Ch 4: Higher Order linear ODE's

4.1 The n^{th} order linear ODE has the general form:

$$y^{(n)} + p_{n-1}(t)\, y^{(n-1)} + p_{n-2}(t)\, y^{(n-2)} + \dots + P_1(t)\, y' + P_0(t)\, y = g(t) \quad (1)$$

- The gen. sol. of (1) is

$$y(t) = y_h(t) + y_p(t)$$

$$= c_1 y_1(t) + c_2 y_2(t) + \dots + c_n y_n(t) + y_p(t)$$

- To find the constants $c_1, c_2, \dots, c_n$ we need n initial conditions:

$$y(t_0) = y_0, \quad y'(t_0) = y_0', \quad \dots, \quad y^{(n-1)}(t_0) = y_0^{(n-1)} \quad (2)$$

Remark: The theory for 2^{nd} order linear ODE fits perfectly well with the n^{th} order linear ODE

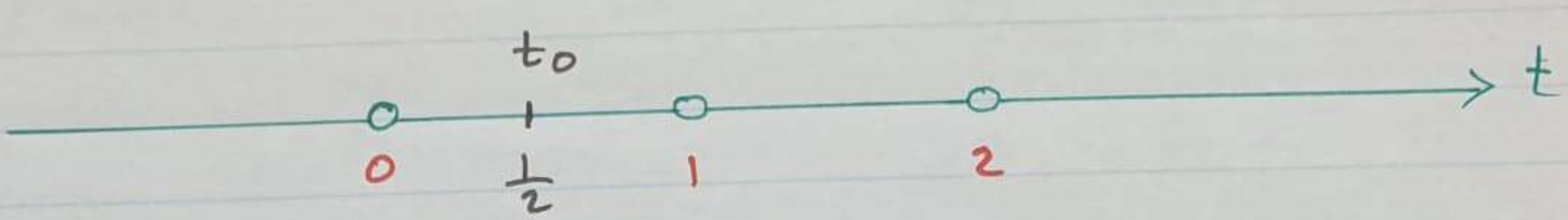
Th Assume $P_0, P_1, \dots, P_{n-1}$ are cont. on an open interval I containing t_0. Then $\exists$ a unique solution $y(t) = \phi(t)$ satisfying (1) and (2) on I.

Exp Find the largest interval in which the solution of the IVP:

$$\ln|t-1|\, y^{(4)} + (t+1)\, y^{(3)} - y = \cos t$$

$$y(\tfrac{1}{2}) = 5, \quad y'(\tfrac{1}{2}) = 3, \quad y''(\tfrac{1}{2}) = \tfrac{1}{2}, \quad y'''(\tfrac{1}{2}) = 4 \text{ is valid}$$

$$y^{(4)} + \frac{t+1}{\ln|t-1|}\,y''' - \frac{1}{\ln|t-1|}\,y = \frac{\cos t}{\ln|t-1|}$$



$P_3(t), P_2(t), P_1(t), P_0(t)$ are all cont. on $\mathbb{R} \setminus \{0,1,2\}$

Since $t_0 = \frac{1}{2} \in (0,1) \Rightarrow I = (0,1)$

Remark • If $y_1, y_2, \dots, y_n$ are solutions for the homogenuous DE :

$$y^{(n)} + P_{n-1}(t)\,y^{(n-1)} + P_{n-2}(t)\,y^{(n-2)} + \dots + P_1(t)\,y' + P_0(t)\,y = 0 \quad \text{--- ③}$$

with IC's as given in ② then the gen. sol. is

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + \dots + c_n y_n(t)$$

• To find $c_1, c_2, \dots, c_n \Rightarrow$ we use IC's from ②

$$\begin{aligned}
c_1 y_1(t_0) + c_2 y_2(t_0) + \dots + c_n y_n(t_0) &= y_0 \\
c_1 y_1'(t_0) + c_2 y_2'(t_0) + \dots + c_n y_n'(t_0) &= y_0' \\
\vdots \qquad\qquad \vdots \qquad\qquad & \vdots \\
c_1 y_1^{(n-1)}(t_0) + c_2 y_2^{(n-1)}(t_0) + \dots + c_n y_n^{(n-1)}(t_0) &= y_0^{(n-1)}
\end{aligned}$$

