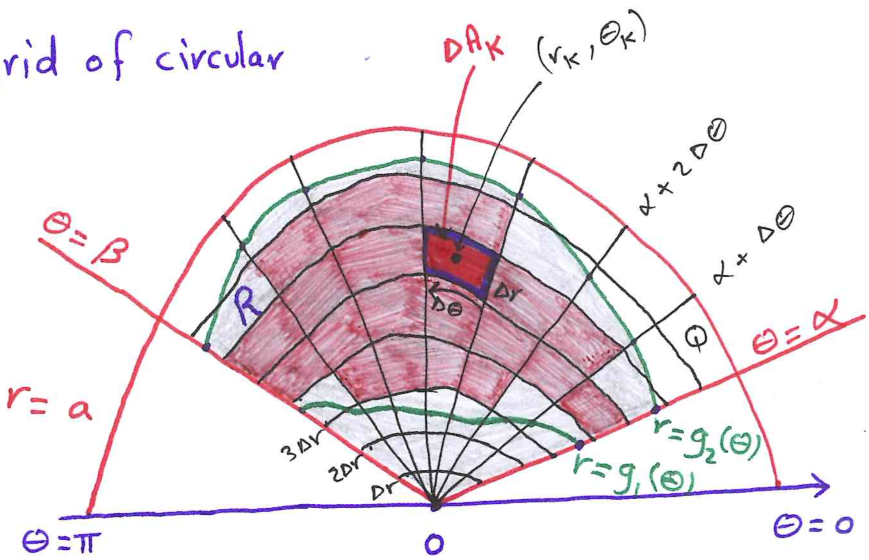
15.4 Double Integrals in Polar Form

How to construct double integral?

- Assume $f(r,\theta)$ is defined on region R: $g_1(\theta) \le r \le g_2(\theta)$, $\alpha \le \theta \le \beta$ contained in the region Q: $0 \le r \le a$, $\alpha \le \theta \le \beta$

  where $g_1(\theta)$ and $g_2(\theta)$ are continuous curves: $0 \le g_1(\theta) \le g_2(\theta) \le a$.

- We cover Q by a grid of circular arcs and rays.



- The arcs are cut from circles centered at origin with radii $\Delta r, 2\Delta r, 3\Delta r, \dots, m\Delta r$ where $\Delta r = \dfrac{a}{m}$
- The rays are: $\theta = \alpha$, $\theta = \alpha + \Delta\theta$, $\theta = \alpha + 2\Delta\theta, \dots, \theta = \beta$
- The arcs and rays partition Q into small polar rectangles.
- We number the polar rectangles that lie inside R with areas:

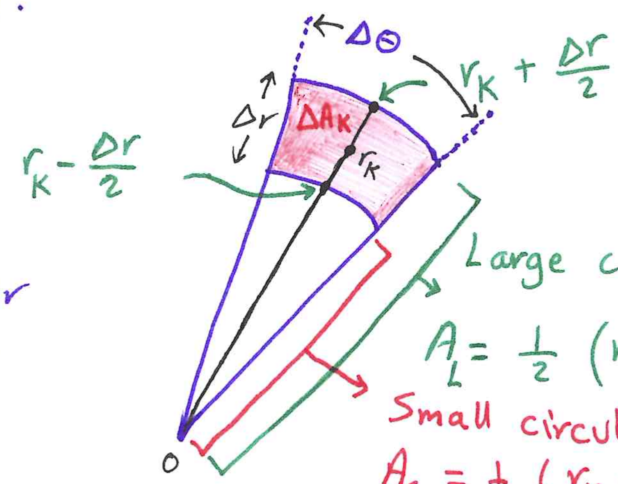
  $$\Delta A_1, \Delta A_2, \dots, \Delta A_n$$

- Let (r_K, θ_K) be any point in the K^{th} polar rectangle whose area is ΔA_K. Then $S_n = \sum_{K=1}^{n} f(r_K, \theta_K)\,\Delta A_K$

If f is continuous, then this sum will approach a limit and as $\Delta r \to 0$ and $\Delta\theta \to 0$, the limit is called the double integral of f over R: $\lim_{n\to\infty} S_n = \iint_R f(r,\theta)\,dA$.

