1.3 : Completeness Axiom

$E \neq \emptyset$

Df ①: let $E \subseteq \mathbb{R}$ be a nonempty set, Then

(i) E is bounded (bdd) above iff $\exists$ an $M \in \mathbb{R}$ s.t $x \leq M$, $\forall x \in E$,
M is called an upper-bound of E. $(-\infty, 1)$, $M=2$, $M=101$
not uniqe
$M=1$

supremum
هو أصغر upper bound
الموجود الأساسي .

(ii) A number β is called a supremum (uniqe) of E iff β is an upperbound and $\beta \leq M$, for all upperbounds M of E.
We say E has a finite supremum and we write $\sup E = \beta$.

every supremum its upperbound, The opposite is not true

RMKs: (1) $\sup E$ (if it exists) is the smallest (least) upperbound.

(2) to prove $\sup E = \beta$, we prove two things:
(i) β is an upperbound of E (i.e, $x \leq \beta$, $\forall x \in E$).
(ii) If M is any an upperbound of E then $\beta \leq M$.

Exp: let $E = [0,1]$, prove that $\sup E = 1$. → By Def of interval

(i) By Def, 1 is an upperbound of E. ($x \leq 1$, $\forall x \in [0,1]$)

(ii) Let M be any upperbound of E, i.e $x \leq M$, $\forall x \in E = [0,1]$.
In particular, take $x = 1 \Rightarrow 1 \leq M$
$\therefore$ 1 is least upperbound of E
$\therefore \sup E = 1$ #

exp: let $E_1 = \mathbb{R}^- = \{x : x < 0\}$

$E_2 = \mathbb{Z}^- = \{\dots, -3, -2, -1\}$.

Then $\sup E_1 = 0$ and $\sup E_2 = -1$. prove that.

① By Def of interval 0 is an upperbound of E ($x \le 0$, $\forall x \in E$).

② we need to show 0 is the smallest upperbound.

let M be any upper bound of E we need to show $0 \le M$

Inparticular, take $x = 0$

implies $0 \le M$ so 0 its least upperbound, $\sup E_1 = 0$.

part 2: $E_2 = \mathbb{Z}^{-1}$, $\sup E_2 = -1$.

① its clear is an upperbound of E. ($x \le -1$, $\forall x \in E$).

② If M is an upperbound of E $\Rightarrow$ $-1 \le M$

let M be an upperbound of E (i.e $x \le M$, $\forall x \in E$)

Inparticular, take $x = -1$ $\rightarrow$ $-1 \le M$

so -1 is least upperbound of E

so $\sup E_2 = -1$

✱ Note: supremum Not always belong to set.

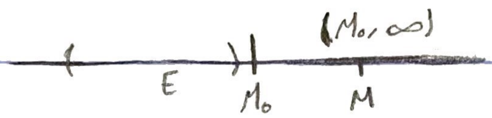
RMK :

① If a set has one upperbound , it has infinitely many upperbounds .

② If $\sup E = \beta$ exists then it is unique.

pf :

(i) If M_0 is an upperbound of a set E, then so is M for any $M > M_0$



② let β_1 and β_2 be two supremum of E, then both β_1 and β_2 are upperbound of E. Hence, by def $\beta_1 \leq \beta_2$ and $\beta_2 \leq \beta_1$.

we conclude , $\beta_1 = \beta_2 \Rightarrow$ uniquence . #

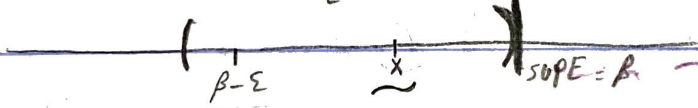
By Trichotomy property

Thm 1 : Approximation property for supremum :

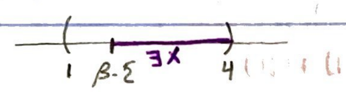
If $\sup E = \beta < +\infty$ and $\varepsilon > 0$ then $\exists$ a point $x \in E$ s.t $\beta - \varepsilon < x \leq \beta$.

pf:

contradiction



$E = (1,4) \to \sup E = 4$



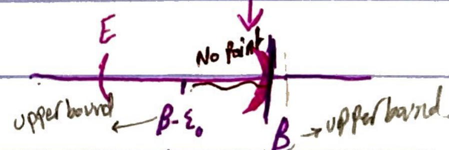
spse the thm is False Then $\exists$ $\varepsilon_0 > 0$ s.t No element of E lies between $\beta - \varepsilon_0$ and β.