• For $c_1, c_2, \dots, c_n$ to make sense we must have $W(y_1, y_2, \dots, y_n)(t_0) \neq 0$, where the

$$W(y_1, y_2, \dots, y_n)(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) & \dots & y_n(t_0) \\ y_1'(t_0) & y_2'(t_0) & \dots & y_n'(t_0) \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)}(t_0) & y_2^{(n-1)}(t_0) & \dots & y_n^{(n-1)}(t_0) \end{vmatrix}$$

- If $W(y_1, y_2, \dots, y_n)(t) \neq 0$ then $y_1, y_2, \dots, y_n$ are Linearly Independent

and since they are solutions for the DE 3 $\Rightarrow$

$\{y_1, y_2, \dots, y_n\}$ form fundemental set of solutions

- If Y_1 and Y_2 are solutions for the nonhomogenuous DE 1 then $Y_1 - Y_2$ is solution for the homogenuous DE 3

Exp show that $\{1, t, t^3\}$ form fundemental set of solutions for the DE: $-t^2 y''' + t y'' = 0$, $t \neq 0$.

- First we show $1, t, t^3$ are solutions $\Rightarrow$

$y_1 = 1 \Rightarrow y_1' = y_1'' = y_1''' = 0 \Rightarrow -t^2 y_1''' + t y_1'' = 0$

$y_2 = t \Rightarrow y_2' = 1$ and $y_2'' = y_2''' = 0 \Rightarrow -t^2 y_2''' + t y_2'' = 0$

$y_3 = t^3 \Rightarrow y_3' = 3t^2$, $y_3'' = 6t$, $y_3''' = 6 \Rightarrow$

$$-t^2 y_3''' + t y'' = -t^2(6) + t(6t) = 0$$

- Now we show $1, t, t^3$ are Linearly Independent $\Rightarrow$

$$W(1, t, t^3)(t) = \begin{vmatrix} 1 & t & t^3 \\ 0 & 1 & 3t^2 \\ 0 & 0 & 6t \end{vmatrix} = 6t \neq 0 \text{ since } t \neq 0$$

Hence, y_1, y_2, y_3 are L. Indep. $\Rightarrow \{1, t, t^3\}$ form fundemental set of solutions

Note that $\begin{vmatrix} 1 & t & t^3 \\ 0 & 1 & 3t^2 \\ 0 & 0 & 6t \end{vmatrix} = (1)\begin{vmatrix} 1 & 3t^2 \\ 0 & 6t \end{vmatrix} - (t)\begin{vmatrix} 0 & 3t^2 \\ 0 & 6t \end{vmatrix} + (t^3)\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 6t$

4.2 Homogenous LDE with Constant Coefficients (for Higher order)

The solution of this kind of DE's follow similarly to the solution of 2^{nd} OLH DE's with CC.

Exp Find the gen. sol. of

① $y^{(4)} - y = 0$, $y(0) = \frac{7}{2}$, $y'(0) = -4$, $y''(0) = \frac{5}{2}$, $y'''(0) = -2$

ch. Eq $r^4 - 1 = 0$

$$(r^2 - 1)(r^2 + 1) = 0$$

$$(r-1)(r+1)(r^2+1) = 0$$

$$r_1 = 1, \; r_2 = -1, \; r_{3,4} = \pm i$$

$\lambda = 0, \mu = 1$

$y_1 = e^{x}$, $y_2 = e^{-x}$, $y_3 = \cos x$, $y_4 = \sin x$ $\Rightarrow$ gen. sol. is

$$y(x) = c_1 e^{x} + c_2 e^{-x} + c_3 \cos x + c_4 \sin x$$

To find c_1, c_2, c_3, c_4 we use IC's

$$y'(x) = c_1 e^{x} - c_2 e^{-x} - c_3 \sin x + c_4 \cos x$$

$$y''(x) = c_1 e^{x} + c_2 e^{-x} - c_3 \cos x - c_4 \sin x$$

$$y'''(x) = c_1 e^{x} - c_2 e^{-x} + c_3 \sin x - c_4 \cos x$$

$$\left.\begin{aligned} \tfrac{7}{2} &= c_1 + c_2 + c_3 \\ -4 &= c_1 - c_2 + c_4 \\ \tfrac{5}{2} &= c_1 + c_2 - c_3 \\ -2 &= c_1 - c_2 - c_4 \end{aligned}\right\} \Rightarrow \quad c_1 = 0, \; c_2 = 3, \quad c_3 = \tfrac{1}{2}, \; c_4 = -1$$

