2+1 \neq 0$

$\Downarrow$

$\nexists$ V. Asy.

③ CP's

$f'(x) = 0 \Rightarrow \dfrac{1-x^2}{(x^2+1)^2} = 0 \Rightarrow 1-x^2 = 0$

$\Rightarrow x = \pm 1$

$x = 1 \in D(f) = \mathbb{R}$ ✓

$x = -1 \in D(f) = \mathbb{R}$ ✓

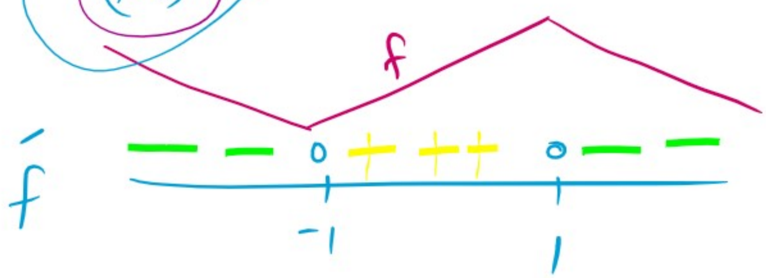
$$f(x) = \frac{x}{x^2+1}$$

$(1, f(1)) = \left(1, \frac{1}{2}\right)$

$(-1, f(1)) = \left(-1, -\frac{1}{2}\right)$

(4) Intervals of $\uparrow$ and $\downarrow$

$f' = 0 \Rightarrow \dfrac{1-x^2}{(x^2+1)^2} = 0 \Rightarrow 1 - x^2 = 0 \Rightarrow x = \pm 1$



$f \uparrow$ on $[-1, 1]$

$f \downarrow$ on $(-\infty, -1] \cup [1, \infty)$

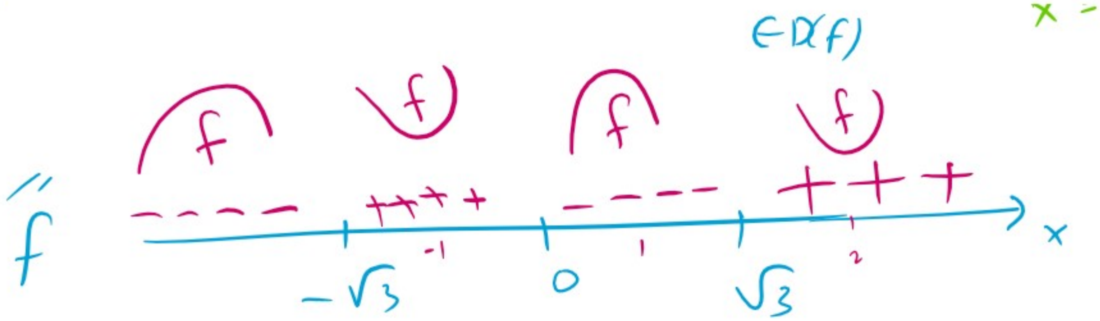
(5) Intervals of concavity

$$f'' = \frac{\overset{-}{\downarrow}2x\,(\overset{+}{\downarrow}x^2-3)}{(x^2+1)^3} = 0 \Rightarrow 2x\,(x^2-3) = 0$$

$(x^2+1)^3$: ×

$2x = 0$: $x = 0 \in D(f)$

$x^2 - 3 = 0$: $x^2 = 3$, $x = \pm\sqrt{3} \in D(f)$



f is concave up on $[-\sqrt{3}, 0] \cup [\sqrt{3}, \infty)$

f is concave down on $(-\infty, -\sqrt{3}] \cup [0, \sqrt{3}]$

6 Inflections Points

$f'' = 0 \Rightarrow$

$\boxed{1}$

$x_1 = 0 \in D$

$x_2 = \sqrt{3} \in D$

$x_3 = -\sqrt{3} \in D$

$\boxed{2}$

f changes concavity about x_1, x_2, x_3

$f(x) = \frac{x}{x^2+1}$

f has tangent at x_1, x_2, x_3

$$\left.\begin{aligned} &(0, f(0)) = (0, 0) \\ &(\sqrt{3}, f(\sqrt{3})) = \left(\sqrt{3}, \frac{\sqrt{3}}{4}\right) \\ &(-\sqrt{3}, f(-\sqrt{3})) = \left(-\sqrt{3}, -\frac{\sqrt{3}}{4}\right) \end{aligned}\right\} \text{Inflection Points}$$

