Exercises :

4.1.0 : $f, g : [a,b] \to R$, True or False .

a. If $f = g^2$ and f is differentiable on $[a,b]$, then g is differntiable on (a,b). False

$g(x) = |x|$ is Not differ. at $x = 0$ But $g^2(x) = x^2$ is ✓

b. If f is diffble. on $[a,b]$ then f is uniformly continuous on $[a,b]$. True.

If f is diffble , then f is continuous on $[a,b]$

since $[a,b]$ is a closed , bounded interval so By Thm 3.4.1

f is uniformly continuous on $[a,b]$ ✓

c. If f is diffble on (a,b) and $f(a)=f(b)=0$ then f is uniformly cont. on $[a,b]$.

False, let $f(x)=\frac{1}{x-1}$ for $x\neq 0$ and $f(0)=0$

Then, f is diffble on $(0,1)$ and $f(0)=f(1)=0$

But is Not even continuous on $[0,1]$.

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d. If f is diffble on $(a,b]$ and $\frac{f(x)}{x-a}\to 1$ as $x\to a^+$, then f is uniformly continuous on $(a,b]$. True.

$$\lim_{x\to a^+} f(x) = \lim_{x\to a^+}\left[\frac{f(x)}{x-a}(x-a)\right] = \lim_{x\to a^+}\frac{f(x)}{x-a}\lim_{x\to a^+}(x-a)$$

$$= 1\cdot 0 = 0 \text{ exists.}$$

$\therefore f$ is continuously extendeds from $(a,b]$ to $[a,b]$

$\therefore f$ is uniformly continuous on $[a,b]$ By (Thm).

4.1.1: use def'n to prove that $f'(a)$ exists: $f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$

a. $f(x)=x^2+x$, $a\in\mathbb{R}$.

$$f'(a)=\lim_{h\to 0}\frac{(a+h)^2+a+h-a^2-a}{h}$$

$(a+h)^2+(a+h)$

$(\cancel{a^2}+2ah+h^2+\cancel{a}+h)-\cancel{a^2}-\cancel{a}$

$$=\lim_{h\to 0}\frac{2ah+h^2+h}{h}=\lim_{h\to 0}\frac{\cancel{h}(2a+h+1)}{\cancel{h}}$$

$$=\lim_{h\to 0} 2a+h+1$$

$$=2a+1$$

b. $f(x) = \sqrt{x}$, $a > 0$.

$$\bar{f}(a) = \lim_{h\to 0} \frac{(\sqrt{a+h} - \sqrt{a})}{h} \left(\frac{\sqrt{a+h} + \sqrt{a}}{\sqrt{a+h} + \sqrt{a}} \right)$$

$$= \lim_{h\to 0} \frac{\cancel{a} + \cancel{h} - \cancel{a}}{\cancel{h}(\sqrt{a+h} + \sqrt{a})} = \lim_{h\to 0} \frac{1}{\sqrt{a+h} + \sqrt{a}} = \frac{1}{2\sqrt{a}},$$

c. $f(x) = \frac{1}{x}$, $a \neq 0$.

$$\frac{1}{a+h} - \frac{1}{a} = \frac{a - (a+h)}{a(a+h)} = \frac{\cancel{a} - \cancel{a} - h}{a(a+h)} = \frac{-h}{a(a+h)}.$$

$$\bar{f}(a) = \lim_{h\to 0} \frac{\frac{1}{a+h} - \frac{1}{a}}{h} = \lim_{h\to 0} \frac{\frac{-h}{a(a+h)}}{\cancel{h}}$$

$$= \lim_{h\to 0} \frac{-1}{a(a+h)} = -\frac{1}{a^2}.$$

4.1.2 : a. prove that $(x^n)' = n x^{n-1}$ for every $n \in \mathbb{N}$ and every $x \in \mathbb{R}$.

let $n \in N$, If $n = 1$ then $\bar{f}(x_0) = 1$ for all $x_0 \in \mathbb{R}$.

If $n > 1$ and $x_0 \neq x$, since $\frac{f(x) - f(x_0)}{x - x_0} = x^{n-1} + \dots + x_0^{n-1} = n x_0^{n-1}$.

it is clear that $\bar{f}(x_0) = n x_0^{n-1}$.

b. prove that $(x^n)' = n x^{n-1}$ for every $n \in -N \cup \{0\}$ and every $x \in (0, \infty)$.