Hence, the gen. sol. becomes

$$y(x) = 3e^{-x} + \frac{1}{2}\cos x - \sin x$$

② $y^{(4)} + y''' - 7y'' - y' + 6y = 0$

Ch. Eq. $r^4 + r^3 - 7r^2 - r + 6 = 0$

$\pm 1, \pm 2$
$\pm 3, \pm 6$

$(r^2 - 1)(r^2 + r - 6) = 0$

- $r_1 = 1$ is root $\Rightarrow$ $r - 1$ is factor
- $r_2 = -1$ is root $\Rightarrow$ $r_2 + 1$ is factor
- Hence, $r^2 - 1$ is factor

$(r-1)(r+1)(r-2)(r+3) = 0$

$$
\begin{array}{r}
r^2 + r - 6 \\
r^2 - 1 \,\overline{\big)\, \cancel{r^4} + r^3 - 7r^2 - r + 6} \\
-\cancel{r^4} + r^2 \\
\hline
\cancel{r^3} - 6r^2 - r + 6 \\
-\cancel{r^3} + r \\
\hline
-6r^2 + 6 \\
+6r^2 \mp 6 \\
\hline
0
\end{array}
$$

$r_1 = 1 \Rightarrow y_1 = e^{x}$
$r_2 = -1 \Rightarrow y_2 = e^{-x}$
$r_3 = 2 \Rightarrow y_3 = e^{2x}$
$r_4 = -3 \Rightarrow y_4 = e^{-3x}$

Hence, the gen. sol. is

$$y(x) = c_1 e^{x} + c_2 e^{-x} + c_3 e^{2x} + c_4 e^{-3x}$$

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③ $y^{(iv)} + 2y'' + y = 0$

Ch. Eq. $r^4 + 2r^2 + 1 = 0$

$(r^2 + 1)(r^2 + 1) = 0$

$r_{1,2} = \pm i \;,\; r_{3,4} = \pm i$ $\qquad \lambda = 0,\ \mu = 1$

$y_1 = e^{\lambda x}\cos \mu x = \cos x$
$y_2 = e^{\lambda x}\sin \mu x = \sin x$

$y_3 = x \cos x$
$y_4 = x \sin x$

Hence, the gen. sol. is

$$y(x) = c_1 \cos x + c_2 \sin x + c_3 x \cos x + c_4 x \sin x$$

④ $y^{(4)} + 2y''' - 13y'' - 14y' + 24y = 0$

$y(0) = 1,\ y'(0) = -1,\ y''(0) = 0,\ y'''(0) = -1$

Ch. Eq $r^4 + 2r^3 - 13r^2 - 14r + 24 = 0$

$r_1 = 1$ is root $\Rightarrow r-1$ is factor
$r_2 = -2$ is root $\Rightarrow r+2$ is factor
$\Rightarrow (r-1)(r+2)$ is factor
$r^2 + r - 2$

$$\begin{array}{r} r^2 + r - 12 \\ r^2+r-2 \overline{\big) r^4 + 2r^3 - 13r^2 - 14r + 24} \\ -r^4 \mp r^3 \pm 2r^2 \\ \hline r^3 - 11r^2 - 14r + 24 \\ -r^3 \mp r^2 \pm 2r \\ \hline -12r^2 - 12r + 24 \\ +12r^2 + 12r \mp 24 \\ \hline 0 \end{array}$$