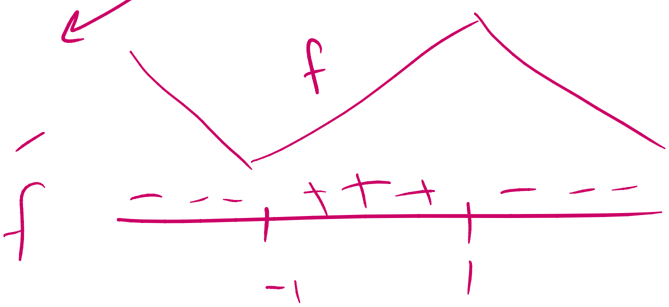
7 EV's

check at critical Point

~~End points~~

Endpoints

we have no endpoints since $D(f) = \mathbb{R} = (-\infty, \infty)$



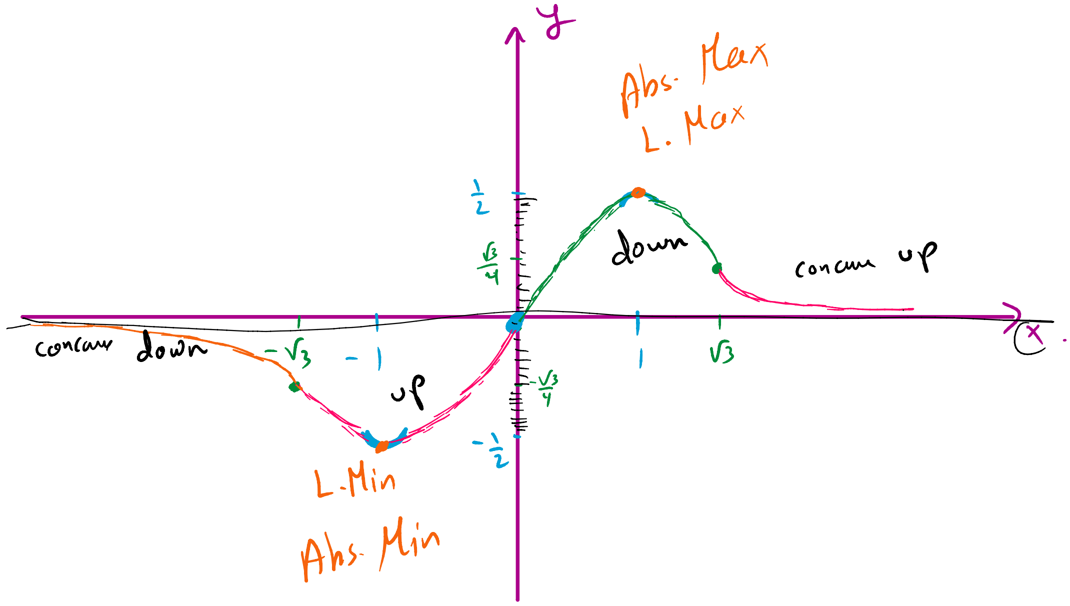
$(-1, f(-1)) = (-1, -\frac{1}{2})$

$(1, f(1)) = (1, \frac{1}{2})$

f has L. Max of $\frac{1}{2}$ at $x = 1$

f has L. Min of $-\frac{1}{2}$ at $x = -1$

8) sketch $f(x)$



$y = 0$ H. Asy

$\lim_{x \to \infty} f(x) = 0$

$y = 0$ H. As.