since $\sup E = \beta$ is an upperbound of E, it follows $x \leq \beta - \varepsilon_0$, $\forall x \in E$.

ie : $\beta - \varepsilon_0$ is an upperbound of E. That is , $\beta \leq \beta - \varepsilon_0$

if follows $\varepsilon_0 \leq 0$, X contradiction

#



$\sup E = \beta$

$x \leq \beta$, $\forall x \in E$

$x \leq \beta - \varepsilon$ $\forall x \in E$

exp. $E = \mathbb{Z}^- = \{\dots, -3, -2, -1\} \to \sup E = -1 \in \mathbb{Z}$

Thm 2 : If $E \subset \mathbb{Z}$ has a supremum, then $\sup E \in \mathbb{Z}$.
Inparticular, if the sup of a set which contains only integers exists, then that sup must be an integers.

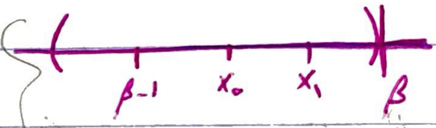
pf : spse that $\sup E := \beta$ and apply the approximation property for supremam,

$$\exists\, x_0 \in E \text{ s.t } \beta - \overset{\varepsilon}{1} < x_0 \le \beta$$

Case 1, If $\beta = x_0$, then $\beta \in E \subset \mathbb{Z} \Rightarrow \beta \in \mathbb{Z}$.

Case 2, $\beta - 1 < x_0 < \beta$, we can apply the approx. property,

$\exists\, x_1 \in E$ s.t $x_0 < x_1 < \beta$



$\exists$ an $x_1 \in E$ s.t
$x_0 < x_1 < \beta$
$-x_0 \quad -x_0 \quad -x_0$

$$\Rightarrow \quad 0 < x_1 - x_0 < \beta - x_0 \quad \dots\dots (i)$$

since $-x_0 < 1 - \beta$ ... (ii) we have
ضربت بسالب

it follows that, $0 < x_1 - x_0 < \beta + (1 - \beta) = 1 \to 0 < x_1 - x_0 < 1$
(i) + (ii)

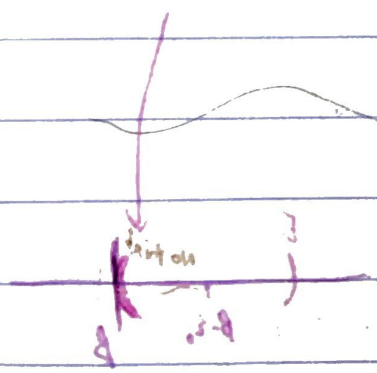
$\Rightarrow x_1 - x_0 \in (0,1)$

and $x_1 - x_0 \in \mathbb{Z}$

Thus, $x_1 - x_0 \in \mathbb{Z} \cap (0,1)$. ✗

We conclude $\beta \in E$

$\beta \in \mathbb{Z}$ #



which is imposible

لأنه $x_1 - x_0$ عبارة عن integer

ولا يوجد integer في الفترة $(0,1)$

So we are done

Postulate 3 (completeness Axiom):

If E is a nonempty subset of $\mathbb{R}$ that is bdd above, then E has a finite supremum.

$1+2 \Rightarrow$

RMK: From postulate 1, 2 and 3, we say that $\mathbb{R}$ is a complete ordered fi

مهم كثير

Thm 3: Archimedean property:

Given $a, b \in \mathbb{R}$ with $a > 0$, $\exists$ an integer $n \in \mathbb{N}$ s.t $b < na$.

$a, b \in \mathbb{R} \rightarrow b < a$ or $b \geq a$.

pf: If $b < a$, set $n = 1 \Rightarrow b < 1 . a$ we are done.

If $b \geq a$ and $a > 0$

consider the set

$E := \{ k \in \mathbb{N} : ka \leq b \}$

$E \neq \phi$ since $1 \in E$ ✓

$(1 . a \leq b$ ✓$)$

Let $k \in E$, That is $ka \leq b$, since $a > 0$

$k \leq \frac{b}{a}$, $\forall\, k \in E$.