* **To find ΔA_K:**

Recall that the area of a circular sector is $A = \frac{1}{2}\theta r^2$



Large circular sector with $A_L = \frac{1}{2}\left(r_K + \frac{\Delta r}{2}\right)^2 \Delta\theta$

Small circular sector with $A_s = \frac{1}{2}\left(r_K - \frac{\Delta r}{2}\right)^2 \Delta\theta$

- We choose r_K to be the average of the radii of the inner and outer arcs bounding the K^{th} polar rectangle.
- $\Delta A_K = A_L - A_s = \frac{\Delta\theta}{2}\left[\left(r_K + \frac{\Delta r}{2}\right)^2 - \left(r_K - \frac{\Delta r}{2}\right)^2\right] = r_K \Delta r \Delta\theta$

Hence, $S_n = \sum_{K=1}^{n} f(r_K, \theta_K)\, r_K\, \Delta r\, \Delta\theta$.

- As $n \to \infty$: $\lim_{n\to\infty} S_n = \iint_R f(r,\theta)\, r\, dr\, d\theta$

. Thus

$$\boxed{\iint_R f(r,\theta)\, dA = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=g_1(\theta)}^{r=g_2(\theta)} f(r,\theta)\, r\, dr\, d\theta} \quad *$$

- Note that if ① f in $*$ is positive then $*$ is volume.

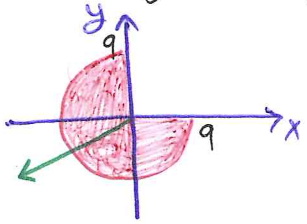
② $f = 1$, then $*$ is the area in Polar coordinate. That is

The area of a closed and bounded region R in the polar coordinate plane is

$$A = \iint_R r\, dr\, d\theta$$

Exp Describe the given region in polar coordinates:

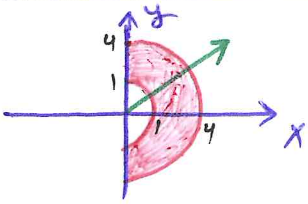
1.



$x^2+y^2=9^2 \iff r^2=9^2 \iff r=9$

$0 \le r \le 9 \quad , \quad \frac{\pi}{2} \le \theta \le 2\pi$

2.

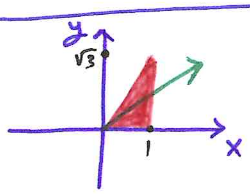


$x^2+y^2=1^2 \iff r=1$

$x^2+y^2=4^2 \iff r=4$

$1 \le r \le 4 \quad , \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

3.



$x=1 \iff r\cos\theta = 1 \iff r=\sec\theta$

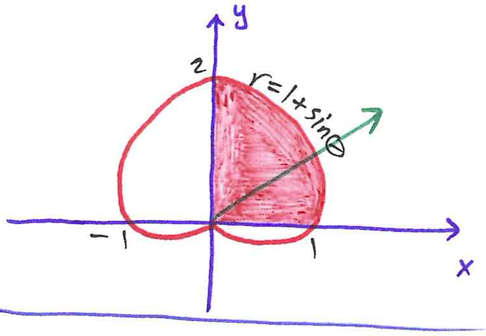
$y=\sqrt{3}\,x \iff \cancel{r}\sin\theta = \sqrt{3}\,\cancel{r}\cos\theta$

$\iff \tan\theta = \sqrt{3} \iff \theta = \frac{\pi}{3}$

$0 \le r \le \sec\theta \quad , \quad 0 \le \theta \le \frac{\pi}{3}$

Exp Find the area of the region cut from the first quadrant by the cardioid $r = 1+\sin\theta$

$$A = \int_0^{\frac{\pi}{2}} \int_0^{1+\sin\theta} r\,dr\,d\theta = \frac{3\pi}{8} + 1$$