$\lim_{x \to -\infty} f(x) = 0$

Abs. min

$R(f) = [-\frac{1}{2}, \frac{1}{2})$

Ex $f(x) = \dfrac{x^2-4}{x-2}$

$D(f) = \mathbb{R} \setminus \{2\}$

$f(2)$ undefined

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{\cancel{(x-2)}(x+2)}{\cancel{x-2}}$$

$$= \lim_{x \to 2} (x+2) = 4$$

$f'(2)$ undefined

$x=2$ CP

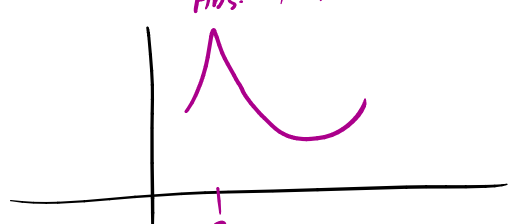
f is not cont. at $x=2$

$$f(2) \neq \lim_{x \to 2} f(x)$$

$$6 \neq 4$$

f' undefined

Abs. Max



$2 \in D(f)$

$f'(2)$ undefined

Exp Sketch $f(x) = \frac{1x^2}{1x^2-1}$ given $f' = \frac{-2x}{(x^2-1)^2}$

$f'' = \frac{6x^2+2}{(x^2-1)^3}$

$f' = 0$

$-2x = 0$

$x = 0$

L. Max

f' + + + | - - -, 0 $\Rightarrow 0 \in D(f)$

$f(0) = 0$

Keypoint $(0, 0)$

$D(f) = \mathbb{R} \setminus \{\pm 1\}$

Asy. $\lim_{x \to \infty} f(x) = \frac{1}{1} = 1 \Rightarrow \boxed{y = 1}$ H. Asy.

$\Downarrow$

$\nexists$ O. Asy.

$\lim_{x \to -\infty} f(x) = 1$

$\boxed{f(-x) = f(x)}$

$f(x)$ is even

V. Asy

$f = \frac{x^2}{x^2-1}$

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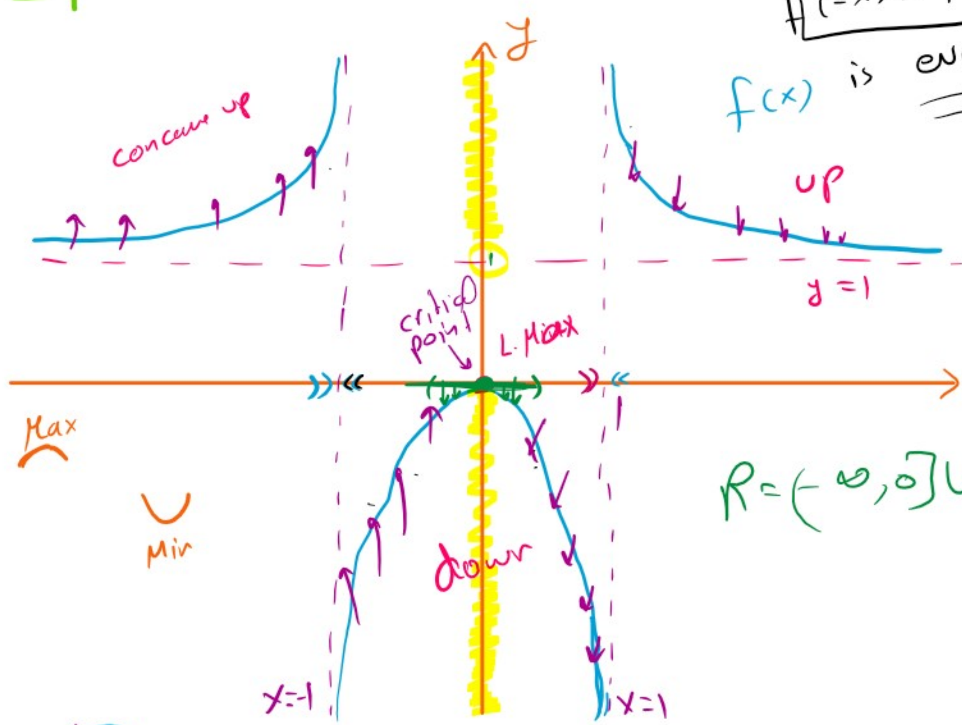
$x = \pm 1$

Check $\boxed{x = 1}$

Max

Min



$R = (-\infty, 0] \cup (1, \infty)$

$\lim_{x \to 1^+} \frac{x^2}{x^2-1} = \frac{1}{small+} = \infty \Rightarrow \boxed{x = 1}$ is V. Asy.