This proves E is bdd above by $\frac{b}{a}$. ✓

$\rightarrow$

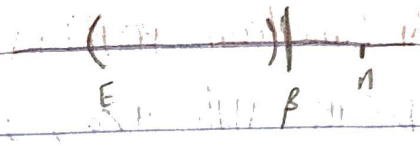
Thus, by the completeness Axiom property, E has a <u>finite sup</u>.

say $\sup E = \beta < +\infty$

By Thm 2 $\Rightarrow$ $\beta \in \mathbb{Z}$

set $n = \beta + 1$ Then $n \in \mathbb{N}$ and $n > \beta$,

it follows that $n \notin \underline{E} = \{ka \le b\}$



Thus, $na > b$ $\Leftarrow$ لانه $n \notin E$ اذا.

$b < na$ $\boxtimes$

RMK: sup E is not always belong to E.

exp: let $A = \{1, \frac{1}{2}, \frac{1}{4}, \ldots\}$, $B = \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\}$ prove that

$\sup A = \sup B = 1$, $\sup A \in A$, $\sup B \notin B$.

pf: $\sup A = 1$:

It is clear that 1 is an upperbound of A. $(x \le 1, \forall x \in A)$

let M be any upperbound of A, i.e, $M \ge a$, $\forall a \in A$ we need to show that $1 \le M$

In particular, take $a = 1$

$1 \le M$

$\therefore$ 1 is the <u>least</u> upperbound $\therefore \sup A = 1$.

$\Rightarrow$

في الخطوة الثانية من الاثبات اذا ما بنقتنع بنثبت باستخدام contradiction .

cont. $\sup B = 1$:

It is clear that 1 is an upperbound of B. since $b \leq 1 \; \forall b \in B$.

let K be any upperbound of B. ($b \leq K, \forall b \in B$)

we have to show $1 \leq K$

By contradiction

Spse not, i.e $K < 1 \to (1-K > 0) \to \frac{1}{1-K} \in \mathbb{R}$.

By Archimedean property, $\to b < na$

$\exists n \in \mathbb{N}$ s.t $\underset{b}{1} < n(1-K)^{a>0}$

i.e, $n > \frac{1}{1-K}$

$1-K > 0 \to \frac{n}{n}(1-K) > \frac{1}{n}$

$1-K > \frac{1}{n}$

$1 - \frac{1}{n} > K$

$\Rightarrow X_0 = 1 - \frac{1}{n} > K$

since $X_0 = 1 - \frac{1}{n} \in B$, this contradicts the assumption that K is an upperbound of B. Thus, $K \geq 1$

أكبر موجودة في B بعوض مكان n ارقام وبتكون موجودة في B

لقينا عنصر أكبر من K إذن $1-\frac{1}{n} > K$ بس K عبارة عن upperbound يعني هي أكبر عنصر بـ B فوصلنا لـ contradiction

and Hence 1 is the least upperbound of B.

$\therefore \sup B = 1$.

#

Thm 4 [Density of Rationals]

The Rational numbers $\mathbb{Q}$ are dense in $\mathbb{R}$. That is,

$\forall a, b \in \mathbb{R}$, with $a<b$ $\exists$ $q \in \mathbb{Q}$: $a < q < b$. $1 < 1.09 < 1.1$

pf :

case 1 : spse that $a > 0$ $(0<a<b)$

since $b-a > 0$, use Archimedean property,

$\exists\, n \in \mathbb{N}$ s.t $\max\left\{\frac{1}{a}, \frac{1}{b-a}\right\} < n$ $\rightarrow a=1$ صح التبرير بنفهم ونعرف كيف نجيبه

$\Rightarrow \frac{1}{a} < n$ and $\boxed{\frac{1}{b-a} < n}$

consider the set,

$E := \{K \in \mathbb{N} : \frac{K}{n} \leq a\} = \{K \in \mathbb{N} : K \leq na\}$

since $1 \in E$, then $E \neq \emptyset$ 1

since $K \leq na$, $\forall K \in E$ then E is bdd above 2

By Thm 2, + Comp axiom [1+2] $K_0 := \sup E$ exists and $K_0 \in E$, in particular $K_0 \in \mathbb{N}$.

set $m = K_0 + 1$ and $q = \frac{m}{n}$. $\frac{m}{n} > a$

since $K_0 = \sup E$, then $\boxed{m \notin E}$ Thus, $q = \frac{m}{n} = \frac{K_0+1}{n} > a$

لأن $m=K_0+1$ فمتكونش ب E
لأن K_0 هي sup

$\Rightarrow a < q$

on the other hand, since $\boxed{K_0 \in E}$, $b = a + (b-a) > \frac{K_0}{n} + \frac{1}{n} = \frac{m}{n} = q$ $a \geq \frac{K_0}{n}$ $\frac{K_0+1}{n}$