$(r^2 + r - 2)(r^2 + r - 12) = 0$
$(r-1)(r+2)(r+4)(r-3) = 0$

$r_1 = 1 \Rightarrow y_1 = e^{t}$
$r_2 = -2 \Rightarrow y_2 = e^{-2t}$
$r_3 = -4 \Rightarrow y_3 = e^{-4t}$
$r_4 = 3 \Rightarrow y_4 = e^{3t}$

gen. sol. $\Rightarrow y(t) = c_1 e^{t} + c_2 e^{-2t} + c_3 e^{-4t} + c_4 e^{3t}$

$y'(t) = c_1 e^{t} - 2c_2 e^{-2t} - 4c_3 e^{-4t} + 3c_4 e^{3t}$
$y''(t) = c_1 e^{t} + 4c_2 e^{-2t} + 16c_3 e^{-4t} + 9c_4 e^{3t}$
$y'''(t) = c_1 e^{t} - 8c_2 e^{-2t} - 48c_3 e^{-4t} + 27c_4 e^{3t}$

$$\left.\begin{aligned} 1 &= c_1 + c_2 + c_3 + c_4 \\ -1 &= c_1 - 2c_2 - 4c_3 + 3c_4 \\ 0 &= c_1 + 4c_2 + 16c_3 + 9c_4 \\ -1 &= c_1 - 8c_2 - 48c_3 + 27c_4 \end{aligned}\right\} \Rightarrow \begin{aligned} c_1 &\approx 0.4 \\ c_2 &\approx 0.9 \\ c_3 &\approx -0.2 \\ c_4 &\approx -0.1 \end{aligned}$$

$y(t) = 0.4e^{t} + 0.9e^{-2t} - 0.2e^{-4t} - 0.1e^{3t}$

⑤ $y^{(4)} + y'' = 0$

Ch. Eq. $r^4 + r^2 = 0 \Rightarrow r^2(r^2+1) = 0 \Rightarrow r_1 = r_2 = 0$

$r_{3,4} = \pm i$

$\lambda = 0,\ \mu = 1$

$y_1 = e^{rt} = e^{0t} = 1$
$y_2 = t\,y_1 = t(1) = t$
$y_3 = e^{\lambda t}\cos\mu t = \cos t$
$y_4 = e^{\lambda t}\sin\mu t = \sin t$

$\Rightarrow$ gen. sol. is

$y(t) = c_1 + c_2 t + c_3\cos t + c_4 \sin t$

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⑥ $y^{(iv)} - y''' - y'' + y' = 0$

Ch. Eq. $r^4 - r^3 - r^2 + r = 0$

$r(r^3 - r^2 - r + 1) = 0$

$r[r^2(r-1) - (r-1)] = 0$

$r(r-1)(r^2-1) = 0$

$r(r-1)(r-1)(r+1) = 0$

$r_1 = 0 \Rightarrow y_1 = 1$

$r_2 = r_3 = 1 \Rightarrow y_2 = e^t$

$\Rightarrow y_3 = t e^t$

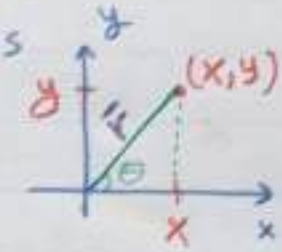
$r_4 = -1 \Rightarrow y_4 = e^{-t}$

gen. sol. $\Rightarrow y(t) = c_1 + c_2 e^t + c_3 t e^t + c_4 e^{-t}$

Exp Express the following complex numbers in the form Euler formula $e^{i\theta} = \cos\theta + i\sin\theta$

① $-1+\sqrt{3}\,i$

Recall that the length of the complex number $z = x+iy$ is $\bar{r} = |z| = \sqrt{x^2+y^2}$



$$\bar{r} = \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2$$

$x = -1 \quad , \quad y = \sqrt{3}$

$x = \bar{r}\cos\theta$ and $y = \bar{r}\sin\theta$

$-1 = 2\cos\theta$ and $\sqrt{3} = 2\sin\theta$

$-\frac{1}{2} = \cos\theta$ and $\frac{\sqrt{3}}{2} = \sin\theta$

$$\theta = \frac{2\pi}{3} + 2\pi m$$

$m = 0, \pm1, \pm2, \dots$

Note that any complex number

$$x+iy = \bar{r}\cos\theta + i\,\bar{r}\sin\theta = \bar{r}(\cos\theta + i\sin\theta) = \bar{r}\,e^{i\theta}$$