$\lim_{x \to 1^-} \frac{x^2}{x^2-1} = \frac{1}{small-} = -\infty$

$$\lim_{x \to 1^-} \frac{\;}{x^2-1} = \frac{\;}{\text{small}-}$$

Check $\boxed{x=-1}$

$$\lim_{x \to -1^+} \frac{x^2}{x^2-1} = \frac{1}{\text{small}-} = -\infty \Rightarrow \boxed{x=-1} \text{ V. Asy}$$

$-1.001 \quad -0.9 \quad -1$

$$\lim_{x \to -1^-} \frac{x^2}{x^2-1} = \frac{1}{\text{small}+} = \infty$$

Exp $\quad f(x) = \dfrac{x^2}{x+1}, \quad f' = \dfrac{x^2+2x}{(x+1)^2}, \quad f'' = \dfrac{2}{(x+1)^3}$

① $D = \mathbb{R} \setminus \{-1\}$

② H. Asy. None

③ O. Asy. Yes

$$f(x) = \frac{x^2}{x+1} = x-1 + \frac{1}{x+1}$$

$$\begin{array}{r|l} & x-1 \\ \hline x+1 & x^2 \\ & \cancel{-x^2} \mp x \\ \hline & -x \\ & \cancel{\mp x} \mp 1 \\ \hline & 1 \end{array}$$

$$\boxed{y = x-1 \quad \text{O. Asy}}$$

④ V. Asy.

$$f(x) = \frac{x^2}{x+1}$$

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Check $\boxed{x = -1}$

$$\lim_{x \to -1^+} \frac{x^2}{x+1} = \frac{1}{\text{small}+} = \infty$$

$\to$ $x = -1$ V. Asy.

-1.001 -0.99

-1

$$\lim_{x \to -1^-} \frac{x^2}{x+1} = \frac{1}{\text{small}-} = -\infty$$

$\lim_{x \to \infty} \frac{2x}{1}$

- $$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{x^2}{x+1} = \infty \qquad \frac{\infty}{\infty}$$

$\lim_{x \to -\infty} \frac{2x}{1}$

- $$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{x^2}{x+1} = -\infty \qquad \frac{\infty}{\infty}$$

$$\lim_{x \to \infty} \frac{x}{1+\frac{1}{x}} = \frac{\infty}{1+0} = \infty$$

$$\lim_{x \to -\infty} \frac{x}{1+\frac{1}{x}} = \frac{-\infty}{1+0} = -\infty$$

• CP's $\Rightarrow f' = 0 \Rightarrow \dfrac{x^2+2x}{(x+1)^2} = 0$ $\quad (x+1)^2 \rightarrow +$