$\Rightarrow b > q$ Rational $0 < a < q < b$

i.e, $\exists\, q \in \mathbb{Q}$: $a < q < b$, $a > 0$.

case 2 : $a \leq 0$ $(-a \geq 0)$

$b < n\,a$

$-a$ K 1

By Arch. prop, $\exists\, K \in \mathbb{N}$ s.t $-a < K$, Then

$0 < K + a < K + b$ and By case 1, $\exists\, r \in \mathbb{Q}$:

بالشق الأول أثبتنا انه طالما يكون في عددين $0<a<b$ حيكون في Rational بينهم

s.t $K + a < r < K + b$

$\Rightarrow$ $a < \underbrace{r - K}_{\text{Rational}} < b$ Therefore,

$q := r - K \in \mathbb{Q}$

and satisfies $a < q < b$ #,

p 1.3.3

H.W: If $a, b \in \mathbb{R}$ with $a < b$, $\exists$ an irrational α s.t $a < \alpha < b$ That is, the irrationals are dense in $\mathbb{R}$.

سؤال بالكتاب

$a < b$ implies $a - \sqrt{2} < b - \sqrt{2}$

choose $r \in \mathbb{Q}$ s.t $a - \sqrt{2} < r < b - \sqrt{2}$ (Thm 4)

Then $a < r + \sqrt{2} < b$

we know $r + \sqrt{2}$ is irrational

Thus, $\alpha = r + \sqrt{2}$

so $a < \alpha < b$

* Infimum of a set :

DF 2 : let $E \subset \mathbb{R}$ be anonempty .

(i) The set E is said to be bounded below iff $\exists\, m \in \mathbb{R}$ s.t $m \leq x$ $\forall x \in E$, in this case, m is said to be a lower bound of E.

(ii) Anumber α is called an infimum of the set E iff α is a lower bound of E and $\alpha \geq \gamma$ for all lower bounds γ of E.

$\alpha \leq x \;, \forall x \in E$.

In this case we sall say that E has an infimum α and write $\boxed{\inf E = \alpha}$.

(iii) E is said to be bounded iff it is bounded above and below.

(That is $\exists\, m, M$ s.t $m \leq x \leq M$, $\forall x \in E$).

OR $\exists\, M > 0$ s.t $|x| \leq M$, $\forall x \in E$).

Note :

$x \leq \beta, \; \forall x \in E$.

$\sup E = \beta$

let M be opperbound show $\beta \leq M$.

$\alpha \leq x \;, \forall x \in E$.

$\inf E = \alpha$

If γ is a lower bound of E, then $\alpha \geq \gamma$.

exp: prove, Abounded nonemty set E has a unique supremum and unique infimum. moreover $\inf E \leq \sup E$ Give necessary and saticient conditions for equality.

→ let s_1 and s_2 be supremum of A.

Then By def, s_1 and s_2 upperbound of set A

So s_1 is the least upperbound i.e $s_1 \leq s_2$

But s_2 is the least upperbound i.e $s_1 \leq s_2 \leq s_1$

$\hookrightarrow s_1 = s_2 = s_1$

so we proved the set has aunique supremum.

→ let m_1 and m_2 be infima of A

Then By def, m_1 and m_2 are Lowerbound of set A

So m_1 is the greatest lowerbound i.e $m_1 \geq m_2$.

But m_2 is the greatest lower bound i.e $m_1 \geq m_2 \geq m_1$

$\hookrightarrow m_1 = m_2 = m_1$

we proved the set has aunique infimum.

→ let $\sup A$ and $\inf A$ Both exsits, and $x \in A$

Then $m \leq x \leq M$, where m is greatest lower bound and M lower upperbound

Then $\inf A \leq x \leq \sup A$

Thus, $\inf A \leq \sup A$

RmK: When a set E contains its supremum, we write $\max E = \sup E$.
similarly, if $\inf E \in E$, we write $\inf E = \min E$.

exp: $E = [0,1]$ $\quad \sup E = \max E = 1$

$\inf E = \min E = 0$.

Thm 5: [Reflection principle] let $E \subseteq \mathbb{R}$ be nonempty:

(i) E has a supremum iff $-E$ has an infimum, in which case $\quad -E = \{-x ; x \in E\}$.

$$\inf(-E) = -\sup E.$$

(ii) E has an infimum iff $-E$ has a supremum, in which case

$$\sup(-E) = -\inf(E).$$

pf: (i)

$\Rightarrow$ spse that $\sup E = \beta$ exists

since β is an upper bound of E, then $x \leq \beta$, $\forall x \in E$.