Hence, $-1+\sqrt{3}\,i = \bar{r}\,e^{i\theta} = 2\,e^{i\left(\frac{2\pi}{3}+2\pi m\right)}$

$$= 2\left[\cos\left(\frac{2\pi}{3}+2\pi m\right) + i\sin\left(\frac{2\pi}{3}+2\pi m\right)\right]$$

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② $-3 \Rightarrow -3 = -3+0i \Rightarrow x = -3 \ , \ y = 0$

$\bar{r} = \sqrt{9+0} = \sqrt{9} = 3$

$x = \bar{r}\cos\theta$	$y = \bar{r}\sin\theta$
$-3 = 3\cos\theta$	$0 = \sin\theta$
$-1 = \cos\theta$	$0 = \sin\theta$

$$\theta = \pi + 2\pi m$$

$m = 0, \pm1, \pm2, \dots$

$$-3 = \bar{r}\,e^{i\theta} = 3\,e^{i(\pi+2\pi m)}$$

$$= 3\left[\cos(\pi+2\pi m) + i\sin(\pi+2\pi m)\right]$$

Exp Show that $(1+i)^8 = 16$

Exp Solve the DE: $y^{(4)} + y = 0$

ch. Eq. $r^4 + 1 = 0 \Rightarrow r^4 = -1 \Rightarrow r = (-1)^{\frac{1}{4}}$
$= (-1+0i)^{\frac{1}{4}}$

$-1 + 0i = \bar{r}\, e^{i\theta}$
$= e^{i(\pi + 2\pi m)}$

$x = -1$ and $y = 0$
$\bar{r} = \sqrt{1+0} = 1$
$x = \bar{r}\cos\theta$ and $y = \bar{r}\sin\theta$
$-1 = \cos\theta$ and $0 = \sin\theta$
$\theta = \pi + 2\pi m$
$m = 0, \pm 1, \pm 2, \dots$

Hence, $r = (-1+0i)^{\frac{1}{4}}$
$= \left[e^{i(\pi + 2\pi m)}\right]^{\frac{1}{4}}$
$= e^{i\left(\frac{\pi}{4} + \frac{\pi m}{2}\right)}$
$= \cos\left(\frac{\pi}{4} + \frac{\pi m}{2}\right) + i \sin\left(\frac{\pi}{4} + \frac{\pi m}{2}\right)$

The four roots are r_1 when $m=0$, r_2 when $m=1$, r_3 when $m=2$, r_4 $m=3$

$m=0 \Rightarrow r_1 = e^{i\frac{\pi}{4}} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$

$m=1 \Rightarrow r_2 = e^{i\left(\frac{\pi}{4}+\frac{\pi}{2}\right)} = \cos\left(\frac{\pi}{4}+\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{4}+\frac{\pi}{2}\right) = \frac{-1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$

$m=2 \Rightarrow r_3 = e^{i\left(\frac{\pi}{4}+\pi\right)} = \cos\left(\frac{\pi}{4}+\pi\right) + i\sin\left(\frac{\pi}{4}+\pi\right) = \frac{-1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$

$m=3 \Rightarrow r_4 = e^{i\left(\frac{\pi}{4}+\frac{3\pi}{2}\right)} = \cos\left(\frac{\pi}{4}+\frac{3\pi}{2}\right) + i\sin\left(\frac{\pi}{4}+\frac{3\pi}{2}\right) = \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$

The four roots are: $r_{1,4} = \underbrace{\frac{1}{\sqrt{2}}}_{\lambda_1} \pm \underbrace{\frac{1}{\sqrt{2}}}_{\mu_1} i$ and $r_{2,3} = \underbrace{\frac{-1}{\sqrt{2}}}_{\lambda_2} \pm \underbrace{\frac{1}{\sqrt{2}}}_{\mu_2} i$