$$x^2+2x = 0$$

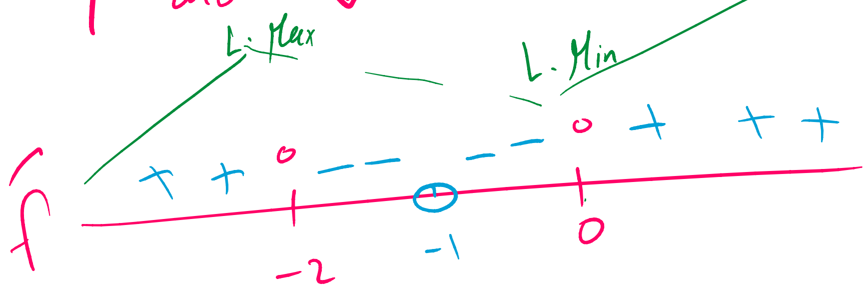
$$x(x+2) = 0$$

$$\boxed{x=0, \quad x=-2} \in D$$

$(0, f(0)) = (0,0)$

$(-2, f(-2)) = (-2,-4)$

• where f ↑ and ↓



f ↑ on $(-\infty, -2] \cup [0, \infty)$

f ↓ on $[-2, -1) \cup (-1, 0]$

• EV's

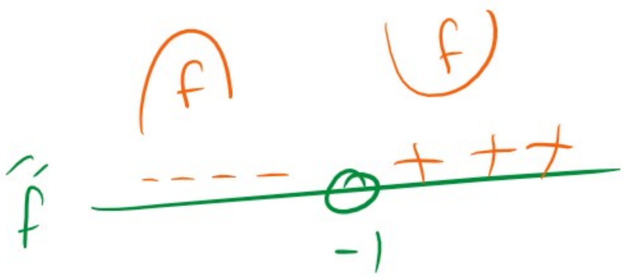
$f(-2) = -4$ is L. Max at $x=-2$

$f(0) = 0$ is L. Min at $x=0$

• Concave up and down

• Concave up and down

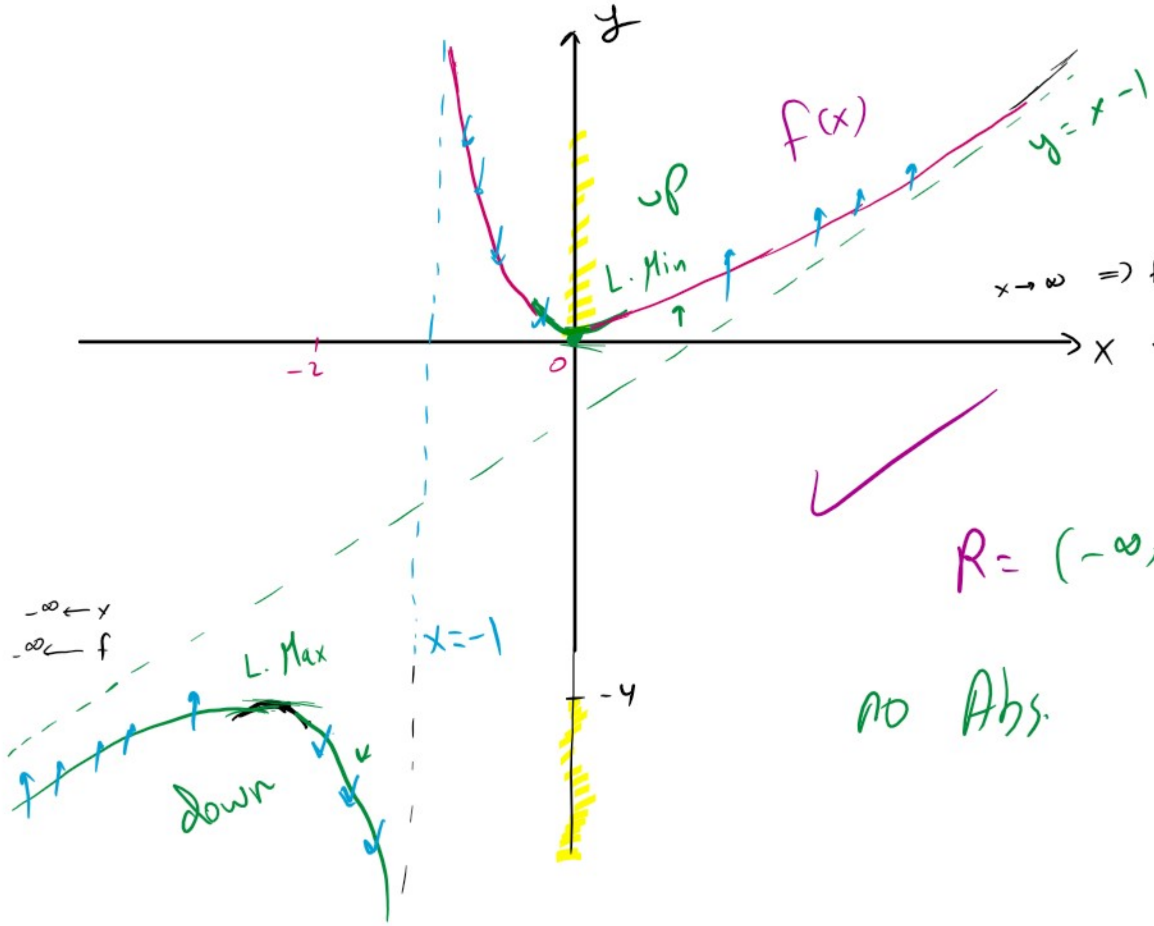
$$f'' = \frac{2}{(x+1)^3}$$



no inflection points

• sketch $f(x) = \dfrac{x^2}{x+1}$

$y = x - 1$ O. Asy.



not even
not odd

$x \to \infty \Rightarrow f \to \infty$

$R = (-\infty, -4] \cup [0, \infty)$

no Abs.