Its clear for $n = 0$.

let $n \in -N$ and $x_0 > 0$, we have $\frac{x^n - x_0^n}{x - x_0} = \frac{x_0^{-n} - x^{-n}}{x - x_0} \cdot x^n x_0^n \to n x_0^{-n-1} \cdot x^{2n} = n x_0^{n-1}$

Thus, the function x^n is diffble at x_0 and $\bar{f}(x_0) = n x_0^{n-1}$.

4.1.5 :

a. Find all points (a,b) on the curve C, given by $y = x + \sin x$ so that the tangent lines to C at (a,b) are parallel to the line $y = x + 15$.

$1 = \dot{y} = 1 + \cos x \longrightarrow$ implies $\cos x = 0$

i.e, $x = \dfrac{(2K+1)\pi}{2}$ for $K \in Z$,

Thus, the points are $(a,b) = \left[\dfrac{(2K+1)\pi}{2}, (-1)^K + \dfrac{(2K+1)\pi}{2}\right]$ for $K \in Z$.

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b. Find all points (a,b) on the curve C, given by $y = 3x^2 + 2$ so that the tangent lines to C at (a,b) pass through the point $(-1,-7)$.

The tangent line at (a,b) is $y = b + 6a(x-a)$.

If it passes through $(-1,-7)$, then $3a^2 + 6a - 9 = 0$

i.e $a = 1, -3$. Thus the points are $(1,5), (-3,29)$.

4.1.6: $f(x) = \begin{cases} x^3, & x \geq 0 \\ 0, & x < 0 \end{cases}$ Find $n \in \mathbb{N}$ s.t $f^{(n)}$ exists on all on $\mathbb{R}$

$\rightarrow$ For $x<0$ we have $f^{(n)}(x) = 0$ for all $n \in N$.

$\rightarrow$ For $x>0$ we have $f(x) = x^3$ on $[0,\infty)$.

$\bar{f}(0) = \lim\limits_{h\to 0} \dfrac{f(h)-f(0)}{h} = \lim\limits_{h\to 0} h^2 = 0$ so $\bar{f}(0) = 0$ exists, $\bar{f}(x) = 3x^2$.

$\bar{\bar{f}}(0) = \lim\limits_{h\to 0} \dfrac{f(h)-f(0)}{h} = \lim \dfrac{3h^2 - 0}{h} = \lim\limits_{h\to 0} 3h = 0$ so $\bar{\bar{f}}(0) = 0$ exists. $\bar{\bar{f}}(x) = 6x$

$\bar{\bar{\bar{f}}}(0) = \lim\limits_{h\to 0} \dfrac{f(h)-f(0)}{h} = \lim\limits_{h\to 0} \dfrac{6h - 0}{h} = 6 \neq 0$ so $\bar{\bar{\bar{f}}}(0)$ DNE.

So $n = 1, 2$ It won't work for $n \geq 4$ either $\bar{\bar{\bar{f}}}$ is not defined at $x = 0$

so No higher derivative exists By def.

4.1.7 : suppose that $f:(0,\infty)\to\mathbb{R}$ satisfies $f(x)-f(y)=f\left(\frac{x}{y}\right)$ for $x,y\in(0,\infty)$ and $f(1)=0$.

a. prove that f is cont. on $(0,\infty)$ iff f is cont. at 1.

let $y_n\to x_0\in(0,\infty)$

If f is cont. at $x=1$ then $|f(x_0)-f(y_n)|=\left|f\left(\frac{x_0}{y_n}\right)\right|\to|f(1)|=0$

as $n\to\infty$, i.e f is cont. at x_0.

b+c : prove that f is diffble on $(0,\infty)$ iff f is diffble at 1.

c. prove that if f is diffble at 1, then $\bar{f}(x)=\frac{\bar{f}(1)}{x}$ for all $x\in(0,\infty)$.

If f is diffble at $x=1$ then for any $x\in(0,\infty)$

$$\frac{f(x+h)-f(x)}{h}=\frac{f\left(\frac{x+h}{x}\right)}{h}=\frac{1}{x}\left(\frac{f\left(1+\frac{h}{x}\right)}{\frac{h}{x}}\right)\to\frac{\bar{f}(1)}{x}\quad\text{as } h\to 0$$

Thus, $\bar{f}(x)$ exists.