The gen. sol. is

$$y(t) = c_1 e^{\frac{1}{\sqrt{2}}t}\cos\left(\tfrac{1}{\sqrt{2}}t\right) + c_2 e^{\frac{1}{\sqrt{2}}t}\sin\left(\tfrac{1}{\sqrt{2}}t\right) + c_3 e^{-\frac{1}{\sqrt{2}}t}\cos\left(\tfrac{1}{\sqrt{2}}t\right) + c_4 e^{-\frac{1}{\sqrt{2}}t}\sin\left(\tfrac{1}{\sqrt{2}}t\right)$$

Exp solve the DE: $\dddot{y} + 8y = 0$

solution1 Ch.Eq. $r^3 + 8 = 0$

$r = -2$ is root $\Rightarrow r+2$ is factor

$r^3 + 8 = 0$

$(r+2)(r^2 - 2r + 4) = 0$

$$\begin{array}{r|l} & r^2 - 2r + 4 \\ \hline r+2 & r^3 + 8 \\ & -r^3 \mp 2r^2 \\ \hline & -2r^2 + 8 \\ & +2r^2 + 4r \\ \hline & 4r + 8 \\ & -4r \mp 8 \\ \hline & 0 \end{array}$$

$r_1 = -2$ and $r_{2,3} = \dfrac{2 \pm \sqrt{4-16}}{2}$

$= \dfrac{2 \pm \sqrt{12}\, i}{2}$

$= 1 \pm \sqrt{3}\, i$

$y_1 = e^{-2t}$ and $y_2 = e^{t} \cos\sqrt{3}t$ and $y_3 = e^{t} \sin\sqrt{3}t$

gen. sol. $y(t) = c_1 e^{-2t} + c_2 e^{t} \cos\sqrt{3}t + c_3 e^{t} \sin\sqrt{3}t$

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solution2 $r^3 + 8 = 0 \Rightarrow r^3 = -8 \Rightarrow r = (-8)^{\frac{1}{3}} = (-8+0i)^{\frac{1}{3}}$

$r = (-8+0i)^{\frac{1}{3}} = \left(\bar{r}\, e^{i\theta}\right)^{\frac{1}{3}}$

$= \left[8\, e^{i(\pi + 2\pi m)}\right]^{\frac{1}{3}}$

$= 2\, e^{i\left(\frac{\pi}{3} + \frac{2\pi m}{3}\right)}$

$= 2\left[\cos\left(\frac{\pi}{3} + \frac{2\pi}{3}m\right) + i \sin\left(\frac{\pi}{3} + \frac{3\pi}{3}m\right)\right]$

$x = -8$ and $y = 0$

$\bar{r} = \sqrt{64+0} = 8$

$x = \bar{r}\cos\theta$ and $y = \bar{r}\sin\theta$

$-1 = \cos\theta$ and $0 = \sin\theta$

$\theta = \pi + 2\pi m$

$m = 0, \pm1, \pm2, \ldots$

when $m=0 \Rightarrow r_2 = 2\left[\cos\frac{\pi}{3} + i \sin\frac{\pi}{3}\right] = 2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = 1 + i\sqrt{3}$

$m=1 \Rightarrow r_1 = 2\left[\cos\left(\frac{\pi}{3} + \frac{2\pi}{3}\right) + i\sin \pi\right] = 2(-1 + 0i) = -2$

$m=2 \Rightarrow r_3 = 2\left[\cos\left(\frac{\pi}{3} + \frac{4\pi}{3}\right) + i\sin\left(\frac{\pi}{3} + \frac{4\pi}{3}\right)\right] = 2\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = 1 - i\sqrt{3}$

gen. sol. $\Rightarrow y(t) = c_1 e^{-2t} + c_2 e^{t}\cos\sqrt{3}t + c_3 e^{t}\sin\sqrt{3}t$

 The Method of Undetermined Coefficients

To solve nonhomogenous linear higher order ODE's

Exp Solve the following DE's:

① $y''' - 3y'' + 3y' - y = 4e^t$

gen. sol. is $y(t) = y_h(t) + y_p(t)$

$y_h(t) \Rightarrow$ Ch. Eq. $\Rightarrow r^3 - 3r^2 + 3r - 1 = 0$

$(r-1)^3 = 0$

$r_1 = r_2 = r_3 = 1$

$y_1 = e^t, \; y_2 = te^t \; y_3 = t^2e^t$

$y_h(t) = c_1 e^t + c_2 t e^t + c_3 t^2 e^t$

$y_p(t) = A e^t t^3$ R* ✓ To find A $\Rightarrow$

$y_p' = At^3e^t + 3At^2e^t$

$y_p'' = At^3e^t + 3t^2Ae^t + 3At^2e^t + 6Ate^t$

$= e^t(At^3 + 6At^2 + 6At)$

$y_p''' = e^t(3At^2 + 12At + 6A) + (At^3 + 6At^2 + 6At)e^t$

Substitute y_p, y_p', y_p'', y_p''' in the nonhomogenous DE above to find $A = \frac{2}{3}$ $\Rightarrow$ $y_p(t) = \frac{2}{3}t^3e^t$

Hence, the gen. sol. becomes

$y(t) = y_h(t) + y_p(t)$

$= c_1e^t + c_2te^t + c_3t^2e^t + \frac{2}{3}t^3e^t$

(2) $y''' + y'' = 10x^2$, $y(0) = y'(0) = y''(0) = 1$

gen. sol. is $y(x) = y_h(x) + y_p(x)$

$y_h(x) \Rightarrow$ Ch. Eq. $r^3 + r^2 = 0$ $\Rightarrow r^2(r+1) = 0$

$\Rightarrow r_1 = r_2 = 0$, $r_3 = -1$

$y_1 = 1$, $y_2 = x$, $y_3 = e^{-x}$

$y_h(x) = c_1 + c_2 x + c_3 e^{-x}$

$y_p(x) = (Ax^2 + Bx + C)x^2$ (R*) ✓

$= Ax^4 + Bx^3 + Cx^2$

substitute y_p'' and y_p''' in the nonhomogenous DE to find

$A = \frac{5}{6}$, $B = -\frac{10}{3}$, $C = 10$

Hence, the gen. sol. becomes:

$y(x) = y_h(x) + y_p(x)$

$y(x) = c_1 + c_2 x + c_3 e^{-x} + \frac{5}{6}x^4 - \frac{10}{3}x^3 + 10x^2$

Using the IC's $\Rightarrow$ we find $c_1 = 20$

$c_2 = -18$

$c_3 = -19$

$y(x) = 20 - 18x - 19e^{-x} + \frac{5}{6}x^4 - \frac{10}{3}x^3 + 10x^2$

③ $y^{(4)} + 8y'' + 16y = 2\sin t - 3\cos t$

gen. sol. is $y(t) = y_h(t) + y_p(t)$

$y_h(t) \Rightarrow$ ch. Eq. $r^4 + 8r^2 + 16 = 0$

$(r^2+4)(r^2+4) = 0$

$r_{1,2} = \pm 2i \quad , \quad r_{3,4} = \pm 2i \qquad \lambda = 0, \mu = 2$

$y_1 = \cos 2t \; , \; y_2 = \sin 2t \; , \; y_3 = t\cos 2t \; , \; y_4 = t\sin 2t$

$y_h(t) = c_1\cos 2t + c_2\sin 2t + c_3 t\cos 2t + c_4 t\sin 2t$

$y_p(t) = A\sin t + B\cos t$ (R*)

substitute $y_p, y_p'', y_p^{(4)}$ in the nonhomogenous DE above

to find $A = \frac{2}{9}$ and $B = -\frac{1}{3}$

Hence, $y(t) = y_h(t) + y_p(t)$

$$y(t) = c_1\cos 2t + c_2\sin 2t + c_3 t\cos 2t + c_4 t\sin 2t + \frac{2}{9}\sin t - \frac{1}{3}\cos t$$

---

④ $y^{(4)} + 8y'' + 16y = 2\sin 2t - 3\cos 2t$

$y_h(t)$ is as above but $y_p(t) = (A\cos 2t + B\sin 2t)t^2$ (R*)

substitute $y_p, y_p'', y_p^{(4)}$ above to find $A = -\frac{1}{16}$, $B = \frac{3}{32}$

$$y(t) = y_h(t) + y_p(t) = c_1\cos 2t + c_2\sin 2t + c_3 t\cos 2t + c_4 t\sin 2t - \frac{1}{16}t^2\cos 2t + \frac{3}{32}t^2\sin 2t$$