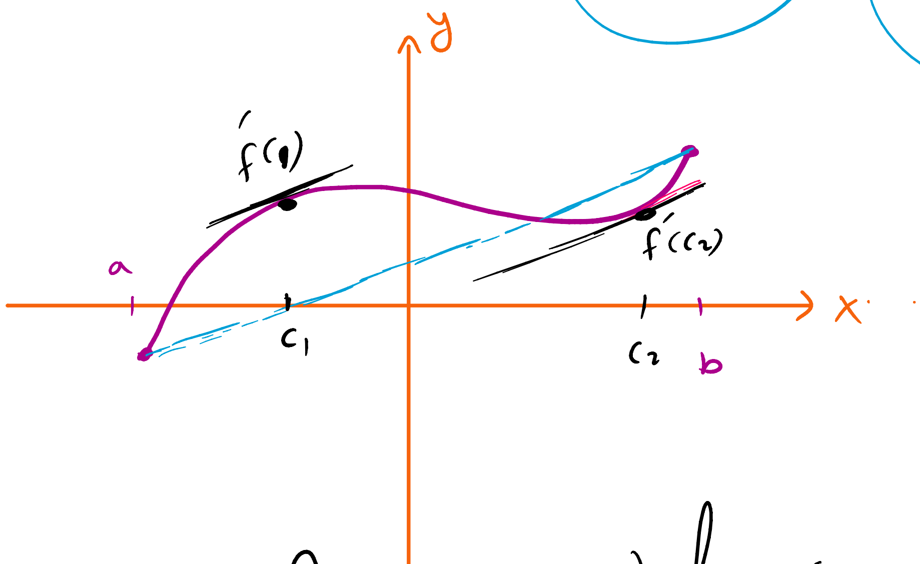
Th (Mean Value Th) نظرية القيمة المتوسطة

f con $[a,b]$ | $\Rightarrow \exists$ at least one number

f con $[a,b]$
f diff (a,b) $\Rightarrow \exists$ at least one number $c \in (a,b)$ s.t

ميل المماس $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

ميل القاطع



Exp Find the constant c that satisfies MVT for $f(x) = x^2$ on $[1,3]$

$$f'(c) = \frac{f(3)-f(1)}{3-1}$$

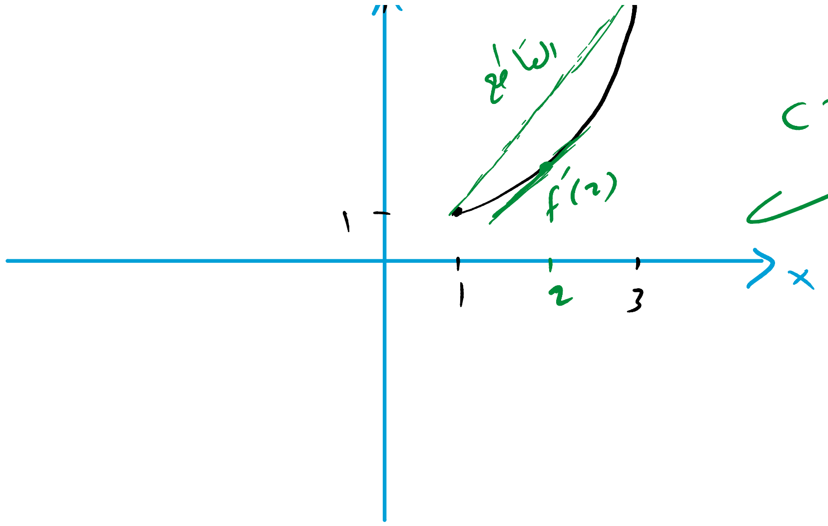
$f' = 2x$

$f'(c) = 2c$

$$2c = \frac{3^2 - 1^2}{2}$$

$$2c = \frac{9-1}{2} = \frac{8}{2} = 4$$

$$\boxed{c = 2}$$



$c = 2$