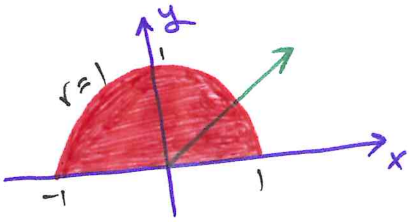
Changing Cartesian Integrals into Polar Integrals:

$$\iint_R f(x,y)\,dx\,dy = \iint_G f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

That is replace
1. x by $r\cos\theta$
2. y by $r\sin\theta$
3. $dx\,dy$ by $r\,dr\,d\theta$

Exp Find $\iint_R e^{x^2+y^2}\,dy\,dx$, where R is the semicircular region bounded by the x-axis and the curve $y=\sqrt{1-x^2}$.

$$\iint_R e^{x^2+y^2}\,dy\,dx = \int_0^{\pi}\int_0^1 e^{r^2}\, r\,dr\,d\theta = \frac{\pi(e-1)}{2}$$



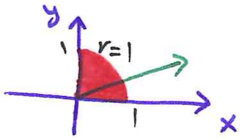
Exp Find $\int_0^1\int_0^{\sqrt{1-x^2}} (x^2+y^2)\,dy\,dx = \int_0^1 \left(x^2\sqrt{1-x^2} + \frac{(1-x^2)^{\frac{3}{2}}}{3}\right)dx$

we can integrate this by trigonometric substitution $x=\sin\theta$ ... but it will take some times. However if we change to polar:

$$\int_0^1\int_0^{\sqrt{1-x^2}} (x^2+y^2)\,dy\,dx = \int_0^{\frac{\pi}{2}}\int_0^1 r^2\, r\,dr\,d\theta$$

since $0 \le x \le 1$
$0 \le y \le \sqrt{1-x^2}$

$$= \int_0^{\frac{\pi}{2}} \frac{d\theta}{4} = \frac{\pi}{8}$$

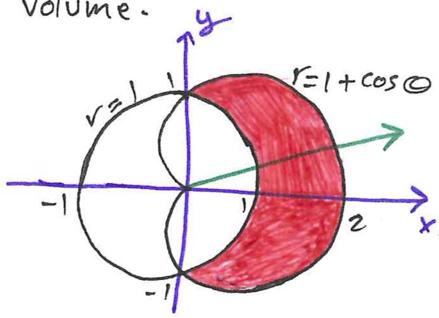


Exp The region that lies inside the cardioid $r=1+\cos\theta$ and outside the circle $r=1$ is the base of a solid right cylinder. The top of the cylinder lies in the plane $z=x$. Find the cylinder's volume.

$$V = 2\int_0^{\frac{\pi}{2}}\int_1^{1+\cos\theta} r\cos\theta\; r\,dr\,d\theta$$

$$= \frac{2}{3}\int_0^{\frac{\pi}{2}} \left[\underbrace{(1+\cos\theta)^3}_{1+3\cos\theta+3\cos^2\theta+\cos^3\theta}-1\right]\cos\theta\,d\theta$$

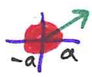
$$= \frac{2}{3}\int_0^{\frac{\pi}{2}} \left[3\cos^2\theta + 3\cos^3\theta + \cos^4\theta\right]d\theta = \frac{4}{3}+\frac{5\pi}{8}$$



The average value of f over R is $av(f) = \frac{1}{Area(R)}\iint_R f(r,\theta)\,r\,dr\,d\theta$

$Area = \iint_R r\,dr\,d\theta$

Exp Find the average height of hemispherical surface $z=\sqrt{a^2-x^2-y^2}$ above the disk $x^2+y^2\le a^2$ in the xy-plane.



$$\text{averge height} = \frac{4}{a^2\pi}\int_0^{\frac{\pi}{2}}\int_0^a r\sqrt{a^2-r^2}\,dr\,d\theta = \frac{4}{3\pi a^2}\int_0^{\frac{\pi}{2}} a^3\,d\theta = \frac{2a}{3}$$