2.4: Cauchy sequences

Df 1: A sequence of points $x_n \in \mathbb{R}$ is said to be cauchy (in $\mathbb{R}$) iff

$\forall \varepsilon > 0$ $\exists$ an $N \in \mathbb{N}$ s.t $n, m \geq N \Rightarrow |x_n - x_m| < \varepsilon$.

RMK 1: If $\{x_n\}$ is convergent, then $\{x_n\}$ is cauchy.

pf: spse that $x_n \to a$ as $n \to \infty$. Then

$\forall \varepsilon > 0$ $\exists$ an $N \in \mathbb{N}$ s.t $|x_n - a| < \frac{\varepsilon}{2}$, $\forall n \geq N$ By Def.

Hence, if $n, m \geq N$, then
$$|x_n - x_m| = |x_n - a + a - x_m|$$
$$\leq |x_n - a| + |x_m - a|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$
$$< \varepsilon$$

The following result shows that the converse of the above remark is also true (for real sequences).

Thm 1: cauchy:

let $\{x_n\}$ be a sequence of real numbers. Then $\{x_n\}$ is cauchy iff $\{x_n\}$ converges (to some point a in $\mathbb{R}$).

→

Thm 1 proof : By RMK1, we need only show every cauchy sequence converges.

spse that $\{x_n\}$ is cauchy.

Given $\varepsilon = 1$, $\exists N \in \mathbb{N}$ s.t $|x_N - x_m| < 1$ for all $m \geq N$.

By triangle inequality

$$|x_m| = |x_m - x_N + x_N|$$
$$\leq |x_N - x_m| + |x_N|$$
$$< 1 + |x_N| \quad \text{for } m \geq N.$$

Also, $|x_m| \leq \max\{|x_1|, |x_2|, \dots, |x_{N-1}|\} := M$ for $m = 1, 2, \dots N-1$.

Therefor, $|x_n| \leq \max\{M, 1 + |x_N|\}$, $\forall n \in \mathbb{N}$.

This means $\{x_n\}$ is bounded.

By the Bolzano-weierstrass theorem, $\{x_n\}$ has a convergent subsequence say $x_{n_k} \to a$ as $k \to \infty$.

let $\varepsilon > 0$, since $\{x_n\}$ is cauchy, $\exists N_1 \in \mathbb{N}$ s.t $n, m \geq N_1 \Rightarrow |x_n - x_m| < \frac{\varepsilon}{2}$.

Since $x_{n_k} \to a$ as $k \to \infty$, $\exists N_2 \in \mathbb{N}$ s.t $k \geq N_2 \Rightarrow |x_{n_k} - a| < \frac{\varepsilon}{2}$.

Fix $k \geq N_2$ s.t $n_k \geq N_1$ Then

$$|x_n - a| = |x_n - x_{n_k} + x_{n_k} - a|$$
$$\leq |x_n - x_{n_k}| + |x_{n_k} - a|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \quad \forall n \geq N_1$$

$\therefore |x_n - a| < \varepsilon$, $\forall n \geq N_1$

Thus, $x_n \to a$ as $n \to \infty$ ∎

RMK 2: This result is extremely useful because it is often easier to show that sequence is cauchy than to show that it converges.

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exp: prove that any real sequence $\{x_n\}$ satisfies $\boxed{|x_n - x_{n+1}| \le \frac{1}{2}, n \in \mathbb{N}}$ * is conv.

proof: If $m > n$, then

$$|x_n - x_m| = |x_n - x_{n+1} + x_{n+1} - x_{n+2} + \cdots + x_{m-1} - x_m|$$

$$\le |x_n - x_{n+1}| + |x_{n+1} - x_{n+2}| + \cdots + |x_{m-1} - x_m|$$

using * $$\le \frac{1}{2^n} + \frac{1}{2^{n+1}} + \cdots + \frac{1}{2^{m-1}}$$

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$$= \frac{1}{2^{n-1}} \underbrace{\left[\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{m-n}}\right]}$$

Geometric series a_1 = First term $= \frac{1}{2}$, r = ratio $= \frac{1}{2}$

$$= \frac{1}{2^{n-1}} \left[\frac{a_1(1 - r^{m-n})}{1-r}\right]$$

$$= \frac{1}{2^{n-1}} \left[\frac{\frac{1}{2}\left(1 - \left(\frac{1}{2}\right)^{m-n}\right)}{1 - \frac{1}{2}}\right]$$

$$= \frac{1}{2^{n-1}} \left[1 - \frac{1}{2^{m-n}}\right] \quad \text{if } m > n$$

$$< \frac{1}{2^{n-1}}$$

It follows that $\boxed{|x_n - x_m| < \frac{1}{2^{n-1}}}$ $\forall\, m > n \ge 1 \rightarrow$ الفرق بين أي حدين

But given $\varepsilon > 0$ we can choose $N \in \mathbb{N}$ so large that $n \ge N \Rightarrow \frac{1}{2^{n-1}} < \varepsilon$.

We have proved that $\{x_n\}$ is cauchy.

By Thm 1, therefore it converges to some real number

∎

RMK 3 : A sequence that satisfies $x_{n+1} - x_n \to 0$ is not necessarily cauchy.

proof : consider the sequence $x_n := \log n$

$x_{n+1} - x_n = \log(n+1) - \log n = \log \frac{n+1}{n} \to \log 1 = 0$ as $n \to \infty$

$\{x_n\}$ cannot be cauchy because it doesnot conv.

$\left(\lim_{n\to\infty} x_n = \lim_{n\to\infty} \log n = \infty \right)$

RMK : Every Caushy seq. is bdd. $\boxed{\text{What about converse?}}$ → Not true

$x_n = (-1)^n$ $\quad |x_n| = 1$ $\quad x_n$ is bdd.

But x_n is not cauchy since it is not converges.