This gives $-\beta \leq -x$, $\forall x \in E$, i.e $-\beta$ is a lower bound of $-E$.

spse that m is any lower bound of $-E$, then $m \leq -x$, $\forall x \in E$

This implies $x \leq -m$, $\forall x \in E$. i.e $-m$ is an upperboun of E.

since $\sup E = \beta$, then $-\beta \geq m$.

Thus, $-\beta$ is then inf. of $-E$. ($-E$ has an inf.)

and $\sup E = \beta = -(-\beta) = -\inf(-E)$

$\Rightarrow \inf(-E) = -\sup E$

$\rightarrow$

$\boxed{\Leftarrow}$ Conversly, spse that $-E$ has an inf. say $\inf(-E) = \alpha$

By Def $\alpha \leq -x \;\; \forall x \in E$. i.e, $x \leq -\alpha$, $\forall x \in E$.

Thus, $-\alpha$ is an upper bound of E, ie E is bounded above.

since $E \neq \phi$, then E has a supremum by the completeness axiom.

#

pf (ii) :

Thm 6: [Monotone property]

suppose that $A \subseteq B$ are nonempty sets of $\mathbb{R}$:

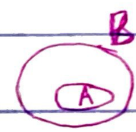
(i) If B has a supremum, then $\sup A \leq \sup B$.

(ii) If B has a infimum, then $\inf A \geq \inf B$.

توضيح exp: $A = [0,1]$, $B = [-1,2]$, $A \subseteq B$:

$\sup A = 1 \leq \sup B = 2$

$\inf A = 0 \geq \inf B = -1$.



pf: (i) since $A \subseteq B$, any upperbound of B is an upperbound of A.

Therefor, $\overset{<\infty}{\underline{\sup B}}$ is an upperbound of A.

It follows that by the completemess Axiom that $\sup A$ exists.

Thus, by def of $\sup A$, $\sup A \leq \sup B$.

(ii) since $A \subseteq B$, then $-A \subseteq -B$.

By part (i), $\sup(-A) \leq \sup(-B)$

By Thm 5, we have $-\inf A \leq -\inf B$

$\Longrightarrow \inf A \geq \inf B$ #.

Thm 7: [Approximation property for infimum]

If a set $E \subseteq \mathbb{R}$ has a finit infimum $\overset{m}{\alpha}$ and $\varepsilon > 0$ is any positive number, then there is a point $x \in E$ s.t

$$\alpha + \varepsilon > x \geq \alpha .$$

Q1.3.6 (a)

pf; let $\varepsilon > 0$ and $m = \inf E$.

Since $\varepsilon + m$ is not a lowerbound of E there is an $a \in E$ such that $m + \varepsilon > a$

Thus $m + \varepsilon > a \geq m$.

* Completeness property for Infimum.

If $E \subseteq \mathbb{R}$ is a nonempty and bold below, then E has a finite infimum. Q1.3.6 (b).

pf:

By Thm 1 there is an $a \in E$ s.t $\sup(-E) - \varepsilon < -a \leq \sup(-E)$

Hence By Thm 5. $\inf E + \varepsilon = -(\sup(-E) - \varepsilon) > a > -\sup(-E) = \inf E$.

[illegible]

* The extended real numbers:

$\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\} = [-\infty, \infty]$.

Thus, x is an extended real number iff $x \in \mathbb{R}$, $x = \infty$ or $x = -\infty$.

* $\emptyset \neq E \subseteq \mathbb{R}$ is unbdd above if it has no upperbound and unbdd below if it has No lowerbound.

* $\emptyset \neq E \subseteq \mathbb{R}$, we define $\sup E = \infty$, if E is unbdd above and $\inf E = -\infty$ if E is unbdd below.

* we define $\sup \emptyset = -\infty$, $\inf \emptyset = \infty$.

exp: $E_1 = \underbrace{(-\infty, 2)}_{\text{bdd above}}$, $E_2 = \underbrace{(2, \infty)}_{\text{bdd below}}$.

$\sup E_1 = 2$, $\inf E_1 = -\infty$

$\sup E_2 = \infty$, $\inf E_2 = 2$.

exp:

$\sup \mathbb{Z} = \infty$, $\inf \mathbb{Z} = -\infty$

$\sup \mathbb{N} = \infty$, $\inf \mathbb{N} = -1$

$\underset{\sup}{\cancel{\inf}}\ \mathbb{R} = \infty$ $\inf \mathbb{R} = -\infty$