Exp Find $y_p(h)$ for the DE

$$y''' - 4y' = h + 3\cos h + e^{-2h}$$

$y_h(h) \Rightarrow$ Ch. Eq. $r^3 - 4r = 0 \Rightarrow r(r^2-4) = 0$

$r_1 = 0,\ r_2 = 2,\ r_3 = -2$

$$y_1 = 1,\quad y_2 = e^{2h},\quad y_3 = e^{-2h}$$

$$y_h(h) = c_1 + c_2 e^{2h} + c_3 e^{-2h}$$

$y_p(h) = y_{p_1}(h) + y_{p_2}(h) + y_{p_3}(h)$

$y_p(h) = Ah^2 + Bh + C\cos h + D\sin h + Ehe^{-2h}$

To find $A, B, C, D, E \Rightarrow$

$y_{p_1}(h) = (Ah + B)h$

$y_{p_2}(h) = C\cos h + D\sin h$

$y_{p_3}(h) = E e^{-2h} h$

$$y_p' = 2Ah + B - C\sin h + D\cos h - 2Ehe^{-2h} + Ee^{-2h}$$
$$y_p'' = 2A - C\cos h - D\sin h + 4Ehe^{-2h} - 2Ee^{-2h} - 2Ee^{-2h}$$
$$y_p''' = C\sin h - D\cos h - 8Ehe^{-2h} + 4Ee^{-2h} + 8Ee^{-2h}$$

$$y_p''' - 4y_p' = h + 3\cos h + e^{-2h}$$

$$C\sin h - D\cos h - 8Ehe^{-2h} + 12Ee^{-2h} - 4(2Ah + B - C\sin h + D\cos h - 2Ehe^{-2h} + Ee^{-2h}) = h + 3\cos h + e^{-2h}$$

$-8A = 1 \Rightarrow A = -\frac{1}{8}$

$-4B = 0 \Rightarrow B = 0$

$C + 4C = 0 \Rightarrow C = 0$

$-D - 4D = 3 \Rightarrow D = -\frac{3}{5}$

$-8E + 8E = 0 \Rightarrow 0 = 0$ ✓

$12E - 4E = 1 \Rightarrow E = \frac{1}{8}$

$$\Rightarrow y_p(h) = -\frac{1}{8}h^2 - \frac{3}{5}\sin h + \frac{1}{8}he^{-2h}$$

Exp Find the particular solution $y_p(t)$ for the following DE (Don't Evaluate Coefficients)

$$y^{(iv)} + 2y''' + 2y'' = 3e^{x} + 2xe^{-x} + e^{-x}\sin x$$

$y_h(x) \Rightarrow$ Ch. Eq. $r^4 + 2r^3 + 2r^2 = 0$

$$r^2(r^2+2r+2) = 0$$

$$r_1 = r_2 = 0 \quad \text{and} \quad r_{3,4} = \frac{-2 \pm \sqrt{4-8}}{2} = \frac{-2 \pm \sqrt{4}\, i}{2}$$

$$= -1 \pm i$$

$\lambda = -1$

$\mu = 1$

$$y_1 = 1,\quad y_2 = x,\quad y_3 = e^{-x}\cos x,\quad y_4 = e^{-x}\sin x$$

$$y_h(x) = c_1 + c_2 x + c_3 e^{-x}\cos x + c_4 e^{-x}\sin x$$

$$y_p(x) = y_{p_1}(x) + y_{p_2}(x) + y_{p_3}(x)$$

R*

$$y_{p_1}(x) = Ae^{x}$$

$$y_{p_2}(x) = (Bx + C)e^{-x}$$

$$y_{p_3}(x) = (D\cos x + E\sin x)e^{-x}\, x$$

$$y_p(x) = Ae^{x} + (Bx+C)e^{-x} + xe^{-x}(D\cos x + E\sin x)$$